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ARITHMETICAL QUADRATIC SURFACES OF GENUS 0, I

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Introduction.

Let A be a Dedekind ring with perfect residue class fields $k(\mathfrak{p})=A/\mathfrak{p}$ for all prime ideals $\mathfrak{p} \in S=\operatorname{Spec}(A)$, $\mathfrak{p} \neq (0)$. Let F be the fraction field of A, and E a finitely generated regular extension of F of genus 0. An S-scheme $\alpha: M \to S$ is a model of E/A if α is proper, dominant, and the induced map $\alpha^*: R(S) \to R(M)$ of the fields of rational functions on S and M is an injection of $F(=R(S))$ into $E(=R(M))$. A model $\alpha: M \to S$ is regular if the scheme M is regular. A regular model $\alpha: M \to S$ is called relatively minimal if every S-morphism $\varphi: M \to M'$, where $\alpha': M' \to S$ is a regular model of E/A, is an isomorphism.

Since A is a Dedekind ring, and the S-models M of E/A have dimension two, they may be called arithmetical surfaces.

The existence of relatively minimal models, when the extension E/F has arbitrary genus, was proved in [28]. If the genus is greater then 0, then all relatively minimal models of E/A are S-isomorphic ([28, p. 155]). The same result is true when the genus of E/F is 0 and A is a discrete valuation ring ([4]). However, for arbitrary Dedekind rings A and extensions E/F of genus 0, there are usually many non-isomorphic relatively minimal models ([9]).

The aim of this paper is to investigate some properties of birational maps between relatively minimal models of the extension E/A, or more generally, such maps between regular quadratic models of E/A, where a regular quadratic model is a regular model $\alpha: M \to S$ such that each fiber $M_{\mathfrak{p}}=M \times_S \operatorname{Spec}(k(\mathfrak{p}))$ is a form of a projective line or a form of two intersecting projective lines over $k(\mathfrak{p})$. Relatively minimal models have in fact this property, and a local description of such models by certain quadratic forms motivates the term "quadratic model" ([1], [2], [3] and Section 3). For a regular quadratic model M, the number of prime idelas $\mathfrak{p}$ of A such that $M_{\mathfrak{p}}$ is a form of two intersecting projective lines over $k(\mathfrak{p})$ is finite ([28, p. 123]). We shall denote by $\mathfrak{d}(M)$ the product of all such $\mathfrak{p}$. Note that two regular quadratic models M and M' are locally isomorphiic if and only if $\mathfrak{d}(M)=\mathfrak{d}(M')$ ([3, Theorem 3] when char $(F) \neq 2$ and Theorem (4.5) of Section 4 in the general case).

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If M and M' are regular models of E/A, then every birational map φ: $M \to M'$ is a composition of a finite number of blowing-ups (and blowing-downs) ([28, p. 55]). If M and M' are regular quadratic models and $\mathfrak{p} \nmid \mathfrak{d}(M)$, then we say that a birational map $\varphi: M \to M'$ (or simply M') is an elementary transformation of M if $\varphi = \tau\sigma^{-1}$, where $\sigma: M^* \to M$ is a blowing-up of M at a $k(\mathfrak{p})$-rational point $x \in M_{\mathfrak{p}}$, and the strict transform of x under σ is contracted to a point of M' by the blowing-up $\tau: M^* \to M'$. If M and M' are relatively minimal models of E/A, we prove (Theorem (4.11)) that there is "a path" of elementary transformations from M to M', which means that there is a sequence of relatively minimal models $M_0 = M, M_1, \ldots, M_n = M'$ such that M_{i+1} is an elementary transform of M_i for $0 \leqq i < n$.

This result and others concerning regular quadratic models of E/A are proved by using a one-to-one correspondence between the isomorphism classes of regular quadratic models of E/A and the isomorphism classes of hereditary A-orders in a quaternion algebra $Q(E/A)$ corresponding to the extension E/A (Section 4). We prove that if a regular quadratic model M of E/A and a hereditary A-order Λ correspond to each other, then for each $\mathfrak{p} \nmid \mathfrak{d}(M)$, there is a one-to-one correspondence between the $k(\mathfrak{p})$-rational points of the fiber $M_{\mathfrak{p}}$ and the integral (left) Λ-ideals with norm $\mathfrak{p}$. In this correspondence, elementary transformations at two points give isomorphic models if and only if the right orders of the left ideals corresponding to these points are isomorphic (Theorem (4.7)). If M is a regular quadratic model and A is a principal ideal ring, this result gives an arithmetical characterization of all prime ideals $\mathfrak{p} \nmid \mathfrak{d}(M)$ such that an elementary transformation at a $k(\mathfrak{p})$-rational point of M leads to a regular quadratic model isomorphic to M: $\mathfrak{p} = (p)$ has this property if there is $r \in A$, $(r) \mid \mathfrak{d}(M)$, such that rp is represented by a certain quaternary quadratic form (Theorem 4.9). If $A = \mathbf{Z}$ is the ring of integers, we get a connection with some old classical problems concerning representations of integers by integral quaternary quadratic forms (Section 5).

The correspondence between regular quadratic models of E/A and hereditary A-orders in $Q(E/A)$ goes via A-lattices on the quadratic space $Q_0(E/A)$ consisting of the quaternions whose trace is equal to 0, and with quadratic structure defined by the reduced norm on $Q(E/A)$ restricted to $Q_0(E/A)$. Various geometrical properties of quadratic models of E/A may be considered as arithmetical properties of A-lattices (quadratic forms) on the quadratic space $Q_0(E/A)$, or algebraical properties of A-orders in the quaternion algebra $Q(E/A)$. The last connection, between A-lattices on $Q_0(E/A)$ and A-orders in $Q(E/A)$, is well-known and has been investigated in many papers (e.g. [18], [5], [20], [11], [21]). We hope that the geometrical point of view will add a new dimension to these classical results.

Section 1 contains some technical facts concerning lattices on ternary quadratic spaces. In Section 2 we discuss relations between geometrical properties of models and arithmetical properties of lattices, and in Section 3 we go from lattices to orders. Section 4 contains proofs of the main results of the paper presented in this Introduction. In the last section we consider the case of global fields and some examples concerning integral quadratic forms.

1. Lattices on ternary quadratic spaces.

Let A be a Dedekind ring with quotient field F. Let (V,q) be a quadratic space over F, that is, $q\colon V \to F$ is a mapping satisfying $q(ax)=a^2q(x)$ for $a \in F$, and $b(x,y)=q(x+y)-q(x)-q(y)$ is a symmetric bilinear form. Note that $b(x,x)=2q(x)$. We shall often write (x,y) instead of $b(x,y)$. Let (V,q) be a ternary quadratic space. If $x_0,x_1,x_2 \in V$, we denote by $d(x_0,x_1,x_2)$ the determinant $\det[(x_i,x_j)]$, and by $d'(x_0,x_1,x_2)$ the determinant $(\frac{1}{2})\det[(x_i,x_j)]$, which in the case of char $(F)=2$ is understood as:

$$(x_0,x_1)(x_0,x_2)(x_1,x_2)+q(x_0)(x_1,x_2)^2+q(x_1)(x_0,x_2)^2+q(x_2)(x_0,x_1)^2 .$$

We say that (V,q) is regular if $d(e_0,e_1,e_2)\neq 0$, and half-regular if $d'(e_0,e_1,e_2)\neq 0$, where e_0,e_1,e_2 is a basis of V ([17, (1.15) and (2.14]). Throughout the whole paper, we shall assume that the ternary quadratic space (V,q) is half-regular.

By an A-lattice on V we mean a finitely generated projective A-module L contained in V and such that $FL=V$. The norm $\mathfrak{n}_A(L)$ of the lattice L is the A-ideal in F generated by $q(x)$ for $x \in L$. The volume $\mathfrak{v}_A(L)$ is the A-ideal in F generated by $d'(x_0,x_1,x_2)$, where x_0,x_1,x_2 are arbitrary elements of L. If L is a (free) lattice over a principal ideal ring A, e_0,e_1,e_2 a basis of L and $\mathfrak{n}_A(L)=(a)$, $a \in F^*$, then we say that the quadratic form:

$$q_L = (1/a)\Big(\sum_i q(e_i)X_i^2+\sum_{i<j} b(e_i,e_j)X_iX_j\Big) \tag{1.1}$$

corresponds to L.

By the reduced determinant of L, we mean the ideal

$$\mathfrak{d}_A(L) = \mathfrak{v}_A(L)\mathfrak{n}_A(L)^{-3} . \tag{1.2}$$

It is easy to see (e.g. by looking at the forms $q_{L_\mathfrak{p}}$ corresponding to the localizations $L_\mathfrak{p}$ of L over $A_\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec}(A)$), that $\mathfrak{d}_A(L)$ is an integral ideal of A.

The A-lattices L and L' on V are isometric if there is an isometry $\sigma\colon V \to V$ (that is, $q(\sigma x)=q(x)$ for $x \in V$) such that $\sigma L=L'$. L and L' are similar of there is an A-ideal $\mathfrak{a}$ in F such that L and $\mathfrak{a}L'$ are isometric. It is easy to check that the ideal $\mathfrak{d}_A(L)$ does not depend on the choice of a lattice in the similarity class of L. Moreover, if (V^a,q^a) denotes the quadratic space (V,q) scaled by $a \in F^*$, that is, $V^a=V$ and $q^a(x)=aq(x)$, then $\mathfrak{d}_A(L^a)=\mathfrak{d}_A(L)$, where L^a denotes the lattice L considered on the quadratic space (V^a,q^a).

If L and L' are A-lattices on V, then $[L\colon L']$ denotes the product of the invariant factors of L' in L, that is, if $\{e_i\}$ is a basis of V such that $L=\bigoplus \mathfrak{a}_i e_i$ and $L'=\bigoplus \mathfrak{a}'_i e_i$, where $\mathfrak{a}_i, \mathfrak{a}'_i$ are A-ideals in F ([19, 81:11]), then $[L\colon L']=\prod \mathfrak{a}'_i \mathfrak{a}_i^{-1}$. It is easy to check that:

$$(1.3) \qquad \mathfrak{v}_A(L') = [L\colon L']^2 \mathfrak{v}_A(L)\,.$$

Note that $L \supset L'$ and $[L\colon L']=A$ imply $L=L'$.

We say that L is a hereditary lattice if $\mathfrak{d}_A(L)$ is square-free. This terminology is justified by the fact that the hereditary A-lattices on (V,q) are in one-to-one correspondence with the hereditary A-orders in the even part of the Clifford algebra $C(V,q)$ of (V,q) (see (3.6)). Sometimes the hereditary order corresponding to such a lattice is maximal, and then it would be motivated to call the lattice maximal. However, according to the accepted terminology, a lattice L on V is maximal if $L' \supset L$ and $\mathfrak{n}_A(L')=\mathfrak{n}_A(L)$ imply $L'=L$. In fact, from (1.2) and (1.3), we get easily:

(1.4) Proposition. *A hereditary lattice is maximal.*

If the residue class fields $k(\mathfrak{p})=A/\mathfrak{p}$ are perfect for all prime ideals $\mathfrak{p} \in \operatorname{Spec}(A)$, $\mathfrak{p} \neq (0)$ then the converse of this statement is also true. If F is a global field and $\operatorname{char}(F) \neq 2$, then using the description of maximal lattices given in [13, Satz 9.5 and Satz 9.7], we get by a direct computation that $\mathfrak{d}_A(L)$ is square-free for any A-maximal lattice L. A similar proof can be given in the general case, but it is easier to give a proof based on the relations between hereditary lattices and hereditary orders developed in Section 3. In the sequel, we shall not use the fact that maximal lattices are hereditary.

In the remaining part of this section (with one clear exception), we shall assume that the ring A is a discrete valuation ring with maximal ideal $\mathfrak{p}=(\pi)$. If (V,q) is a quadratic space, we write (x,y) instead of $b(x,y)$. We shall omit the subscript A in $\mathfrak{n}_A, \mathfrak{v}_A, \mathfrak{d}_A$.

(1.5) Lemma. *Let $L^0 \subset L$ be two A-lattices on the quadratic space (V,q). Let $L^0 \not\subset \pi L$ and $\mathfrak{d}(L)=A$.*

(a) *If $\mathfrak{d}(L^0)=(\pi)$, then there is a basis e_0,e_1,e_2 of L and a natural number $a \geqq 1$ such that $L^0=Ae_0+A\pi^a e_1+A\pi^{2a-1}e_2$, $q(e_0) \in \pi^{2a-1}\mathfrak{n}(L)$ and $(e_0,e_1) \in \pi^{a-1}\mathfrak{n}(L)$. Moreover, $\mathfrak{n}(L^0)=\pi^{2a-1}\mathfrak{n}(L)$.*

(b) *If e_0,e_1,e_2 is a basis of L such that $q(e_0) \in \pi^{2a-1}\mathfrak{n}(L)$ and $(e_0,e_1) \in \pi^{a-1}\mathfrak{n}(L)$, then $L^0=Ae_0+A\pi^a e_1+A\pi^{2a-1}e_2$ is a hereditary sublattice of L and $\mathfrak{d}(L^0)=(\pi)$.*

Proof. (a) We can choose a basis e_0,e_1,e_2 of V such that

$$L = Ae_0+Ae_1+Ae_2, \quad L^0 = Ae_0+A\pi^a e_1+A\pi^b e_2 \quad \text{and} \quad a \leqq b$$

([19, 81:11]). Then $\mathfrak{v}(L^0)=\pi^{2(a+b)}\mathfrak{v}(L)$. Let $\mathfrak{n}(L^0)=\pi^c\mathfrak{n}(L)$, $c\geqq 0$. By (1.2), $\mathfrak{d}(L^0)=\pi^{2a+2b-3c}\mathfrak{d}(L)$, so $2a+2b=3c+1$. Since the possible values of c are $0, a, b, 2a, 2b$ or $a+b$, we get, using the last equality, that $c=b$. Hence $b=c=2a-1$ and $a\geqq 1$. Now $\mathfrak{n}(L^0)=\pi^{2a-1}\mathfrak{n}(L)$ implies $q(e_0)\in\pi^{2a-1}\mathfrak{n}(L)$ and $(e_0, e_1)\in\pi^{a-1}\mathfrak{n}(L)$.

