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# ON THE EQUATION $a(x^m-1)/(x-1)=b(y^n-1)/(y-1)$

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## 1.

The equation

$$\frac{x^m-1}{x-1} = \frac{y^n-1}{y-1}$$

has finitely many solutions in integers $m>1$, $n>1$, $x>1$, $y>1$ with $x \neq y$ if

(a) $x, y$ fixed (Thue [10], Siegel [9], Baker [1])
(b) $x, n$ fixed (Siegel [8], Schinzel [7], Coates [3])
(c) $m, n$ fixed and $(m-1, n-1)>2$ (Schinzel [6]).
(d) $m, n$ fixed (Davenport, Lewis and Schinzel [4]).

Let $P \geqq 2$. Denote by $S$ the set of all positive integers composed of primes not exceeding $P$. Then we shall generalise (a) as follows:

THEOREM 1. *The equation*

$$a\frac{x^m-1}{x-1} = b\frac{y^n-1}{y-1} \tag{1}$$

*has finitely many solutions in integers $m>1$, $n>1$, $x>1$, $y>1$, $a \geqq 1$, $b \geqq 1$ with $x$, $y$, $a$, $b$ in $S$, $(a,b)=1$ and $a(y-1) \neq b(x-1)$. Further bounds for $m$, $n$, $x$, $y$, $a$ and $b$ can be determined explicitly in terms of $P$.*

Thus there are only finitely many integers with all the digits equal to $a$ in their $x$-adic expansions and all the digits equal to $b$ in their $y$-adic expansions. For integers with digits equal to 0 and 1 that occur periodically in their $g$-adic expansions, theorem 1 gives the following:

COROLLARY. *Let $g>1$ and $g_1>1$ be multiplicatively independent integers. Then the equation*

$$\frac{g^{uv}-1}{g^u-1} = \frac{g_1^{u_1 v_1}-1}{g_1^{u_1}-1}$$

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*has finitely many solutions in integers* $u \geqq 1$, $u_1 \geqq 1$, $v > 1$, $v_1 > 1$. *Further bounds for* $u$, $u_1$, $v$, $v_1$ *can be given explicitly in terms of the greatest prime factor of* $gg_1$.

## 2.

The proof of theorem 1 depends on linear forms in logarithms. Let $\alpha_1, \ldots, \alpha_n$ be non zero rational numbers of heights not exceeding $A_1, \ldots, A_n$ respectively, where we assume that $A_j \geqq 3$ for $1 \leqq j \leqq n$. (The height of a rational number $m/n$ with $(m,n)=1$ is defined as $\max(|m|, |n|)$). Put

$$\Omega' = \prod_{j=1}^{n-1} \log A_j \quad \text{and} \quad \Omega = \Omega' \log A_n .$$

THEOREM A (Baker [2]). *There exist effectively computable absolute constants* $c_1 > 0$ *and* $c_2 > 0$ *such that the inequalities*

$$0 < |\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1| < \exp(-(c_1 n)^{c_2 n} \Omega \log \Omega' \log B)$$

*have no solution in rational integers* $b_1, \ldots, b_n$ *with absolute values at most* $B$ $(\geqq 2)$.

THEOREM B (van der Poorten [5]). *Let* $p > 0$ *be a prime number. There exist effectively computable absolute constants* $c_3 > 0$ *and* $c_4 > 0$ *such that the inequalities*

$$\infty > \operatorname{ord}_p(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1) > (c_3 n)^{c_4 n} \Omega (\log B)^2 \frac{p}{\log p}$$

*have no solution in rational integers* $b_1, \ldots, b_n$ *with absolute values at most* $B$ $(\geqq 2)$.

We note that we could have used older results in place of theorem A and B. In fact we shall apply theorem A with $n, A_1, \ldots, A_{n-1}$ fixed and theorem B with $n, p, A_1, \ldots, A_n$ fixed. Further we remark that we shall use theorem A thrice and theorem B twice in the proof of theorem 1.