(b) Let $L^0=Ae_0+A\pi^a e_1+A\pi^{2a-1}e_2$. Then $\mathfrak{v}(L^0)=\pi^{2(3a-1)}\mathfrak{v}(L)$ and $\mathfrak{n}(L^0)=\pi^c\mathfrak{n}(L)$, where, by the assumptions, $c\geqq 2a-1$. Since $\mathfrak{d}(L^0)=\pi^{6a-2-3c}\mathfrak{d}(L)$, we have $c\leqq 2a-1$. Therefore $c=2a-1$, so $\mathfrak{d}(L^0)=(\pi)$.

(1.6) Corollary. *Let A be complete relative to the $\mathfrak{p}$-adic topology. If there are hereditary A-lattices L and L^0 on V such that $\mathfrak{d}(L)=A$ and $\mathfrak{d}(L^0)=(\pi)$, then (V, q) is isotropic.*

Proof. Multiplying the lattice L^0 by a suitable A-ideal in F and scaling the quadratic space (V, q) by an element of F^*, we may assume that $L^0\not\subseteq\pi L$ and $\mathfrak{n}(L)=A$. By (1.5),

$$L = Ae_0+Ae_1+Ae_2, \qquad L^0 = Ae_0+A\pi^a e_1+A\pi^{2a-1}e_2,$$

where $a\geqq 1$, $q(e_0)\in(\pi^{2a-1})$, $(e_0, e_1)\in(\pi^{a-1})$ and $\mathfrak{n}(L^0)=(\pi^{2a-1})$. Let $q_L=\sum a_{ij}X_iX_j$, where $0\leqq i\leqq j\leqq 2$, $a_{ij}=(e_i, e_j)$ for $i\neq j$ and $a_{ii}=q(e_i)$. If $q_{L^0}=\sum b_{ij}X_iX_j$, then $b_{00}=\pi^{-(2a-1)}a_{00}$, $b_{11}=\pi a_{11}$, $b_{22}=\pi^{2a-1}a_{22}$, $b_{01}=\pi^{-(a-1)}a_{01}$, $b_{02}=a_{02}$ and $b_{12}=\pi^a a_{12}$. Suppose $\pi\mid b_{01}$ and $\pi\mid b_{02}$. Then $\pi\mid a_{0i}$ for $i=0,1,2$, and the assumption $\mathfrak{d}(L)=A$ gives a contradiction. Therefore $\pi\nmid b_{01}$ or $\pi\nmid b_{02}$. Since

$$q_{L^0} = b_{00}X_0^2+b_{01}X_0X_1+b_{02}X_0X_2 \pmod{\pi},$$

q_{L^0} has non-trivial zeros $(0,1,0)$ and $(0,0,1)$ over $k(\mathfrak{p})$. By the Hensel lemma, at least one of these zeros can be lifted to a non-trivial zero of q_{L^0} over A.

If L is an A-lattice on the quadratic space (V, q), denote by $P^2(L/\mathfrak{p}L)$ the projective $k(\mathfrak{p})$-space defined by the linear $k(\mathfrak{p})$-space $L/\mathfrak{p}L$. The image of $x\in L$ in $L/\mathfrak{p}L$ will be denoted by x^*, and if $x\in L\setminus\mathfrak{p}L$, the corresponding point of $P^2(L/\mathfrak{p}L)$ by $[x^*]$. There is a quadratic structure on the linear space $L/\mathfrak{p}L$ defined by $q_{\mathfrak{p}}(x^*)=(1/\pi^r)q(x) \pmod{\pi}$, where $(\pi^r)=\mathfrak{n}(L)$. We say that a point $[x^*]$ of $P^2(L/\mathfrak{p}L)$ is isotropic if x^* is a (non-trivial) zero of the form $q_{\mathfrak{p}}$. Note that $x\in L$ defines an isotropic point of $P^2(L/\mathfrak{p}L)$ if and only if $x\in L\setminus\mathfrak{p}L$ and $q(x)\in\mathfrak{p}\mathfrak{n}(L)$.

(1.7) Proposition. *Let L be a hereditary lattice on the quadratic space (V, q) and $\mathfrak{d}(L)=A$.*

(a) *If L^0 is a hereditary sublattice of L and $[L: L^0]=\mathfrak{p}^2$, then there is a basis e_0, e_1, e_2 of L such that $L^0=Ae_0+A\pi e_1+A\pi e_2$ and $q(e_0)\in\mathfrak{p}\mathfrak{n}(L)$. Moreover, $\mathfrak{n}(L^0)=\mathfrak{p}\mathfrak{n}(L)$ and $\mathfrak{d}(L^0)=\mathfrak{p}$.*

(b) *If* $L = Ae_0 + Ae_1 + Ae_2$ *and* $q(e_0) \in \mathfrak{p}\mathfrak{n}(L)$, *then* $L^0 = Ae_0 + A\pi e_1 + A\pi e_2$ *is a hereditary sublattice of* L *and* $[L\colon L^0] = \mathfrak{p}^2$.

(c) (a) *and* (b) *define a one-to-one correspondence between the hereditary sublattices* L^0 *of* L *such that* $[L\colon L^0] = \mathfrak{p}^2$ *and the isotropic points of* $P^2(L/\mathfrak{p}L)$.

PROOF. (a) Let e_0, e_1, e_2 be a basis of L such that $L^0 = Ae_0 + A\pi^a e_1 + A\pi^b e_2$, where $a \leqq b$ and $a + b = 2$. If $\mathfrak{n}(L) = \pi^c \mathfrak{n}(L^0)$, then $\mathfrak{d}(L^0) = \pi^{4-3c}\mathfrak{d}(L)$. Hence $c = 1$, so we can apply the first part of Lemma (1.5).

(b) Follows immediately from the second part of Lemma (1.5).

(c) If L^0 is a hereditary sublattice of L, then by (a), e_0 defines an isotropic point of $P^2(L/\mathfrak{p}L)$. It is easy to check that another choice of bases of L and L^0 defines the same point of $P^2(L/\mathfrak{p}L)$.

Conversely, if $[x^*]$ is an isotropic point of $P^2(L/\mathfrak{p}L)$, then $x \in L \setminus \mathfrak{p}L$ and $q(x) \in \mathfrak{p}\mathfrak{n}(L)$. Hence there is a basis $e_0 = x, e_1, e_2$ of L and by (b), e_0 defines a hereditary sublattice L^0 of L. We check that L^0 does not depend on the choice of e_1 and e_2, as well as on the choice of another element $x' \in L$ defining the point $[x^*]$ of $P^2(L/\mathfrak{p}L)$.

Now the desired result follows easily.

We end this section with a result concerning ternary lattices over arbitrary Dedekind rings. We already use this result in the next section, but we prove it later (see (3.12)).

(1.8) LEMMA. *Let* $L \supset L^0$ *be hereditary* A*-lattices on the quadratic space* (V, q). *If* $[L\colon L^0] \doteq \mathfrak{p}^2$, *then there is precisely one hereditary lattice* $L' \neq L$ *such that* $L' \supset L$ *and* $[L'\colon L^0] = \mathfrak{p}^2$. *Moreover,* $\mathfrak{d}(L') = \mathfrak{d}(L)$.

2. Lattices and models.

Let A be a Dedekind ring with quotient field F, and (V, q) a half-regular ternary quadratic space over F. We denote by $\tilde{X}$ the sheaf of $\tilde{A}$-modules associated with an A-module X, where $\tilde{A}$ denotes the structure sheaf on $S = \operatorname{Spec}(A)$. Let L be an A-lattice on (V, q), and $\mathfrak{a}$ an A-ideal in F. If U is an open subset of S, we may assume that

$$\tilde{L}(U) = \bigcap_{\mathfrak{p} \in U} L_{\mathfrak{p}} \quad \text{and} \quad \tilde{\mathfrak{a}}(U) = \bigcap_{\mathfrak{p} \in U} \mathfrak{a}_{\mathfrak{p}},$$

where all the localizations $L_{\mathfrak{p}}$ (respectively $\mathfrak{a}_{\mathfrak{p}}$) are contained in V (respectively F).

The lattice L defines an isomorphisms class of projective S-schemes in the following way. We choose an open covering of S such that for each open set U of this covering both $\tilde{L}(U)$ and $n(L)\tilde{}(U)$ are free $\tilde{A}(U)$-modules. The quadratic form q_U, corresponding by (1.1) to a basis of $\tilde{L}(U)$ over $\tilde{A}(U)$, belongs to the polynomial ring $\tilde{A}(U)[X]$, where X stands for X_0, X_1, X_2, and defines a projective $\tilde{A}(U)$-scheme.

$$\text{(2.1)} \qquad \alpha_U\colon \operatorname{Proj}(\tilde{A}(U)[X]/(q_U)) \to \operatorname{Spec}(\tilde{A}(U)),$$

where α_U is induced by the natural injection $\tilde{A}(U) \to \tilde{A}(U)[X]/(q_U)$. If U and U' are two arbitrary open sets of the covering, there is a natural $\tilde{A}(U \cap U')$-isomorphism $\varphi_{UU'}$ of $\operatorname{Proj}(\tilde{A}(U \cap U')[X]/(q_U))$ onto $\operatorname{Proj}(\tilde{A}(U \cap U')[X]/(q_{U'}))$ induced by the equivalence of the quadratic forms q_U and $q_{U'}$ over $\tilde{A}(U \cap U')$. Now we can glue the morphisms (2.1) along the isomorphisms $\varphi_{UU'}$. We shall denote by

$$\text{(2.2)} \qquad \alpha_L\colon M(L) \to \operatorname{Spec}(A)$$

any $\operatorname{Spec}(A)$-scheme obtained in this way.

Let us note that if (V,q) is a half-regular ternary quadratic space, the field $E = R(M(L))$ of rational functions on $M(L)$ is a regular finitely generated extension of genus 0 of the field F, and (2.2) is a model of E/A.

The S-scheme $M(L)$ need not be regular, but we have:

(2.3) PROPOSITION. *$M(L)$ is regular if and only if L is hereditary.*

PROOF. [3, Theorem 2].

Now let L_1 and L_2 be two A-lattices on (V,q) and let $L_1 \supset L_2$. We may choose an open covering of S such that for each open set U of this covering, $\tilde{L}_i(U)$ and $\mathfrak{n}(L_i)\tilde{}(U)$, where $i = 1, 2$, are free $\tilde{A}(U)$-lattices. Let $M(L_i)$ be the S-scheme obtained by glueing the projective schemes corresponding to a choice of a basis of $\tilde{L}_i(U)$ for each U. If U_0 is an open set of the covering, the matrix expressing the chosen basis of $\tilde{L}_2(U_0)$ by the chosen basis of $\tilde{L}_1(U_0)$ defines a graded homomorphism of graded rings, $\tilde{A}(U_0)[X]/(q^{(1)}_{U_0})$ into $\tilde{A}(U_0)[X]/(q^{(2)}_{U_0})$, and hence a birational map

$$\text{(2.4)} \qquad \operatorname{Proj}(\tilde{A}(U_0)[X]/(q^{(2)}_{U_0})) \to \operatorname{Proj}(\tilde{A}(U_0)[X]/(q^{(1)}_{U_0})).$$

We shall say that the birational S-map from $M(L_2)$ to $M(L_1)$, defined in this way, corresponds to the pair of lattices L_1, L_2.

(2.5) PROPOSITION. *Let $M(L)$ be a regular S-scheme defined by a hereditary A-lattice L on the ternary quadratic space (V,q). Let $\mathfrak{p} \in S$, $\mathfrak{p} \neq 0$ and $\mathfrak{p} \nmid \mathfrak{d}_A(L)$.*

(a) *There is a one-to-one correspondence between the $k(\mathfrak{p})$-rational points of the fiber $M(L)_\mathfrak{p}$ and the hereditary sublattices $L^0 \subset L$ such that $[L\colon L^0] = \mathfrak{p}^2$.*

(b) *If L^0 is a hereditary sublattice of L corresponding by* (a) *to a $k(\mathfrak{p})$-rational point x of $M(L)_\mathfrak{p}$, then $M(L^0)$ is a blowing-up of $M(L)$ at the point x.*

(c) *Let L' be the second hereditary lattice on (V, q) such that $L' \neq L$, $L' \supset L^0$ and $[L'\colon L^0] = \mathfrak{p}^2$* (see (1.8)). *Let $\sigma\colon M(L^0) \to M(L)$ and $\sigma'\colon M(L^0) \to M(L')$ be blowing-ups corresponding by* (b) *to the pairs of lattices L^0, L and L^0, L'. Then $\sigma'\sigma^{-1}$ is an elementary transformation of $M(L)$ at a $k(\mathfrak{p})$-rational point of $M(L)_\mathfrak{p}$.*

PROOF. (a) Let $q_{L_\mathfrak{p}}$ be a quadratic form corresponding to $L_\mathfrak{p}$. Since $M(L)_\mathfrak{p} \cong \operatorname{Proj}(k(p)[X]/(q^*_{L_\mathfrak{p}}))$, where $q^*_{L_\mathfrak{p}}$ is the image of $q_{L_\mathfrak{p}}$ under the canonical homomorphism $A[X] \to k(\mathfrak{p})[X]$, there is a one-to-one correspondence between the $k(\mathfrak{p})$-rational points of $M(L)_\mathfrak{p}$ and the isotropic points of $P^2(L_\mathfrak{p}/\mathfrak{p}_{L_8})$. By (1.7) (c) the isotropic points of $P^2(L_\mathfrak{p}/\mathfrak{p}_{L_8})$ are in one-to-one correspondence with the hereditary sublattices $L^0_{(\mathfrak{p})} \subset L_\mathfrak{p}$ such that $[L_\mathfrak{p}\colon L^0_{(\mathfrak{p})}] = \mathfrak{p}^2 A_\mathfrak{p}$. Now, $L^0_{(\mathfrak{p})}$ over $A_\mathfrak{p}$ uniquely defines a lattice L^0 over A such that $L^0_\mathfrak{p} = L_{(\mathfrak{p})}$ and $L^0_\mathfrak{q} = L_\mathfrak{q}$ for $\mathfrak{q} \in S$, $\mathfrak{q} \neq \mathfrak{p}$. Thus L^0 is a hereditary A-lattice and $[L\colon L^0] = \mathfrak{p}^2$. Conversely, each hereditary sublattice L^0 of L such that $[L\colon L^0] = \mathfrak{p}^2$ (uniquely) defines its localization $L^0_\mathfrak{p} \subset L_\mathfrak{p}$, which is a hereditary sublattice of $L_\mathfrak{p}$ such that $[L_\mathfrak{p}\colon L^0_\mathfrak{p}] = \mathfrak{p}^2 A_\mathfrak{p}$.

(b) Let e_0, e_1, e_2 be a basis of V such that $L = \mathfrak{a}_0 e_0 + \mathfrak{a}_1 e_1 + \mathfrak{a}_2 e_2$ and $L^0 = \mathfrak{a}_0 e_0 + \mathfrak{a}_1 \mathfrak{p} e_1 + \mathfrak{a}_2 \mathfrak{p} e_2$ ([19, Theorem 81:11] and Proposition (1.7)). We may assume that $L_\mathfrak{p} = A_\mathfrak{p} e_0 + A_\mathfrak{p} e_1 + A_\mathfrak{p} e_2$, and then,

$$L^0_\mathfrak{p} = A_\mathfrak{p} e_0 + A_\mathfrak{p} \pi e_1 + A_\mathfrak{p} \pi e_2, \quad \text{where } \mathfrak{p} A_\mathfrak{p} = (\pi).$$

Define $\alpha_L\colon M(L) \to S$ and $\alpha_{L^0}\colon M(L^0) \to S$ by an open covering of S containing a neighbourhood U_0 of $\mathfrak{p}$ such that e_0, e_1, e_2 is a basis of $\tilde{L}(U_0)$, and $e_0, \pi e_1, \pi e_2$ is a basis of $\tilde{L}^0(U_0)$. Let $\sigma\colon M(L^0) \to M(L)$ be the S-birational map defined on an open subset of $\alpha_{L^0}^{-1}(U_0)$ by the chosen bases of $\tilde{L}(U_0)$ and $\tilde{L}^0(U_0)$. If $\mathfrak{q} \in S$, $\mathfrak{q} \neq \mathfrak{p}$, then $L^0_\mathfrak{q} = L_\mathfrak{q}$, so there is a neighbourhood $U_\mathfrak{q}$ of $\mathfrak{q}$ such that $\sigma \mid \alpha_{L^0}^{-1}(U_\mathfrak{q})$ is represented by an isomorphism of $\alpha_{L^0}^{-1}(U_\mathfrak{q})$ onto $\alpha_L^{-1}(U_\mathfrak{q})$. We have to examine σ in a neighbourhood of $\alpha_{L^0}^{-1}(\mathfrak{p})$, so we may assume that A is a discrete valuation ring, $L = Ae_0 + Ae_1 + Ae_2$ and $L^0 = Ae_0 + A\pi e_1 + A\pi e_2$, where $\mathfrak{p} = (\pi)$ is the maximal ideal of A. Let

$$M(L) = \operatorname{Proj}(A[x_0, x_1, x_2]) = \operatorname{Proj}(A[X_0, X_1, X_2]/(q_L))$$

and

$$M(L^0) = \operatorname{Proj}(A[y_0, y_1, y_2]) = \operatorname{Proj}(A[Y_0, Y_1, Y_2]/(q_{L^0})),$$

where q_L and q_{L^0} are the quadratic forms corresponding to the chosen bases of L and L^0. We have to show that the birational map $\sigma\colon M(L^0) \to M(L)$ induced by the homomorphism $\sigma_0\colon A[x_0, x_1, x_2] \to A[y_0, y_1, y_2]$ such that

$$(2.6) \qquad \sigma_0(x_0) = y_0, \quad \sigma_0(x_1) = \pi y_1, \quad \sigma_0(x_2) = \pi y_2$$

is a blowing-up of $M(L)$ at the point $x \in M(L)_\mathfrak{p}$ corresponding to the sublattice L^0 of L.

Let $a_{ij}=b(e_i,e_j)$, for $i\neq j$, and $a_{ii}=q(e_i)$. It follows from (1.7) (a) that $\pi\mid a_{00}$. Let $a_{00}=\pi b_{00}$, $b_{00}\in A$. We have

$$q_{L^0} = b_{00}Y_0^2+a_{01}Y_0Y_1+a_{02}Y_0Y_2+\pi g(Y_1,Y_2)\,, \tag{2.7}$$

where $g\in A[Y_0,Y_1,Y_2]$. Since $(\pi)\nmid \mathfrak{d}_A(L)$, we get $\pi\nmid a_{01}$ or $\pi\nmid a_{02}$. Thus the fiber $M(L)_\mathfrak{p}$ is an intersection of two projective lines corresponding to the ideals (π,y_0) and $(\pi,b_{00}y_0+a_{01}y_1+a_{02}y_2)$ of $A[y_0,y_1,y_2]$.

It is easily verified that σ is regular at all points of $M(L^0)$ not on the line (π,y_0). Using (2.6), the equality $y_0(b_{00}y_0+a_{01}y_1+a_{02}y_2)=\pi g(y_1,y_2)$ in $A[y_0,y_1,y_2]$, and the fact that $(\pi)\nmid\mathfrak{d}_A(L)$, it is a matter of easy computations to show that σ is also regular on that line.

Now we check that there is only one point at which σ^{-1} is not regular, namely, the point (π,x_1,x_2), and that this point is just the image under σ of the projective line $(\pi,b_{00}y_0+a_{01}y_1+a_{02}y_2)$.

It follows from (1.7) (c) that the $k(\mathfrak{p})$-rational point $x\in M(L)_\mathfrak{p}$ corresponding to L^0 is defined by the zero $(1,0,0)$ of q_L^*, that is, by the ideal (π,x_1,x_2) of $A[x_0,x_1,x_2]$. Hence σ is a blowing-up of $M(L)$ at the $k(\mathfrak{p})$-rational point $x\in M(L)_\mathfrak{p}$ corresponding to L^0.

(c) Let $L'=Ae_0'+Ae_1'+Ae_2'$, $L^0=Ae_0'+A\pi e_1'+A\pi e_2'$ and

$$(e_0,\pi e_1,\pi e_2) = (e_0',\pi e_1',\pi e_2')P\,, \tag{2.8}$$

where P is an invertible matrix with elements in A. In the basis $e_0,\pi e_1,\pi e_2$, the two primes in the fiber $M(L^0)_\mathfrak{p}$ are described by the points (y_0,y_1,y_2) such that $b_{00}y_0+a_{01}y_1+a_{02}y_2=0$ (the prime blown down to a point of $M(L)$) and $y_0=0$ (the second prime). In the basis $e_0',\pi e_1',\pi e_2'$, the same two primes are described by the points (y_0',y_1',y_2') such that $b_{00}'y_0'+a_{01}'y_1'+a_{02}'y_2'=0$ and $y_0'=0$, where a_{ij}', b_{00}' are defined for $L^0\subset L'$ as a_{ij}, b_{00} in (2.7) for $L^0\subset L$. We want to show that the prime blown down to a point by σ' is not the same as the one blown down to a point by σ. If they are equal, the sets of points of L characterized by $y_0=0$ and $y_0'=0$ are equal. Then, by elementary considerations, we get from (2.8) that e_i can be expressed over A by e_i', that is, $L\subset L'$. By symmetry, we get $L=L'$, which is impossible.

3. Lattices and orders.

Recall that if (V,q) is a quadratic space over the field F, then the Clifford algebra $C(V,q)$ of (V,q) is the F-algebra $T(V)/I$, where $T(V)=\oplus_{i\geqq 0}T^i(V)$ is the tensor algebra of V, and I is the ideal of $T(V)$ generated by the elements $x\otimes x-q(x)$, $x\in V$. We are interested in the subalgebra $C_0(V,q)$ of $C(V,q)$ which is generated by $1\in T^0(V)=F$ and the images in $C(V,q)$ of the products $x_1\otimes\ldots\otimes x_{2r}\in T^{2r}(V)$, for $r>0$, which will be denoted by $[x_1,\ldots,x_{2r}]$.

Let, as earlier, A be a Dedekind ring with quotient field F and L an A-lattice on the quadratic space (V, q) over F. We shall denote by $\mathfrak{O}(L)$ the A-order in $C_0(V, q)$ (that is, a subring of $C_0(V, q)$ containing A an finitely generated as an A-module) corresponding to L. $\mathfrak{O}(L)$ is generated as an A-module by the elements 1 and $a[x_1, \ldots, x_{2r}]$, where $x_i \in L$, $a \in F$ and $a\mathfrak{n}_A(L)^r \subset A$ ([13, Satz 14.1]).

(3.1) Proposition. *If L and L' are two lattices on (V, q), then $\mathfrak{O}(L) = \mathfrak{O}(L')$ if and only if $L' = \mathfrak{a}L$, where $\mathfrak{a}$ is an A-ideal in F.*

Proof. [13, Satz 14.1].

If (V, q) is a half-regular ternary quadratic space over F, then $Q = C_0(V, q)$ is a generalized quaternion algebra over F, that is, Q is a central simple F-algebra of dimension four ([17, (6.11) and (5.21)]). Q has an involution $a \mapsto a^*$ such that the trace $T(a) = a + a^*$ and the (reduced) norm $N(a) = aa^*$ are elements of F([17, (5.9)]). If X is an A-lattice in Q, we denote by $N(X)$ the A-ideal in F generated by the norms $N(x)$, where $x \in X$.

Let Λ be an A-order in Q. Since the discriminant of any basis of Q is a square of an element of F, the discriminant of Λ is a square of an ideal of A ([24, pp. 218, 221]). Denote by $\mathfrak{d}_A(\Lambda)$ (or simply $\mathfrak{d}(\Lambda)$) the squareroot of the discriminant of Λ. By $\mathfrak{n}_A(\Lambda)$ (or simply $\mathfrak{n}(\Lambda)$) we denote the ideal $N(\Lambda^{\#})^{-1}$, where $\Lambda^{\#}$ is the complementary ideal of Λ, that is, the set of $x \in Q$ such that $T(\Lambda x) \subseteq A$.

If Λ and Λ' are A-orders in Q, then

$$\mathfrak{d}(\Lambda) = [\Lambda' \colon \Lambda]\mathfrak{d}(\Lambda') \tag{3.2}$$

where $[\Lambda' \colon \Lambda]$ denotes, as earlier in (1.3), the product of invariant factors of Λ in Λ', both considered as A-lattices.

Let Q_0 be the subspace of Q consisting of all $x \in Q$ such that $T(x) = 0$. We shall consider Q_0 with quadratic structure defined by the norme N restricted to Q_0. It is easy to check that (Q_0, N) is half-regular, and $C_0(Q_0, N) \cong Q$, where an isomorphism is given by $[x, y] \to xy^*$ ([21, p. 343]).

If Λ is an A-lattice on Q_0, then $\mathfrak{O}(L)$ is an A-order in Q and

$$\mathfrak{d}(\mathfrak{O}(L)) = \mathfrak{n}(\mathfrak{O}(L)) = \mathfrak{d}(L) \tag{3.3}$$

([21, Satz 7] and [29]).

If Λ is an A-order in Q, we define the corresponding A-lattice on Q_0 by

$$\mathfrak{L}(\Lambda) = \mathfrak{n}(\Lambda)(Q_0 \cap \Lambda^{\#})\,. \tag{3.4}$$

It is easy to check that

$$\mathfrak{n}(\mathfrak{L}(\Lambda)) = \mathfrak{n}(\Lambda) \quad \text{and} \quad \mathfrak{d}(\mathfrak{L}(\Lambda)) = \mathfrak{n}(\Lambda)^3\mathfrak{d}(\Lambda)^{-2}\,. \tag{3.5}$$

Recall that Λ is a hereditary order if every (left) ideal of Λ is Λ-projective.