## 3.

In this section, we shall prove theorem 1. Let $m, n, x, y, a, b$ be as in the theorem. Assume that they satisfy (1). Write

$$x = p_1^{x_1} \ldots p_s^{x_s}, \quad y = p_1^{y_1} \ldots p_s^{y_s},$$

$$a = p_1^{a_1} \ldots p_s^{a_s}, \quad b = p_1^{b_1} \ldots p_s^{b_s}.$$

Here $p_1,\ldots,p_s$ are primes $\leqq P$ and the exponents of $p_1,\ldots,p_s$ in the factorisation of $x, y, a, b$ are non negative integers not exceeding $2\log x$, $2\log y$, $2\log a$, $2\log b$ respectively. It is no loss of generality to assume that $m\geqq n$. Further the equation (1) gives

$$(2) \qquad ax^{m-1} < nby^{n-1},\ by^{n-1} < max^{m-1}\ .$$

Denote by $c_5, c_6, \ldots$ effectively computable positive constants depending only on $P$. Then we have:

LEMMA 1. $\max(\log a, \log b) \leqq c_5(\log(m\log x))^2$ .

PROOF. First we prove the inequality for $\log b$. Suppose that a prime $p$ divides $b$. Then, by (1), we have

$$\operatorname{ord}_p(b) \leqq \operatorname{ord}_p\left(a\frac{x^m-1}{x-1}\right) \leqq \operatorname{ord}_p(x^m-1)$$

$$= \operatorname{ord}_p(p_1^{mx_1}\ldots p_s^{mx_s}-1)\ .$$

Now we apply theorem B with $n=s\leqq P$, $A_1=A_2=\ldots=A_s=P$ and $B=2m\log x$ to the right hand side of the above inequality. We obtain

$$\operatorname{ord}_p(b) \leqq c_6(\log(m\log x))^2\ ,$$

and hence

$$\log b = \sum_{p\mid b}\operatorname{ord}_p(b)\log p \leqq c_7(\log(m\log x))^2\ .$$

Similarly

$$\log a \leqq c_8(\log(n\log y))^2\ .$$

In view of (2), the lemma follows immediately.

LEMMA 2.

$$(3) \qquad \min(\log x, \log y) \leqq c_9\log m\ .$$

PROOF. We prove the lemma when $x\leqq y$. The proof is similar for the case $x\geqq y$. Let $\delta$ be the smallest positive integer such that $ax^{m-\delta}\neq by^{n-\delta}$. Observe that $\delta\leqq n$, since $(a,b)=1$. Now it follows from (1) and (2) that

$$0 < |ax^{m-\delta}-by^{n-\delta}| \leqq max^{m-1-\delta}+nby^{n-1-\delta} \leqq 2m^2ax^{m-1-\delta}\ .$$

Thus we have

$$(4) \qquad 0 < |p_1^{u_1}\ldots p_s^{u_s}-1| \leqq 2m^2x^{-1}$$

where $u_i = b_i - a_i + (n-\delta)y_i - (m-\delta)x_i$ for $i = 1, \ldots, s$. By lemma 1 and (2), we find that the integers $|u_i|$ do not exceed $c_{10} m \log x$. Now apply theorem A with $n = s \leqq P$, $A_1 = A_2 = \ldots = A_s = P$ and $B = c_{10} m \log x$ to obtain

$$|p_1^{u_1} \ldots p_s^{u_s} - 1| > (m \log x)^{-c_{11}} . \tag{5}$$

Now by combining (5) and (4), we get (3). This completes the proof of lemma 2.

LEMMA 3. $\max(\log x, \log y) \leqq c_{12} (\log m)^2$.