(3.6) PROPOSITION. (a) *The maps*

$$L \to \mathfrak{O}(L) \quad \textit{and} \quad \Lambda \to \mathfrak{L}(\Lambda)$$

define a one-to-one correspondence between the A-lattices on Q_0 such that $\mathfrak{d}(L) = \mathfrak{n}(L)$ and the A-orders in Q such that $\mathfrak{d}(\Lambda) = \mathfrak{n}(\Lambda)$.

(b) *If Λ is a hereditary order in Q, then $\mathfrak{d}(\Lambda) = \mathfrak{n}(\Lambda)$. Moreover, $\mathfrak{d}(\Lambda)$ is a square-free ideal of A.*

(c) *A lattice L on Q_0 is hereditary if and only if the order $\mathfrak{O}(L)$ in Q is hereditary.*

(3.7) REMARK. If Λ is an A-order in Q, then $\mathfrak{d}(\Lambda) = \mathfrak{n}(\Lambda)$ if and only if Λ is a Gorenstein order ([29, Proposition 2.4]).

(3.8) REMARK. Since the discriminant of any basis of Q is a square of an element of F, the volume $\mathfrak{v}(L)$ of an A-lattice L on Q_0, defined in Section 1, is a square of an A-ideal in F. Using this fact and the definition of $\mathfrak{d}(L)$, we get easily that for each A-lattice L on Q_0, there is an A-ideal $\mathfrak{a}$ in F such that $\mathfrak{d}(\mathfrak{a}L) = \mathfrak{n}(\mathfrak{a}L)$.

PROOF. (a) If L is an A-lattice, then $\mathfrak{O}(L)$ is an A-order satisfying (3.3). If L and L' are two A-lattices such that $\mathfrak{d}(L) = \mathfrak{n}(L)$ and $\mathfrak{d}(L') = \mathfrak{n}(L')$, then by (3.1), the equality $\mathfrak{O}(L') = \mathfrak{O}(L)$ implies $L' = \mathfrak{a}L$, where $\mathfrak{a}$ is an A-ideal in F. Since $\mathfrak{d}(L') = \mathfrak{n}(L') = \mathfrak{a}^2\mathfrak{n}(L) = \mathfrak{a}^2\mathfrak{d}(L)$ and $\mathfrak{d}(L') = \mathfrak{d}(L)$, we get $\mathfrak{a} = A$, that is, $L' = L$. If Λ is an A-order such that $\mathfrak{d}(\Lambda) = \mathfrak{n}(\Lambda)$, then $L = \mathfrak{n}(L)(Q_0 \cap \Lambda^*)$ is an A-lattice on Q_0, and $\mathfrak{d}(L) = \mathfrak{n}(L)$ by (3.5). Moreover, $\Lambda = \mathfrak{O}(L)$ by Theorem 3.4 of [29].

(b) We may suppose that A is a complete discrete valuation ring with maximal ideal $\mathfrak{p} = (\pi)$ ([24, (40.5)). If Λ is maximal, then $\mathfrak{d}(\Lambda) = \mathfrak{n}(\Lambda) = A$ or (π) ([24, (25.2) and (25.4)]). If Λ is not maximal, then $\Lambda \subset \Lambda'$, where Λ' is a maximal order. In that case, Q can not be a skewfield, so Q is isomorphic to the matrix algebra $M_2(F)$. Since Λ is a hereditary, non-maximal order, Λ is isomorphic to the order Λ^0 consisting of all the matrices

$$\begin{bmatrix} a & b \\ \pi c & d \end{bmatrix}, \tag{3.9}$$

where $a,b,c,d \in A$ ([24, (39.14)]). By an easy computation, we get $\mathfrak{d}(\Lambda^0) = \mathfrak{n}(\Lambda^0) = (\pi)$, so the same is true for Λ. Note that $\mathfrak{d}(\Lambda)$ is square-free.

(c) If $\mathfrak{O}(L)$ is hereditary, then $\mathfrak{d}(\mathfrak{O}(L))$ is square-free by (b). But by (3.3), $\mathfrak{d}(L) = \mathfrak{d}(\mathfrak{O}(L))$, so L is a hereditary lattice.

Conversely, let L be a hereditary lattice. We may assume that $\mathfrak{d}(L) = \mathfrak{n}(L)$

((3.8)) and that A is a complete discrete valuation ring with maximal ideal $\mathfrak{p}=(\pi)$ ([24, (40.5)]). Note that $\mathfrak{d}(\Lambda)=A$ or (π), where $\Lambda=\mathfrak{O}(L)$. If Λ is maximal, then it is hereditary. Assume Λ is not maximal, and Λ' is a maximal order containing it. By (3.2), $\mathfrak{d}(\Lambda')=A$ and $\mathfrak{d}(\Lambda)=(\pi)$. Let $L'=\mathfrak{L}(\Lambda')$. We have $\mathfrak{d}(L')=A$ and $\mathfrak{d}(L)=(\pi)$. By (1.6), (Q_0,N) is isotropic, so Q is isomorphic to the matrix algebra $M_2(F)$ ([17, (5.21)]). Let Λ^0 be the hereditary order of all matrices (3.9), and $L^0=\mathfrak{L}(\Lambda^0)$. We have $\mathfrak{d}(L)=\mathfrak{n}(L)=(\pi)$ and $\mathfrak{d}(L^0)=\mathfrak{n}(L^0)=(\pi)$, so the lattices L and L^0 are maximal (in the sense of (1.4)) and have the same norm. Therefore L and L^0 are isometric ([17, (15.6)]), which implies that the orders $\mathfrak{O}(L)$ and $\mathfrak{O}(L^0)$ are isomorphic. This proves that $\mathfrak{O}(L)$ is hereditary.

(3.10) Corollary. *Two A-lattices L and L' on Q_0 are similar if and only if the A-orders $\mathfrak{O}(L)$ and $\mathfrak{O}(L')$ in Q are isomorphic.*

Proof. If L and L' are similar, then $\mathfrak{O}(L)$ *and* $\mathfrak{O}(L')$ are isomorphic by (3.1) and [13, Satz 14.3].

Conversely, let $\Lambda=\mathfrak{O}(L)$ and $\Lambda'=\mathfrak{O}(L')$ be two isomorphic orders. By (3.1) and (3.8), we may assume that $\mathfrak{d}(L)=\mathfrak{n}(L)$ and $\mathfrak{d}(L')=\mathfrak{n}(L')$. If $\Lambda'=a\Lambda a^{-1}$, then $L'=aLa^{-1}$ by (3.4) and (3.6) (a). Since the mapping $x\to axa^{-1}$, $x\in Q$ induces an isometry of (Q_0,N), we get the desired result.

(3.11) Proposition. (a) *If L^0 and L are hereditary A-lattices on Q_0 such that $L^0\subset L$ and $[L\colon L^0]=\mathfrak{p}^2$, then $\mathfrak{O}(L^0)\subset\mathfrak{O}(L)$ and $[\mathfrak{O}(L)\colon \mathfrak{O}(L^0)]=\mathfrak{p}$,*

(b) *If Λ^0 and Λ are hereditary A-orders in Q such that $\Lambda^0\subset\Lambda$ and $[\Lambda\colon \Lambda^0]=\mathfrak{p}$, then $\mathfrak{L}(\Lambda^0)\subset\mathfrak{L}(\Lambda)$ and $[\mathfrak{L}(\Lambda)\colon \mathfrak{L}(\Lambda^0)]=\mathfrak{p}^2$.*

Proof. (a) Since $\mathfrak{d}(L^0)=\mathfrak{d}(L)\mathfrak{p}$ by (1.7) (a), the equality $[\mathfrak{O}(L)\colon \mathfrak{O}(L^0)]=\mathfrak{p}$ follows from (3.2) and (3.3). By (3.8) we can assume that $\mathfrak{d}(L)=\mathfrak{n}(L)$. Since $L^0_\mathfrak{q}=L_\mathfrak{q}$ when $\mathfrak{q}\neq\mathfrak{p}$, we have only to prove that $\mathfrak{O}(L^0)_\mathfrak{p}\subset\mathfrak{O}(L)_\mathfrak{p}$, so we may assume that A is a discrete valuation ring with maximal ideal $\mathfrak{p}=(\pi)$. Then by (1.7) (a),

$$L=Ae_0+Ae_1+Ae_2,\qquad L^0=Ae_0+A\pi e_1+A\pi e_2\,,$$

$\mathfrak{n}(L)=A$ and $\mathfrak{n}(L^0)=(\pi)$. It is easy to see that the order $\mathfrak{O}(L)\subset C_0(Q_0,N)$ is generated, as an A-module, by 1, $[e_0,e_1]$, $[e_0,e_2]$ and $[e_1,e_2]$. Denote $f_0=e_0$, $f_1=\pi e_1$, $f_2=\pi e_2$. Then, since $\mathfrak{n}(L^0)=(\pi)$, $\mathfrak{O}(L^0)$ is generated as an A-module by the elements 1 and $(1/\pi)[f_i,f_j]$, where $0\leqq i\leqq j\leqq 2$. But $(1/\pi)[f_i,f_j]\in\mathfrak{O}(L)$ if $i\neq 0$ or $j\neq 0$, while $(1/\pi)[f_0,f_0]=(1/\pi)[e_0,e_0]=(1/\pi)N(e_0)\in A$ by (1.7) (a).

(b) By (3.2), $\mathfrak{d}(\Lambda^0)=\mathfrak{d}(\Lambda)\mathfrak{p}$. Therefore the equality $[\mathfrak{L}(\Lambda)\colon \mathfrak{L}(\Lambda^0)]=\mathfrak{p}^2$ follows from (1.2), (1.3), (3.3) and (3.6). Since $\Lambda^0\subset\Lambda$ and $\mathfrak{p}\Lambda\subset\Lambda^0$, we get $\mathfrak{p}(\Lambda^0)^\#\subset\Lambda^\#$. By (3.5), $\mathfrak{n}(\Lambda^0)=\mathfrak{d}(\Lambda^0)=\mathfrak{d}(\Lambda)\mathfrak{p}=\mathfrak{n}(\Lambda)\mathfrak{p}$, so we have

$$\mathfrak{n}(\Lambda)\mathfrak{p}(Q_0 \cap (\Lambda^0)^{\#}) \subset \mathfrak{n}(\Lambda)(Q_0 \cap \Lambda^{\#}),$$

that is, $\mathfrak{L}(\Lambda^0) \subset \mathfrak{L}(\Lambda)$.

Now, as an application, we shall prove Lemma (1.8).

(3.12) Proof of Lemma (1.8). If $Q = C_0(V, q)$, then (Q_0, N) is similar to (V, q), which means that there is $a \in F^*$ such that (Q_0, N) and (V^a, q^a) are F-isometric ([17, (5.20)]). Therefore we can assume that L and L^0 are hereditary A-lattices on Q_0 such that $[L: L^0] = \mathfrak{p}^2$. Replacing L and L^0 by $\mathfrak{a}L$ and $\mathfrak{a}L^0$ for a suitable A-ideal $\mathfrak{a}$ in F, we can assume that $\mathfrak{d}(L) = \mathfrak{n}(L)$, and then, $\mathfrak{d}(L^0) = \mathfrak{n}(L^0)$ by (1.7) (a). Since $\Lambda^0 = \mathfrak{O}(L^0)$ is a hereditary, non-maximal A-order in the central simple F-algebra Q, there are exactly two hereditary A-orders Λ and Λ' containing it and such that $[\Lambda: \Lambda^0] = [\Lambda': \Lambda^0] = \mathfrak{p}$ ([24, (40.8)]). $\mathfrak{O}(L)$ is one of these orders, say Λ. Let $L' = \mathfrak{L}(\Lambda')$. Then $L' \neq L$, $L' \supset L^0$ and $[L': L^0] = \mathfrak{p}^2$ by (3.11) (b). It follows from (3.6) (a) that besides L, L' is the only A-lattice satisfying these properties.

4. Models and orders.

Throughout this section, A denotes a Dedekind ring with perfect residue class fields $k(\mathfrak{p}) = A/\mathfrak{p}$ for all prime ideals $\mathfrak{p} \in S^* = \operatorname{Spec}(A) \setminus (0)$, F is the fraction field of A, and E is a finitely generated regular extension of F of genus 0. E is the field of rational functions on a non-singular conic $q(X_0, X_1, X_2) = 0$ in $P^2(F)$, where q is a (half-regular) quadratic form over F ([28, p. 68]). We may assume that

$$q(X_0, X_1, X_2) = aX_1^2 + bX_1X_2 + cX_2^2 - X_0^2,$$

where $a, b, c \in F$ and $b^2 - 4ac \in F^*$ ([17, (2.16)]). Two non-singular conics $q_1 = 0$ and $q_2 = 0$ are F-isomorphic if and only if the quadratic forms q_1 and q_2 are similar, that is, equivalent over F up to a constant factor ([28, p. 70]). Since two half-regular ternary quadratic spaces (F^3, q_1) and (F^3, q_2) are similar if and only if the algebras $C_0(F^3, q_1)$ and $C_0(F^3, q_2)$ are isomorphic ([17, (5.20)]), the extension E/F defines an isomorphism class of quaternion algebras. We shall denote by $Q(E/A)$ a representative of this class. Note that if $E = F(x, y)$, where $ax^2 + bxy + cy^2 = 1$ and $b^2 - 4ac \in F^*$, then the algebra $Q(E/A)$ is isomorphic to the Clifford algebra of the binary regular quadratic form $aX^2 + bXY + cY^2$ ([17, (5.22)]).

As in Section 3, let $Q_0(E/A)$ be the subspace of $Q(E/A)$ consisting of all $x \in Q(E/A)$ such that $T(x) = 0$, and considered with quadratic structure defined by the norm N restricted to $Q_0(E/A)$. If $E = F(x, y)$, $ax^2 + bxy + cy^2 = 1$ and $b^2 - 4ac \in F^*$, then the $S = \operatorname{Spec}(A)$-scheme

$$M = \operatorname{Proj}\,(A[X_0, X_1, X_2]/(q_0))\,,$$

where $q_0 = a_1X_1^2 + a_{12}X_1X_2 + a_2X_2^2 + a_0X_0^2$, $a_0, a_1, a_2, a_{12} \in A$ and $a_1/a_0 = -a$, $a_{12}/a_0 = -b$, $a_2/a_0 = -c$, is a model of E/A. Since the quadratic form q_0 is similar to the quadratic forms corresponding to $Q_0(E/A)$ ([17, (5.20)]), each S-scheme $M(L)$ constructed for an A-lattice L on $Q_0(E/A)$ is a model of E/A.

The next Proposition was proved in [9] (when char $(F) \neq 2$) using the results of [1] and [2]. The present proof is based on the properties of hereditary orders and does not depend on these three papers.

(4.1) PROPOSITION. *Let Λ be a maximal A-order in $Q(E/A)$, and $M(\mathfrak{L}(\Lambda))$ an S-scheme defined by the lattice $\mathfrak{L}(\Lambda)$. Then $M(\mathfrak{L}(\Lambda))$ is a relatively minimal model of E/A and $\mathfrak{d}(M(\mathfrak{L}(\Lambda))) = \mathfrak{d}(\mathfrak{L}(\Lambda)) = \mathfrak{d}(\Lambda)$.*

PROOF. Since Λ is a maximal order and the residue class fields of A are perfect, the ideal $\mathfrak{d}(\Lambda) = \mathfrak{n}(\Lambda)$ is a product of all prime ideals $\mathfrak{p}$ of A such that $Q(E/A)$ is ramified at $\mathfrak{p}$ ([24, (25.4)]). Let $\mathfrak{L}(\Lambda) = L$. Since the ideal $\mathfrak{d}(L) = \mathfrak{d}(\Lambda)$ is square-free, $M(L)_\mathfrak{p}$ is a regular S-model of E/A by (3.1). It suffices to show that for each $\mathfrak{p} \in S^*$, the fiber $M(L)_\mathfrak{p}$ is a form of the projective line $P^1(k(\mathfrak{p}))$ or a non-trivial form of two intersecting copies of $P^1(k(\mathfrak{p}))$ ([28, p. 155]). We have

$$M(L)_\mathfrak{p} \cong \operatorname{Proj}\,(k(\mathfrak{p})[X_0, X_1, X_2]/(q^*_{L_\mathfrak{p}}))$$

where $q^*_{L_\mathfrak{p}}$ is the image under the canonical homomorphism $A[X_0, X_1, X_2] \to k(\mathfrak{p})[X_0, X_1, X_2]$ of a quadratic form $q_{L_\mathfrak{p}}$ corresponding to the $A_\mathfrak{p}$-lattice $L_\mathfrak{p}$. If $\mathfrak{p} \nmid \mathfrak{d}(L)$, then $M(L)_\mathfrak{p}$ is a form (trivial or non-trivial) of the projective line $P^1(k(\mathfrak{p}))$, since then $q^*_{L_\mathfrak{p}}$ has rank 3. Let $\mathfrak{p} \mid \mathfrak{d}(L) = \mathfrak{d}(\Lambda)$. We want to show that the quadratic form $q^*_{L_\mathfrak{p}}$ is irreducible over $k(\mathfrak{p})$, and is a product of two distinct linear factors over a non-trivial extension of $k(\mathfrak{p})$. In order to do this, we can assume that A is a complete discrete valuation ring, $\mathfrak{p}$ its maximal ideal, and the quaternion algebra $Q(E/A)$ is a skewfield. We have $\mathfrak{p}\Lambda = \mathfrak{P}^2$, where $\mathfrak{P}$ is the maximal ideal of Λ, and $\Lambda/\mathfrak{P} = K$ is a quadratic separable extension of $A/\mathfrak{p} = k(\mathfrak{p})$. If ξ is an element of Λ, whose image generates K over $k(\mathfrak{p})$, then $A^* = A[\xi]$ is a non-ramified extension of A and $F^* = F(\xi)$ is a splitting field for $Q(E/A)$. Consider the A^*-lattice $L^* = L \otimes_A A^*$ on the quadratic F^*-space $Q_0(E/A) \otimes_F F^*$. Since $\mathfrak{d}(L^*) = \mathfrak{p}A^*$ is the maximal ideal of A^*, the lattice L^* is hereditary, and consequently, the order $\mathfrak{O}(L^*)$ corresponding to it in the matrix F^*-algebra $Q(E/A) \otimes_F F^*$ is hereditary and non-maximal. Hence $\mathfrak{O}(L^*)$ is contained in a maximal order which defines a hereditary lattice L' on $Q_0(E/A) \otimes_F F^*$ such that $L' \supset L^*$ and $[L'\colon L^*] = \mathfrak{p}^2 A^*$. Let $M(L^*)$ and $M(L')$ be Spec (A^*)-schemes defined by L^* and L'. Since $M(L^*)$ is a blowing-up of $M(L')$ at a $k(\mathfrak{p}A^*)$-rational point of the fiber $M(L')_{\mathfrak{p}A^*}$, the fiber $M(L^*)_{\mathfrak{p}A^*}$, consists of two intersecting projective lines over $k(\mathfrak{p}A^*)$. This means that the quadratic form $q_{L^*}=q_L$ modulo $\mathfrak{p}A^*$ is a product of two distinct linear factors over $k(\mathfrak{p}A^*)$. To end the proof, let us note that such a factorization is not possible over $k(\mathfrak{p})$. Otherwise, the quadratic form q_L would have a non-trivial zero over F by the Hensel lemma, which would imply that the quaternion algebra $Q(E/A)$ is not a skewfield ([17, (5.21)]). The last statement of the Proposition follows directly from the proof.

(4.2) Corollary ([1], [2], [4]). *All relatively minimal models of E/A are locally isomorphic.*

Proof. We may assume that A is a discrete valuation ring with maximal ideal $\mathfrak{p}$. Let L be an A-lattice on the quadratic space $Q_0(E/A)$ such that $M(L)$ is a relatively minimal model of E/A and $\mathfrak{d}(L)=\mathfrak{n}(L)$. Using the same inductive argument as in [1, p. 303], it suffices to show that if M' is obtained from $M(L)$ by an elementary transformation at a $k(\mathfrak{p})$-rational point of the fiber $M(L)_\mathfrak{p}$, then M' is isomorphic to $M(L)$. If $x \in M(L)_\mathfrak{p}$ is such a point, and L^0 the hereditary sublattice of L corresponding to it by (2.5) (a), then by (2.5) (c), $M' \cong M(L')$, where L' is the second hereditary lattice on $Q_0(E/A)$ containing L^0 and such that $[L'\colon L^0]=\mathfrak{p}^2$. By (3.2) and (3.11), $\mathfrak{O}(L)$ and $\mathfrak{O}(L')$ are two maximal A-orders in the quaternion algebra $Q(E/A)$. Since A is a discrete valuation ring, the orders $\mathfrak{O}(L)$ and $\mathfrak{O}(L')$ are A-isomorphic ([24, (18.7)]). Therefore the lattices L and L' are similar (even isometric), and hence, the models $M(L)$ and $M(L')\cong M'$ are A-isomorphic.

Since all relatively mininal models of E/A are locally isomorphic, the ideal $\mathfrak{d}(M)$ does not depend on the choice of a relatively minimal model M of E/A. We shall denote this ideal by $\mathfrak{d}_{E/A}$. The last statement of Proposition (4.1) and Theorem (25.4) of [24] imply:

(4.3) Corollary. *$\mathfrak{d}_{E/A}$ is the ground ideal of $Q(E/A)$, that is, the product of the prime ideals of A such that $Q(E/A)$ is ramified at $\mathfrak{p}$.*

(4.4) Corollary. *Let L be a hereditary lattice on $Q_0(E/A)$. Then $\mathfrak{d}(M(L))=\mathfrak{d}(L)$. Moreover, $\mathfrak{d}(M(L))=\mathfrak{d}_{E/A}\mathfrak{m}_0$, where $\mathfrak{m}_0$ is the product of all prime ideals $\mathfrak{p} \in S^*$ such that the hereditary $A_\mathfrak{p}$-order $\mathfrak{O}(L)_\mathfrak{p}$ is not maximal.*

Proof. It suffices to prove the equality when A is a discrete valuation ring. We may assume that $\mathfrak{d}(L)=\mathfrak{n}(L)$. If L is a hereditary lattice on $Q_0(E/A)$ such that $\mathfrak{O}(L)$ is a maximal order, then $M(L)$ is a relatively minimal model and $\mathfrak{d}(M(L))=\mathfrak{d}(L)=\mathfrak{d}_{E/A}$ by (4.1). If $\Lambda=\mathfrak{O}(L)$ is not a maximal order, then there is a maximal order $\Lambda' \supset \Lambda$ such that $[\Lambda'\colon \Lambda]=\mathfrak{p}$, where $\mathfrak{p}$ is the maximal ideal of

A. Let $L' = \mathfrak{L}(\Lambda')$. Then $L' \supset L$ and $[L' : L] = \mathfrak{p}^2$. Thus the S-model $M(L)$ is a blowing-up of $M(L')$ at a $k(\mathfrak{p})$-rational point of the fiber $M(L')_{\mathfrak{p}}$ by (2.5) (b). Since $\mathfrak{d}(M(L')) = \mathfrak{d}(L') = \mathfrak{d}_{E/A}$, $\mathfrak{d}(M(L)) = \mathfrak{d}(M(L'))\mathfrak{p}$ and $\mathfrak{d}(L) = \mathfrak{d}(L')\mathfrak{p}$, we get $\mathfrak{d}(M(L)) = \mathfrak{d}(L) = \mathfrak{d}_{E/A}\mathfrak{p}$.

Now we can generalize Proposition (4.1), Corollary (4.2) and Theorem 1 of [8]:

(4.5) THEOREM. (a) *Each regular quadratic S-model of E/A is isomorphic to an S-model $M(L)$ for some hereditary A-lattice L on $Q_0(E/A)$.*

(b) *Let* $\mathfrak{m}$ *be a square-free ideal of A such that* $\mathfrak{d}_{E/A} \mid \mathfrak{m}$. *There is a one-to-one correspondence between the isomorphism classes of regular quadratic S-models M of E/A such that* $\mathfrak{d}(M) = \mathfrak{m}$, *and the similarity classes of hereditary A-lattices L on $Q_0(E/A)$ such that* $\mathfrak{d}(L) = \mathfrak{m}$. *Hence, by Proposition* (3.6), *there is a one-to-one correspondence between the isomorphism classes of regular quadratic S-models M of E/A such that* $\mathfrak{d}(M) = \mathfrak{m}$, *and the isomorphism classes of hereditary A-orders in $Q(E/A)$ such that* $\mathfrak{d}(\Lambda) = \mathfrak{m}$.

(c) *All regular quadratic models of E/A with the same invariant* $\mathfrak{d}$ *are locally isomorphic.*

PROOF. (a) Let M be a regular quadratic model of E/A. Then there is an S-morphism $\varphi\colon M \to M^*$, where M^* is a relatively minimal model of E/A ([28, p. 131]). By Theorem 3 of [7] (the argument there does not depend on the characteristic of F) and (2.1), $M^* = M(L^*)$, where L^* is a hereditary A-lattice on $Q_0(E/A)$. The morphism φ is a composition of blowing-ups at closed points ([28, p. 55]), so by induction, it suffices to show that if $M(L)$ is a regular quadratic model and a blowing-up of $M(L)$ at a closed point also gives a quadratic model, then this model is isomorphic to a model $M(L^0)$ for some hereditary A-lattice L^0. This follows directly from (2.5) (b), since a blowing-up of $M(L)$ at a point x of a fiber $M(L)_{\mathfrak{p}}$ gives a quadratic model if and only if $M(L)_{\mathfrak{p}} \cong P^1(k(\mathfrak{p}))$ and x is a $k(\mathfrak{p})$-rational point.

(b) If $M(L_1)$ and $M(L_2)$ are two isomorphic regular quadratic S-models of E/A, where L_1 and L_2 are hereditary A-lattices on $Q_0(E/A)$, then $\mathfrak{d}(L_1) = \mathfrak{d}(L_2)$ by (4.4). Replacing L_1 and L_2 by similar lattices, we can assume that $\mathfrak{d}(L_i) = \mathfrak{n}(L_i)$, for $i = 1, 2$, and then, it follows from Theorem 1 of [7] that L_1 and L_2 are isometric. Since similar hereditary A-lattices L on $Q_0(E/A)$ define S-isomorphic models $M(L)$, and every regular quadratic S-model of E/A is isomorphic to a model of this form by (a), the proof is completed.