PROOF. We prove the lemma when $x \leqq y$. The proof is similar for the case $x \geqq y$. From (1), lemma 1 and (2), we have

$$0 \neq \frac{ax^m}{x-1} - by^{n-1} = \frac{a}{x-1} + b(y^{n-2} + \ldots + 1)$$
$$\leqq a + nby^{n-2} \leqq y^{n-2} \exp\left(c_{13} (\log(n \log y))^2\right) .$$

Thus

$$0 \neq (p_1^{v_1} \ldots p_s^{v_s} (x-1)^{-1} - 1) \leqq y^{-1} \exp\left(c_{13} (\log(n \log y))^2\right) , \tag{6}$$

where $v_i = a_i - b_i + mx_i - (n-1)y_i$ for $i = 1, \ldots s$. From lemma 1 and (2), we observe that the absolute values of $v_i$ with $i = 1, \ldots, s$ do not exceed $c_{14} m \log x$ which, in view of lemma 2, is less than $c_{15} m \log m$. Now apply theorem A with $n = s+1 \leqq P+1$, $A_1 = A_2 = \ldots = A_s = P$, $A_{s+1} = x \leqq m^{c_9}$ and $B = c_{15} m \log m$ to conclude that

$$(p_1^{v_1} \ldots p_s^{v_s} (x-1)^{-1} - 1) \geqq \exp\left(-c_{16} (\log m)^2\right) . \tag{7}$$

Observe that we have used lemma 2 for a bound for $A_{s+1}$. Now the lemma follows immediately from (6) and (7).

PROOF OF THEOREM 1. From (1) and lemma 1, we have

$$0 \neq \left|\frac{ax^m}{x-1} - \frac{by^n}{y-1}\right| = \left|\frac{a}{x-1} - \frac{b}{y-1}\right| \leqq \exp\left(c_{17} (\log(m \log x))^2\right) .$$

Thus

$$0 < \left|p_1^{w_1} \ldots p_s^{w_s} \frac{x-1}{y-1} - 1\right| \leqq c_{18} x^{-m+\frac{3}{2}} , \tag{8}$$

where $w_i = b_i - a_i + ny_i - mx_i$. Observe that the absolute values of the integers $w_i$ do not exceed $c_{19} m \log m$. Now apply theorem A with $n = s+1 \leqq P+1$, $A_1 = A_2 = \ldots = A_s = P$, $B = c_{19} m \log m$ and $A_{s+1} = \max(x-1, y-1) \leqq \exp\left(c_{12} (\log m)^2\right)$.

The last inequality follows from lemma 3. We obtain

$$(9)\qquad \left|p_1^{w_1}\dots p_s^{w_s}\frac{x-1}{y-1}-1\right| > \exp\left(-c_{20}(\log m)^3\right).$$

Combining (9) and (8), we find that $m \leqq c_{21}$. Now the proof of the theorem is complete in view of lemma 2 and lemma 1.

REMARK. The proof of theorem 1 depends on three approximations that the equation (1) provides. If $a=1$, $b=1$, it is sufficient to use only one approximation. Rewriting the equation

$$x\frac{x^{m-1}-1}{x-1} = y\frac{y^{n-1}-1}{y-1},$$

observe that $\operatorname{ord}_p(x)=\operatorname{ord}_p(y)$ for every prime $p$ dividing $(x,y)$. Thus we can write $x=dx_1$, $y=dy_1$ with $(x_1,y_1)=(d,x_1)=(d,y_1)=1$. From the equation

$$x^{m-1}+\dots+x^n = (y-x)\left(\frac{y^{n-1}-x^{n-1}}{y-x}+\dots+1\right)$$

and theorem B, we show that $\log d \leqq c_{22}(\log\log y)^2$. Now using the equation

$$x_1\frac{x^{m-1}-1}{x-1} = y_1\frac{y^{n-1}-1}{y-1}$$

and theorem B along with the above bound for $\log d$, we find that $\log y \leqq c_{23}(\log m)^2$. This is lemma 3. Now use an approximation, as in theorem 1, to complete the proof.

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