(c) We can assume that A is a discrete valuation ring. Let M_1 and M_2 be two regular quadratic models and let $M_i = M(L_i)$, for $i = 1, 2$, where L_i is a hereditary A-lattice. If $\mathfrak{d}(M_1) = \mathfrak{d}(M_2)$, then $\mathfrak{d}(L_1) = \mathfrak{d}(L_2)$ by (4.4), so $\mathfrak{d}(\mathfrak{O}(L_1))$

$=\mathfrak{d}(\mathfrak{O}(L_2))$ by (3.3). Hence the hereditary orders $\mathfrak{O}(L_1)$ and $\mathfrak{O}(L_2)$ are both maximal or both hereditary non-maximal, which means that the $\mathfrak{p}$-adic completions of $\mathfrak{O}(L_1)$ and $\mathfrak{O}(L_2)$ are of the same type ([24, p. 360]). By Theorem 4.2,4) of [14], these orders are A-isomorphic. Hence the lattices L_1 and L_2 are similar, and consequently, the S-models $M(L_1)$ and $M(L_2)$ are S-isomorphic.

(4.6) REMARK. A more "geometrical proof" of Theorem (4.5) (c) was given in [3, Theorem 3] when char $(F)\neq 2$. In our "algebraical proof" we use the property of hereditary orders in quaternion algebras which says that two hereditary orders of the same local type are locally isomorphic. The third possibility is "an arithmetical proof" using the fact that two maximal lattices (in the sense of (1.14)) of the same norm are locally isometric ([17, (156)]). In fact, with the same notations as in (c) above, we may assume that $\mathfrak{d}(L_i)=\mathfrak{n}(L_i)$ for $i=1,2$ by (3.8). Then the lattices L_1 and L_2 are maximal by (1.4), and have the same norm. Hence they are isometric, so the models $M(L_1)$ and $M(L_2)$ are S-isomorphic.

Let I be an A-lattice on $Q(E/A)$. The right order of I is defined as

$$O_r(I) = \{x \in Q(E/A) \mid Ix \subset I\} .$$

This is an A-order in $Q(E/A)$. Similarly, $O_l(I)$ denotes the left order of I. If Λ and Λ' are two A-orders in $Q(E/A)$, we define the right distance ideal from Λ' to Λ as

$$\mathfrak{D}_r(\Lambda', \Lambda) = \{x \in Q(E/A) \mid \Lambda' x \subset \Lambda\} .$$

This is the largest ideal in the set of all Λ'-left and Λ-right ideals contained in Λ. Similarly, $\mathfrak{D}_l(\Lambda', \Lambda)$ denotes the left distance ideal from Λ' to Λ. We say that a left (or right) Λ-ideal $\mathfrak{P}$ is integral if $\mathfrak{P}$ is contained in Λ. $\mathfrak{P}$ is called Λ-regular if $O_l(P)=\Lambda$.

If M is a regular quadratic model of E/A and $M\cong M(L)$, where L is a hereditary A-lattice on $Q_0(E/A)$, we shall say that the hereditary order $\Lambda=\mathfrak{O}(L)$ corresponds to M.

(4.7) THEOREM. *Let M be a regular quadratic S-model of E/A, and let Λ be a hereditary A-order in $Q(E/A)$ corresponding to M. Let $\mathfrak{p} \in S^*$, $\mathfrak{p} \nmid \mathfrak{d}(M)$. Then there is a one-to-one correspondence between the $k(\mathfrak{p})$-rational points of the fiber $M_{\mathfrak{p}}$ and the integral left Λ-ideals with norm equal to $\mathfrak{p}$ such that elementary transformations at two $k(\mathfrak{p})$-rational points of $M_{\mathfrak{p}}$ give isomorphic models if and only if the right orders of the left ideals corresponding to these points are isomorphic (as A-algebras).*

Moreover, the right order of the left ideal corresponding to a $k(\mathfrak{p})$-rational point of M corresponds to any model obtained by an elementary transformation of M at this point.

PROOF. Let $x \in M_{\mathfrak{p}}$ be a $k(\mathfrak{p})$-rational point and let L^0 be the hereditary sublattice of $L = \mathfrak{L}(\Lambda)$ corresponding to the point x by (2.5) (a). Let L' be the second hereditary lattice on $Q_0(E/A)$ containing L^0 and such that $[L': L^0] = \mathfrak{p}^2$ (see (1.8)). If $\Lambda^0 = \mathfrak{O}(L^0)$ and $\Lambda' = \mathfrak{O}(L')$, then $[\Lambda: \Lambda^0] = [\Lambda': \Lambda^0] = \mathfrak{p}$ and $\Lambda^0 = \Lambda \cap \Lambda'$ by (3.11). Let $\mathfrak{P}$ be the left distance ideal from Λ' to Λ. We have

$$\text{(4.8)} \qquad \mathfrak{P} = (\Lambda'\Lambda)^{-1} \quad \text{and} \quad N((\Lambda'\Lambda)^{-1}) = [\Lambda: \Lambda \cap \Lambda'] .$$

Both equalities can be checked locally. If $\mathfrak{q} \in S^*$ and $\mathfrak{q} \neq \mathfrak{p}$, then $\Lambda_{\mathfrak{q}} = \Lambda'_{\mathfrak{q}} = \Lambda^0_{\mathfrak{q}}$, while $\Lambda_{\mathfrak{p}}$ and $\Lambda'_{\mathfrak{p}}$ are maximal $A_{\mathfrak{p}}$-orders, and in this case the equalities can be easily proved ([10, VI, § 2, Satz 15]). Now we see that $N(\mathfrak{P}) = [\Lambda: \Lambda^0] = \mathfrak{p}$. Let $\mathfrak{P}$ be the integral left Λ-ideal corresponding to the $k(\mathfrak{p})$-rational point $x \in M_{\mathfrak{p}}$.

Conversely, let $\mathfrak{P}$ be an integral left Λ-ideal such that $N(\mathfrak{P}) = \mathfrak{p}$. Let $\Lambda' = O_r(\mathfrak{P})$ be the right order of $\mathfrak{P}$. We want to show that both Λ' and $\Lambda^0 = \Lambda \cap \Lambda'$ are hereditary orders in $Q(E/A)$ and $[\Lambda: \Lambda^0] = [\Lambda': \Lambda^0] = \mathfrak{p}$. If $\mathfrak{q} \in S^*$ and $\mathfrak{q} \neq \mathfrak{p}$, then $\Lambda'_{\mathfrak{q}} = \Lambda_{\mathfrak{q}}$. Since $N(\mathfrak{P}) = \mathfrak{p} \nmid \mathfrak{d}(\Lambda)$, $\mathfrak{P}_{\mathfrak{p}}$ is not a two-sided ideal of Λ ([24, (17.3)]), so $\Lambda_{\mathfrak{p}}$ and $\Lambda'_{\mathfrak{p}}$ are two distinct maximal $A_{\mathfrak{p}}$-orders. Therefore Λ' is a hereditary order ([24, (40.5)]), $\Lambda' \neq \Lambda$ and the left distance ideal from Λ' to Λ is equal to $(\Lambda'\Lambda)^{-1}$. Since $\mathfrak{P} \subset (\Lambda'\Lambda)^{-1}$ and $N(\mathfrak{P}) = \mathfrak{p}$, we get $\mathfrak{P} = (\Lambda'\Lambda)^{-1}$. Hence by (4.8), $[\Lambda: \Lambda^0] = [\Lambda': \Lambda^0] = \mathfrak{p}$, where $\Lambda^0 = \Lambda \cap \Lambda'$. Since $\mathfrak{d}(\Lambda^0) = \mathfrak{d}(\Lambda)\mathfrak{p}$ and $\mathfrak{p} \nmid \mathfrak{d}(\Lambda)$, Λ^0 is a hereditary suborder of Λ by (3.6). Now, let the $k(\mathfrak{p})$-rational point x of $M_{\mathfrak{p}}$ corresponding to the integral left Λ-ideal $\mathfrak{P}$ be defined by the hereditary sublattice $L^0 = \mathfrak{L}(\Lambda^0)$ of L.

Note that $\Lambda' = \mathfrak{O}(L')$, where L' is the second hereditary lattice on $Q_0(E/A)$ such that $L' \supset L^0$ and $[L': L^0] = \mathfrak{p}^2$.

It is easy to see that distinct $k(\mathfrak{p})$-rational points of $M_{\mathfrak{p}}$ define distinct integral left Λ-ideals, so it remains to show that elementary transformations at two $k(\mathfrak{p})$-rational points $x_1, x_2 \in M_{\mathfrak{p}}$ give isomorphic models if and only if the right orders of the integral left Λ-ideals $\mathfrak{P}_1$ and $\mathfrak{P}_2$ corresponding to these points are isomorphic. Let L_i^0, where $i = 1, 2$, be the hereditary sublattice of L corresponding to x_i. If M'_i is a model of E/A obtained by an elementary transformation of M at x_i, then $M'_i \cong M(L'_i)$, where L'_i is the second hereditary lattice on $Q_0(E/A)$ such that $[L'_i: L_i^0] = \mathfrak{p}^2$ and $L'_i \neq L$.

Now, if $M'_1 \cong M'_2$, then the lattices L'_1 and L'_2 are similar (even isometric) by (4.5) (b), and hence, the orders $\mathfrak{O}(L'_1) = \Lambda'_1$ and $\mathfrak{O}(L'_2) = \Lambda'_2$ are isomorphic by (3.10). Thus the right orders of the ideals $\mathfrak{P}_1$ och $\mathfrak{P}_2$ are isomorphic.

Conversely, if $O_r(\mathfrak{P}_1) \cong O_r(\mathfrak{P}_2)$, then the lattices $L'_1 = \mathfrak{L}(O_r(\mathfrak{P}_1))$ and $L'_2 = \mathfrak{L}(O_r(\mathfrak{P}_2))$ are similar (and even isometric) by (3.10). Since L'_i is the second hereditary lattice on $Q_0(E/A)$ containing L_i^0 and such that $[L_i': L_i^0] = \mathfrak{p}^2$, $M(L_i')$ can be obtained by an elementary transformation of M at the $k(\mathfrak{p})$-rational point $x_i \in M_\mathfrak{p}$ by (2.5) (c). Hence the elementary transformations of M at the points $x_1, x_2 \in M_\mathfrak{p}$ give isomorphic models.

The last statement of the Theorem follows directly from the proof.

In the next theorem we want to characterize the fibers $M_\mathfrak{p}$ of a regular quadratic model M of E/A such that an elementary transformation at a $k(\mathfrak{p})$-rational point of this fiber gives a model isomorphic to M. We have to assume that the Dedekind ring A is a principal ideal ring.

(4.9) THEOREM. *Let M be a regular quadratic model of E/A, where the ring A is a principal ideal ring. Let Λ be a hereditary A-order corresponding to M in $Q(E/A)$, and $\mathfrak{p} \in S^*$, $\mathfrak{p} \nmid \mathfrak{d}(M)$. If an elementary transformation of M at a $k(\mathfrak{p})$-rational point of $M_\mathfrak{p}$ gives a model isomorphic to M, then there is an ideal $\mathfrak{r} \mid \mathfrak{d}(M)$ and an element $\alpha \in \Lambda$ such that*

$$\mathfrak{r}\mathfrak{p} = N(\Lambda\alpha)\,. \tag{4.10}$$

The converse is true if $\mathfrak{r} \mid \mathfrak{d}_{E/A}$.

PROOF. Note that in a hereditary A-order Λ, for each $\mathfrak{p} \in S^*$, there is exactly one two-sided regular integral prime ideal whose norm is a power of $\mathfrak{p}$ ([24, (17.3)] and [12, Satz 5]). If $\mathfrak{p} \nmid \mathfrak{d}(\Lambda)$ then the prime ideal corresponding to $\mathfrak{p}$ is $\Lambda\mathfrak{p}$ and its norm is equal to $\mathfrak{p}^2$. If $\mathfrak{p} \mid \mathfrak{d}(\Lambda)$, then the corresponding prime ideal has the norm equal to $\mathfrak{p}$.

Let $x \in M_\mathfrak{p}$ be a $k(\mathfrak{p})$-rational point such that an elementary transformation of M at x gives a regular quadratic model isomorphic to M, and let $\mathfrak{P}$ be the integral left Λ-ideal corresponding to this point. By (4.7), $O_r(\mathfrak{P})$ is isomorphic to Λ. Since $\mathfrak{P}$ is a regular Λ-ideal, there is a two-sided regular Λ-ideal $\mathfrak{Q}$ and $\alpha \in \Lambda$ such that $\mathfrak{P} = \mathfrak{Q}\alpha$ ([15, Proposition 4.1]). Multiplying $\mathfrak{Q}$ by an element of A, we can assume that $\alpha \in \Lambda$, and the ideal $\Lambda\alpha$ is primitive, that is, $(1/a)\Lambda\alpha \subset \Lambda$ and $a \in A$ imply $a \in A^*$. Let $\mathfrak{Q} = \mathfrak{R}mn^{-1}$, where $m, n \in A$ $(m,n) = A$, and $\mathfrak{R}$ is a square-free integral Λ-ideal such that $N(\mathfrak{R})$ divides $\mathfrak{d}(\Lambda)$. Hence $n\mathfrak{P} = m\mathfrak{R}\alpha$. By taking the norms, we get $(m) = A$ and $\Lambda n = \mathfrak{R}^2$, so $\mathfrak{R}\mathfrak{P} = \Lambda\alpha$. Therefore $\mathfrak{r}\mathfrak{p} = N(\Lambda\alpha)$, where $\mathfrak{r} = N(\mathfrak{R}) \mid \mathfrak{d}(M)$.

Conversely, if $\mathfrak{r}\mathfrak{p} = N(\Lambda\alpha)$ and $\mathfrak{r} \mid \mathfrak{d}_{E/A}$, then $\mathfrak{P} = \mathfrak{R}^{-1}\Lambda\alpha$ is an integral left Λ-ideal, where $\mathfrak{R}$ is the only two-sided regular Λ-ideal whose norm is equal to $\mathfrak{r}$. Since $N(\mathfrak{P}) = \mathfrak{p}$ and $O_r(\mathfrak{P}) = \alpha^{-1}\Lambda\alpha$, any elementary transformation of M at the $k(\mathfrak{p})$-rational point of $M_\mathfrak{p}$ corresponding to $\mathfrak{P}$ gives a regular quadratic model isomorphic to M by (4.7).

In general, an elementary transformation of a regular quadratic model may give a model from another isomorphism class. We end this section with a result which shows that it is always possible to go from one isomorphism class of relatively minimal models to another by a finite number of elementary transformations.

(4.11) THEOREM. *If M and M' are two relatively minimal models of E/A, then there is a sequence of relatively minimal models $M_0=M, M_1, \ldots, M_n=M'$ such that M_{i+1} is an elementary transform of M_i for $0 \leqq i < n$.*

PROOF. Let Λ and Λ' be maximal orders corresponding to the models M and M'. We shall construct "a path" of maximal orders from Λ to Λ' ([15, Theorem 3.1]). If $\mathfrak{D}=(\Lambda'\Lambda)^{-1}$ is the left distance ideal from Λ' to Λ, then $\mathfrak{D}=\mathfrak{P}_0\mathfrak{P}_1 \ldots \mathfrak{P}_{n-1}$, where $\mathfrak{P}_i$ are integral normal maximal ideals and the product is proper ([24, p. 181 and p. 183]). Let $\Lambda_i=O_l(\mathfrak{P}_i)$ for $0 \leqq i \leqq n-1$, and $\Lambda_n=O_r(\mathfrak{P}_{n-1})$. Λ_i are maximal orders, $\Lambda_0=\Lambda$ and $\Lambda_n=\Lambda'$. Let M_i be a relatively minimal model of E/A defined by $\mathfrak{L}(\Lambda_i)$. Since $\mathfrak{P}_i$ is an integral left Λ_i-ideal such that $N(\mathfrak{P}_i)=\mathfrak{p}_i \mid \mathfrak{d}_{E/A}$ and $O_r(\mathfrak{P}_i)=\Lambda_{i+1}$, the model M_{i+1} is an elementary transfom of M_i by (4.7).

5. Global fields. Examples.

Let F be a global field, Ω the set of all non-trivial spots on F, and S a subset of Ω such that $\Omega \setminus S$ is finite and contains all archimedean spots on F. If (V,q) is a half-regular quadratic space over F, then we say that (V,q) is S-indefinite if there is a spot $\mathfrak{p} \in \Omega \setminus S$ such that $(V_\mathfrak{p}=V \otimes_F F_\mathfrak{p}, q_\mathfrak{p})$ is isotropic, where $F_\mathfrak{p}$ denotes the $\mathfrak{p}$-adic completion of F and $q_\mathfrak{p}$ the natural extension of q from V to $V_\mathfrak{p}$. If $(V_\mathfrak{p}, q_\mathfrak{p})$ is anisotropic for each $\mathfrak{p} \in \Omega \setminus S$, then (V,q) is called S-definite. Let $A(S)=A$ be the ring of all elements of F which are integral with respect to all $\mathfrak{p} \in S$, and let E/A be an extension satisfying the usual assumptions. If $V=Q_0(E/A)$, where $Q(E/A)$ is a quaternion algebra corresponding to the extension E/A, then $Q_0(E/A)$ is S-indefinite if and only if there is $\mathfrak{p} \in \Omega \setminus S$ such that $E \otimes_F F_\mathfrak{p} \cong F_\mathfrak{p}(x)$ is a purely transcendental extension of $F_\mathfrak{p}$, or, in terms of algebras, $Q(E/A)$ is ramified at $\mathfrak{p}$. The last condition says that the algebra $Q(E/A)$ satisfies the Eichler condition with respect to A ([24, (34.3)]). We shall say that E/A satisfies the Eichler condition (or is S-indefinite) if the corresponding algebra $Q(E/A)$ (or the quadratic space $Q_0(E/A)$) satisfies this condition (is S-indefinite).

Let $\mathrm{Cl}_Q(A)$ be the ray class group, that is, the multiplicative group of classes of fractional A-ideals in F, where two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are in the same class if $\mathfrak{b}=\mathfrak{a}N(\alpha)$ for some $\alpha \in Q$. The class of $\mathfrak{a}$ in $\mathrm{Cl}_Q(A)$ will be denoted by $[\mathfrak{a}]$.

(5.1) THEOREM. *If E/A satisfies the Eichler condition, and M is a regular quadratic S-model of E/A, then the number of isomorphism classes of regular quadratic S-models locally isomorphic to M is equal to the order of the group $\mathrm{Cl}_Q(A)/G_1G_2$, where G_1 is the subgroup of $\mathrm{Cl}_Q(A)$ generated by the ideal classes $[\mathfrak{p}]$ such that $\mathfrak{p}$ divides $\mathfrak{d}(M)$, and G_2 is the subgroup of $\mathrm{Cl}_Q(A)$ generated by the ideal classes $[\mathfrak{p}^2]$ such that $\mathfrak{p}$ does not divide $\mathfrak{d}(M)$.*

PROOF. Let Λ be a hereditary order in $Q(E/A)$ corresponding to M. We have to show that there is a one-to-one correspondence between the isomorphism classes of hereditary orders locally isomorphic to Λ and the elements of the group $\mathrm{Cl}_Q(A)/G_1G_2$. To prove this, we need some rather general facts.

(5.2) Each hereditary order locally isomorphic to Λ is equal to the right order of a regular left Λ-ideal ([15, Proposition 3.2]). Moreover, for every regular left Λ-ideal I, the order $O_r(I)$ is hereditary and locally isomorphic to Λ. The right orders of two regular left Λ-ideals I_1 and I_2 are isomorphic if and only if there is a regular two-sided Λ-ideal $\mathfrak{Q}$ and $\alpha \in Q(E/A)$ such that

$$I_2 = \mathfrak{Q}I_1\alpha \tag{5.3}$$

([15, Proposition 4.1]).

This means that we have to compute the number of classes of regular left Λ-ideals with respect to the relation (5.3).

(5.4) Eichler's Theorem ([24, (34.9)]) holds for regular left Λ-ideals, that is, if I is such an ideal and $N(I)=Aa$, $a \in F$, then $I=\Lambda\alpha$ for some $\alpha \in Q$. By Theorem (40.22) of [24], we have to check that I and Λ are in the same genus (that is, the $\mathfrak{p}$-adic completions $I_\mathfrak{p}$ and $\Lambda_\mathfrak{p}$ are $\Lambda_\mathfrak{p}$-isomorphic for each $\mathfrak{p} \in S$), and $\Lambda' I$ is Λ'-isomorphic to Λ' for an A-order Λ' containing Λ. Since I is a regular left Λ-ideal and Λ is hereditary, I is locally free ([16, Theorem 2]). Thus $I_\mathfrak{p} \cong \Lambda_\mathfrak{p}$ for $\mathfrak{p} \in S$. Using this fact, we check easily that $N(\Lambda' I)=N(I)=Aa$ for a maximal A-order Λ' containing Λ. Hence the left Λ'-ideal $\Lambda' I$ satisfies the assumptions of Eichler's Theorem for maximal orders. Therefore $\Lambda' I \cong \Lambda'$, and the desired result follows from Theorem (40.22) of [24].

As a direct consequence, we get that if I_1 and I_2 are two regular left Λ-ideals and the elements of $\mathrm{Cl}_Q(A)$ defined by the norms of these ideals are equal, then I_1 and I_2 are Λ-isomorphic. In fact, the product $I_1^{-1}I_2$ is proper (concordant in the terminology of [16, pp. 221]), and $O_r(I_1^{-1}I_2)=O_r(I_1)$ is a hereditary order locally isomorphic to Λ. Since $N(I_1^{-1}I_2)=Aa$, for some $a \in F$ ([16, Theorem 3]), we get $I_1^{-1}I_2=O_r(I_1)\alpha$, for some $\alpha \in Q$, that is, $I_2=I_1\alpha$.

(5.5) We already know that the norms of regular integral prime Λ-ideals are the prime ideals $\mathfrak{p} \in S$ such that $\mathfrak{p} \mid \mathfrak{d}(\Lambda)$ and the squares $\mathfrak{p}^2$ of the prime ideals $\mathfrak{p} \in S$ such that $\mathfrak{p} \nmid \mathfrak{d}(\Lambda)$ (see the proof of (4.9)). Moreover, each ideal of A is a norm of a regular left Λ-ideal. This is clear for prime ideals of A ([24, (24.13)]).

If $\mathfrak{a}=\mathfrak{p}_1\mathfrak{p}_2\ldots,\mathfrak{p}_i\in S$, then we can choose a regular left Λ-ideal $\mathfrak{P}_1$ such that $N(\mathfrak{P}_1)=\mathfrak{p}_1$, then a regular left $O_r(\mathfrak{P}_1)$-ideal $\mathfrak{P}_2$ such that $N(\mathfrak{P}_2)=\mathfrak{p}_2$ and so on. The product of the ideals $\mathfrak{P}_i$ is proper and its norm is equal to $\mathfrak{a}$ ([16, Theorem 3]).

Now, let Λ' be an A-order locally isomorphic to Λ and let I be a regular Λ-left and Λ'-right ideal. If we map the ideal I on the element defined by its norm in $\mathrm{Cl}_Q(A)/G_1G_2$, then this element corresponds to the whole class of hereditary orders A-isomorphic to Λ' by (5.3) and (5.5). By (5.5) the mapping is surjective, while by (5.4) and (5.5) it is injective.

(5.6) Remark. If $A=\mathbf{Z}$, then Theorem (5.1) says that in an indefinite quaternion algebra over the field of rational numbers two locally isomorphic hereditary $\mathbf{Z}$-orders are isomorphic, or in the language of quadratic forms, that there is only one class in the genus of indefinite integral ternary quadratic forms with square-free discriminant. In particular, all maximal $\mathbf{Z}$-orders in an indefinite quaternion algebra over the field of rational numbers are isomorphic, or, in the terms of quadratic forms, there is only one class in the genus of indefinite integral ternary stemforms (that is, integral quadratic forms with minimal discriminant in its similarity class). These two classical results (see e.g. [18], [5]), were extended in [11, Satz 3] (see also [19, 104:10]) to arbitrary indefinite sets of spots on the field of rational numbers. Theorem (5.1) gives, in the ternary case, an expected generalization of these results to arbitrary global fields ([11, p. 219]).

(5.7) Remark. Theorem (5.1) remains true if A is an arbitrary Dedekind ring, and E is a purely transcendental extension of F. In this case $Q(E/A)$ is the matrix algebra $M_2(F)$, $\mathrm{Cl}_Q(A)=\mathrm{Cl}(A)$ the class group of A, and the number of isomorphism classes of relatively minimal models of E/A is equal to the order of $\mathrm{Cl}(A)/\mathrm{Cl}(A)^2$. This result was announced in [28], p. 158 and proved in [2] by methods of Galois cohomology of sheaves. Using the correspondence between the isomorphism classes of relatively minimal models and the isomorphism classes of maximal orders, the result follows directly from a theorem of Chevalley ([27, § 3, Satz 3]) which says that the number of isomorphism classes of maximal A-orders in the matrix algebra $M_2(F)$ is equal to the order of $\mathrm{Cl}(A)/\mathrm{Cl}(A)^2$. A slightly modified proof works for arbitrary hereditary orders in $M_2(F)$ ([27, p. 385–6]) giving the result of Theorem (5.1) also in this case.

If E/A is a definite extension the situation is much more complicated. Let $t(\mathfrak{d}_{E/A},\mathfrak{m})$ denotes the number of isomorphism classes of regular quadratic A-models M of E/A such that $\mathfrak{d}(M)=\mathfrak{d}_{E/A}\mathfrak{m}$. The number $t(\mathfrak{d}_{E/A},A)$ is called the type number of $Q(E/A)$, and is denoted by t. If $A=\mathbf{Z}$ and $\mathfrak{d}_{E/\mathbf{Z}}=(d_{E/\mathbf{Z}})$, $\mathfrak{m}=(m)$, where $d_{E/\mathbf{Z}}>0$ and $m>0$, we shall write $t(d_{E/\mathbf{Z}},m)$ and $d(M)=d_{E/\mathbf{Z}}m$ instead of $t(\mathfrak{d}_{E/\mathbf{Z}},\mathfrak{m})$ and $\mathfrak{d}(M)$.

(5.8) THEOREM. *If $E/\mathbf{Z}$ is definite, then $t(d_{E/\mathbf{Z}},m)=1$ if and only if $(d_{E/\mathbf{Z}},m)$* $=(2,1)$, $(2,3)$, $(2,5)$, $(2,7)$, $(2,11)$, $(2,23)$, $(2,15)$, $(3,1)$, $(3,2)$, $(3,5)$, $(3,11)$, $(5,1)$, $(5,2)$, $(7,1)$, $(7,3)$, $(13,1)$, $(30,1)$, $(42,1)$, $(70,1)$, $(78,1)$.

Hence there are exactly 9 definite extensions $E/\mathbf{Z}$ which have only one isomorphism class of relatively minimal models $(m=1)$. In the language of algebras, this means that there are exactly 9 definite quaternion algebras over the field of rational numbers with only one isomorphism class of maximal orders.

In the language of quadratic forms, Theorem (5.8) says that there are exactly 20 genera of definite integral ternary quadratic forms with square-free discriminant and only one class in the genus.

Theorem (5.8) follows directly if we apply the formulas of Theorems 16 and 26 of [23] to obtain a number N_0 such that $t(d_{E/\mathbf{Z}},m)=1$ implies $d_{E/\mathbf{Z}}\leqq N_0$ (e.g. $N_0=400$). Then using the table [6] of integral reduced ternary quadratic forms, we get all the cases. If we denote the quadratic form $q=\sum a_{ij}X_iX_j$, $0\leqq i\leqq j\leqq 2$ by:

$$\begin{bmatrix} a_{00} & a_{11} & a_{22} \\ a_{12} & a_{02} & a_{01} \end{bmatrix}^{d_{E/\mathbf{Z}},m}$$

then the twenty regular quadratic models $M=\mathrm{Proj}\,(\mathbf{Z}[x_0,x_1,x_2]/(q))$ of Theorem (5.8) are:

$$\begin{bmatrix} 1&1&1\\0&1&1 \end{bmatrix}^{2,1}, \begin{bmatrix} 1&1&2\\1&1&0 \end{bmatrix}^{2,3}, \begin{bmatrix} 1&1&3\\1&1&0 \end{bmatrix}^{2,5}, \begin{bmatrix} 1&1&5\\0&1&1 \end{bmatrix}^{2,7}, \begin{bmatrix} 1&2&3\\0&1&0 \end{bmatrix}^{2,11}, \begin{bmatrix} 1&3&5\\3&0&1 \end{bmatrix}^{2,23},$$

$$\begin{bmatrix} 1&3&3\\0&1&1 \end{bmatrix}^{2,15}, \begin{bmatrix} 1&1&1\\0&0&1 \end{bmatrix}^{3,1}, \begin{bmatrix} 1&1&2\\0&0&1 \end{bmatrix}^{3,2}, \begin{bmatrix} 1&1&4\\0&1&0 \end{bmatrix}^{3,5}, \begin{bmatrix} 1&2&5\\0&1&1 \end{bmatrix}^{3,11}, \begin{bmatrix} 1&1&2\\0&1&1 \end{bmatrix}^{5,1},$$

$$\begin{bmatrix} 1&2&2\\2&0&1 \end{bmatrix}^{5,2}, \begin{bmatrix} 1&1&2\\0&1&0 \end{bmatrix}^{7,1}, \begin{bmatrix} 1&2&3\\1&1&0 \end{bmatrix}^{7,3}, \begin{bmatrix} 1&2&2\\1&0&1 \end{bmatrix}^{13,1},$$

$$\begin{bmatrix} 1&1&10\\0&1&0 \end{bmatrix}^{30,1}, \begin{bmatrix} 1&1&11\\1&1&0 \end{bmatrix}^{42,1}, \begin{bmatrix} 1&2&9\\0&1&0 \end{bmatrix}^{70,1}, \begin{bmatrix} 1&5&5\\-3&1&1 \end{bmatrix}^{78,1}.$$

Let us note that the results of computations in [22] give the number of isomorphism classes of relatively minimal models for $d_{E/\mathbf{Z}}m\leqq 210$. The table [6] gives this number for $d_{E/\mathbf{Z}}m\leqq 1000$.

Theorem (4.9) shows that there is a very close connection between the existance of an elementary transformation of a regular quadratic model M of $E/\mathbf{Z}$ at a $\mathbf{Z}/(p)$-rational point of the fiber M_p and the possibility to represent a multiple rp, where $r \mid d(M) = d_{E/\mathbf{Z}}m$, by an integral quaternary quadratic form — a norm-form of a hereditary order corresponding to M. Here, by a norm-form of a hereditary order Λ we mean each quadratic form $N(\sum_{i=0}^{3} x_i e_i)$, where e_0, e_1, e_2, e_3 is a $\mathbf{Z}$-basis of Λ. If $E/\mathbf{Z}$ is indefinite, then the class number h of each hereditary $\mathbf{Z}$-order in $Q(E/\mathbf{Z})$ is equal to 1 (e.g. by (5.4)), and the norm-forms are universal, that is, they represent all the integers. If $E/\mathbf{Z}$ is definite the situation is again more complicated. We have:

(5.9) THEOREM. *Let $E/\mathbf{Z}$ be a definite extension and let M be a regular quadratic $\mathbf{Z}$-model of $E/\mathbf{Z}$. Then the set $P(M)$ of prime numbers $p \nmid d(M)$ such that each elementary transformation at any $\mathbf{Z}/(p)$-rational point of the fiber M_p gives a model non-isomorphic to M is finite (maybe empty).*

PROOF. Let Λ be a hereditary order corresponding to the $\mathbf{Z}$-model M. We claim that for each prime number p, each non-negative integer is locally the norm of an element of $\Lambda_p = \Lambda \otimes_{\mathbf{Z}} \mathbf{Z}_p$, where $\mathbf{Z}_p$ denotes the p-adic integers. This is clear when Λ_p is a hereditary order in a matrix algebra. Since a norm-form, as a quaternary quadratic form, is universal over the field of p-adic numbers, each element of $\mathbf{Z}_p$ is the norm of an element of Λ_p if Λ_p is a maximal order in a skewfield ([24, (12.5)]). Now Theorem 1 of [26] implies that each sufficiently large prime number is represented by the norm-forms of Λ. By (4.9), if $p \in P(M)$, then for each $r \mid d_{E/\mathbf{Z}}$, rp is not represented by the norm-forms of Λ. Hence $P(M)$ is finite.

If $t(d_{E/\mathbf{Z}}, m) = 1$ and Λ is a hereditary order such that $d(\Lambda) = d_{E/\mathbf{Z}}m$, then for each prime number $p \nmid d_{E/\mathbf{Z}}$ some multiple rp, where $r \mid d_{E/\mathbf{Z}}$, is represented by the norm-forms of Λ. As we know, there are 20 such cases. For 10 of them, not only the type number $t = 1$, but also the class number $h = 1$, so the norm-forms corresponding to these orders represent all non-negative integers. It seems that if $p \mid d_{E/\mathbf{Z}}$, it is also possible to take $r = 1$ for the remaining 10 orders (for which $h = 2$).

Let us look at some examples.

(5.10) EXAMPLE. If $E = \mathbf{Q}(x, y)$, where $x^2 + 11y^2 = -1$, then $d_{E/\mathbf{Z}} = 11$, and $t(11, 1) = 2$. The two isomorphism classes of relatively minimal models are represented by $M_i = \operatorname{Proj}(\mathbf{Z}[x_0, x_1, x_2]/(q_i))$, $i = 1, 2$, where

$$q_1 = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \end{bmatrix}^{11,1} \quad \text{and} \quad q_2 = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \end{bmatrix}^{11,1}.$$

The corresponding norm-forms are e.g. (see [20, p. 286]):

$$N_1 = x_0^2+3x_1^2+3x_2^2+x_3^2+x_0x_2-x_1x_3 ,$$

$$N_2 = x_0^2+4x_1^2+4x_2^2+x_3^2+x_0x_2+x_0x_3-4x_1x_2-x_1x_3+x_2x_3 .$$

It is easy to see that $2 \in P(M_1)$ and $2 \notin P(M_2)$.

(5.11) EXAMPLE. If $E=\mathbf{Q}(x,y)$, where $6x^2+11y^2=-1$ then $d_{E/\mathbf{Z}}=66$ and $t(66,1)=2$. The two isomorphism classes of relatively minimal models are represented by $M_i=\operatorname{Proj}(\mathbf{Z}[x_0,x_1,x_2]/(q_i))$, $i=1,2$, where

$$q_1 = \begin{bmatrix} 1 & 1 & 17 \\ 1 & 1 & 0 \end{bmatrix}^{66,1} \quad \text{and} \quad q_2 = \begin{bmatrix} 1 & 1 & 22 \\ 0 & 0 & 1 \end{bmatrix}^{66,1} .$$

The corresponding norms-forms are e.g.:

$$N_1 = x_0^2+17x_1^2+17x_2^2+x_3^2+x_0x_1+x_0x_1+x_0x_2+x_1x_2-x_1x_3-x_2x_3 ,$$

$$N_2 = x_0^2+22x_1^2+22x_2^2+x_3^2+x_0x_3-22x_1x_2 .$$

It is easy to check that N_1 represents $3\cdot 7$, while 7 is not represented by N_1. Note that $5 \notin P(M_1)$ and $5 \in P(M_2)$.

In order to discuss more classical, diagonal forms, we may extend the ground ring A.

(5.12) EXAMPLE. If $E=\mathbf{Q}(x,y)$, where $x^2+y^2=-1$ and $A=\mathbf{Z}[1/2]$, then $d_{E/A}=A$ and $M=\operatorname{Proj}(A[x_0,x_1,x_2]/(x_0^2+x_1^2+x_2^2))$ represents the only isomorphism class of relatively minimal A-models of E/A. As a corresponding norm-form we may take $N_0=x_0^2+x_1^2+x_2^2+x_3^2$. By (4.9), N_0 represents 2^np for each prime number p and for some $n \geqq 0$. Now the well-known Euler trick gives that N_0 represents all non-negative integers.

The sum of four squares is a norm-form of a non-hereditary order (in fact, a Bass order) corresponding to the singular $\mathbf{Z}$-scheme $\operatorname{Proj}(\mathbf{Z}[x_0,x_1,x_2]/(x_0^2+x_1^2+x_2^2))$. This suggests that another possibility to look at the problem of integral representations by arbitrary ternary and quaternary quadratic forms is to investigate singular models. This problem will be discussed in the second part of the paper.

REFERENCES

1. A. Białynicki-Birula, *Remarks on relatively minimal models of fields of genus* 0, I, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 15 (1967), 301–307.
2. A. Białynicki-Birula, *Remarks on relatively minimal models of fields of genus* 0, II, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), 81–85.

3. A. Białynicki-Birula, *Remarks on relatively minimal models of fields of genus* 0, III, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 17 (1969), 419–243.
4. A. Białynicki-Birula, *A note on deformations of Severi–Brauer varieties and relatively minimal models of fields of genus* 0, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 18 (1970), 175–176.
5. H. Brandt, *Zur Zahlentheorie der Quaternionen*, Jber. Deutsch. Math.-Verein. 53 (1943), 23–57.
6. H. Brandt und O. Intrau, *Tabellen reduzierter, positiver ternärer quadratischer Formen*, Akademie-Verlag, Berlin, 1958.
7. J. Brzezinski, *Lattices and models of fields of genus* 0, Math. Scand. 32 (1973), 22–30.
8. J. Brzezinski, *Remarks on relations between maximal lattices and relatively minimal models*, Math. Scand. 35 (1974), 25–28.
9. J. Brzezinski, *On the number of relatively minimal models*, J. London Math. Soc. 14 (1976), 216–220.
10. M. Deuring, *Algebren* (Ergebnisse Math. 41), Springer-Verlag, Berlin - Heidelberg - New York, 1968.
11. M. Eichler, *Die Ähnlichkeitsklassen indefiniter Gitter*, Math. Z. 55 (1952), 216–252.
12. M. Eichler, *Zur Zahlentheorie der Quaternionen-Algebren*, J. Reine Angew. Math. 195 (1955), 127–151.
13. M. Eichler, *Quadratische Formen und Ortogonale Gruppen* (Grundlehren Math. Wissensch. 63), Springer-Verlag, Berlin - Heidelberg - New York, 1974.
14. M. Harada, *Structure of hereditary orders*, J. Math. Osaka City Univ. 14 (1963), 1–22.
15. M. Harada, *Multiplicative ideal theory in hereditary orders*, J. Math. Osaka City Univ. 14 (1963), 83–106.
16. I. Kaplansky, *Submodules of quaternion algebras*, Proc. London Math. Soc. 19 (1969), 219–232.
17. M. Kneser, *Quadratische Formen*, Göttingen, 1973/74.
18. C. G. Latimer, *The classes of integral sets in a quaternion algebra*, Duke Math. J. 3 (1937), 237–247.
19. O. T. O'Meara, *Introduction to Quadratic Forms* (Grundlehren Math. Wissensch. 117), Springer-Verlag, Berlin - Heidelberg - New York, 1973.
20. G. Pall, *On generalized quaternions*, Trans. Amer. Math. Soc. 59 (1946), 280–332.
21. M. Peters, *Ternäre und quaternäre quadratische Formen und Quaternionen-Algebren*, Acta. Arith. 15 (1969), 329–365.
22. A. Pizer, *Type Numbers of Eichler Orders*, J. Reine Angew. Math. 264 (1973), 76–102.
23. A. Pizer, *On the arithmetic of quaternion algebras*, Acta Arith. 31 (1976), 61–89.
24. I. Reiner, *Maximal Orders*, Academic-Press, London - New York - San Francisco, 1975.
25. K. W. Roggenkamp, *Lattices over Orders* II (Lecture Notes in Math. 142), Springer-Verlag, Berlin - Heidelberg - New York 1970.
26. A. E. Roos and G. Pall, *An extension of a problem of Kloosterman*, Amer. J. Math. 68 (1946), 59–65.
27. O. Schilling, *Über gewisse Beziehungen zwischen der Arithmetik hyperkomplexer Zahlensysteme und algebraische Zahlkörper*, Math. Ann. 111 (1935), 372–389.
28. R. Shafarevich, *Lectures on minimal models and birational transformations of two dimensional schemes*, Tata Institute of Fundamental Research, Bombay, 1966.
29. J. Brzezinski, *A characterization of Gorenstein orders in quaternion algebras*, Chalmers University of Technology, Preprint, 1980.

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