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# LARGE HOMOMORPHISMS OF LOCAL RINGS

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In [1], L. Avramov introduced the idea of a small homomorphism of local rings. Namely, if $(R,m,k)$ and $(S,m,k)$ are local rings and $f\colon R \to S$ is a homomorphism which makes the triangle $k \leftarrow R \to S \to k$ commute, then $f$ is small if $f_*\colon \operatorname{Tor}^R(k,k) \to \operatorname{Tor}^S(k,k)$ is injective. We present here a dual notion. Namely $f$ will be called large if $f_*$ is surjective. The main result is Theorem 1.1 which gives several equivalent characterizations of surjective large homomorphisms. Besides providing a dual to small, large homomorphisms include the homomorphisms $R \to R/(x)$ where $x$ is either a non zero-divisor or a member of $(0:m)$. They also occur in recent work of Herzog [4] and Schoeller [6]. Section 1 is devoted to proving the main result and Section 2 applies this to produce the examples cited.

## 1.

Let $(R,m,k)$ be a local ring and $M$ a finitely generated $R$-module. The Poincaré series $P_R^M$ is the formal power series $\sum B_i z^i$ where $B_i = \dim_k \operatorname{Tor}_i^R(M,k)$. Also, if $S$ is any $R$-algebra, $\operatorname{Tor}^R(S,k)$ has the structure of a graded algebra via the standard external product [5, p. 221].

THEOREM 1.1. *Let $(R,m,k)$ and $(S,n,k)$ are local rings and $f\colon R \to S$ a local homomorphism which is surjective. Then the following are equivalent.*

1. *The homomorphism $f$ is large, i.e. $f_*\colon \operatorname{Tor}^R(k,k) \to \operatorname{Tor}^S(k,k)$ is surjective.*
2. *For any finitely generated $S$-module $M$, considered as an $R$-module via $f$,*

$$P_R^M = P_S^M P_R^S .$$

3. *The homomorphism $p_*\colon \operatorname{Tor}^R(S,k) \to \operatorname{Tor}^R(k,k)$ induced by the canonical map $p\colon S \to k$ is injective.*
4. *For any finitely generated $S$-module $M$, regarded as an $R$-module via $f$, the induced homomorphisms $\operatorname{Tor}^R(M,k) \to \operatorname{Tor}^S(M,k)$ are surjective.*
5. *There is an exact sequence of algebras*

$$k \to \operatorname{Tor}^R(S,k) \to \operatorname{Tor}^R(k,k) \to \operatorname{Tor}^S(k,k) \to k .$$

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PROOF.

1 ⇒ 3. We consider the change of rings spectral sequence

$$E^2_{p,q} \cong \mathrm{Tor}^S_p(k,k)\otimes\mathrm{Tor}^R_q(S,k) \Rightarrow \mathrm{Tor}^R_{p+q}(k,k)\,.$$

By [2, p. 348], the edge homomorphisms

$$\mathrm{Tor}^R_p(k,k) \to \mathrm{Tor}^S_p(k,k) \cong E^2_{p,0}$$

and

$$E^2_{0,q} \cong \mathrm{Tor}^R_q(S,k) \to \mathrm{Tor}^R_q(k,k)$$

are precisely the induced homomorphisms $f_*$ and $p_*$.

Hence $f_*$ is surjective if and only if $E^2_{p,0}=E^\infty_{p,0}$ for all $p\geqq 0$ and $p_*$ is injective if and only if $E^2_{0,q}=E^\infty_{0,q}$. We will prove the latter by showing that, in fact, all the $d^r_{p,q}$ for $r\geqq 2$ are zero.

The spectral sequence is derived from the double complex

$$C_{p,q} = Y_p\otimes_R X_q$$

where $Y$ is a free resolution of $k$ over $S$ and $X$ is a free resolution of $k$ over $R$. Choosing $Y$ and $X$ to be algebra resolutions makes $C$ a DG algebra and then each $E^r, d^r$ is also a DG algebra. With respect to this algebra structure

$$E^2_{p,q} \cong \mathrm{Tor}^S_p(k,k)\otimes\mathrm{Tor}^R_q(S,k) = E^2_{p,0}E^2_{0,q}\,.$$

Since $f$ is surjective, $d^2_{p,0}\colon E^2_{p,0} \to E^2_{p-2,1}$ is zero and $d^2_{0,q}\colon E^2_{0,q} \to 0$ so since $d^2$ is a derivation on $E^2$, $d^2_{p,q}=0$ for all $p\geqq 0$ and $q\geqq 0$. Now $E^3=E^2$ so $E^3_{p,q}=E^3_{p,0}E^3_{0,q}$. For every $p\geqq 0$, $d^3_{p,0}=0$ since $f_*$ is surjective so the argument may be continued to give $d^r_{p,q}=0$ for all $r\geqq 2$, $p\geqq 0$, $q\geqq 0$.

3 ⇒ 2. Let $Y$ be a minimal resolution of $M$ over $S$ and $X$ a minimal resolution of $k$ over $R$. We consider two double complexes

$$C_{p,q} = Y_p\otimes_R X_q \quad\text{and}\quad C'_{p,q} = Y_p\otimes_S k\otimes_R X_q$$

with canonical maps

$$h_{p,q}\colon C_{p,q} \to C'_{p,q}\,.$$

Since $Y$ and $X$ are both minimal, the differential in $C'$ is zero and $h$ is clearly a mapping of complexes. Define

$$F_pC = \sum_{i\leqq p} Y_i\otimes_R X \quad\text{and}\quad F_pC' = \sum_{i\leqq p} Y_i\otimes_S k\otimes_R X\,.$$

With these filtrations $C$ gives rise to the change of rings spectral sequence $E^r_{p,q}$ with

$$E^2_{p,q} \cong \mathrm{Tor}^S_p(M,k) \otimes \mathrm{Tor}^R_q(S,k) \Rightarrow \mathrm{Tor}^R_{p+q}(M,k)$$

and $C'$ gives rise to a spectral sequence $E'^r_{p,q}$ with

$$E'^2_{p,q} \cong \mathrm{Tor}^S_p(M,k) \otimes \mathrm{Tor}^R_q(k,k)$$

and $d'^r_{p,q}=0$ for all $r \geqq 0$.

The mapping $h$ then induces a mapping of spectral sequences

$$h^r_{p,q}\colon E^r_{p,q} \to E'^r_{p,q}$$

with $h^2_{p,q}$ simply given by

$$1 \otimes p_*\colon \mathrm{Tor}^S_p(M,k) \otimes \mathrm{Tor}^R_q(S,k) \to \mathrm{Tor}^S_p(M,k) \otimes \mathrm{Tor}^R_q(k,k)$$

where $p$ is, as above, the homomorphism induced by the canonical map $p\colon S \to k$. By hypothesis, $p_*$ and thus $h^2_{p,q}$ is injective.

Then the commutative diagram

$$\begin{array}{ccc} E^2_{p,q} & \xrightarrow{d^2} & E^2_{p-2,q+1} \\ {\scriptstyle h^2}\downarrow & & \downarrow{\scriptstyle h^2} \\ E'^2_{p,q} & \xrightarrow{0} & E'^2_{p-2,q+1} \end{array}$$

shows that $d^2_{p,q}=0$. Then $E^3=E^2$ and $h^3=h^2$ is injective so $d^3_{p,q}=0$, etc. Thus, in the change of rings spectral sequence $E^2=E^\infty$ showing that

$$P^M_R = P^M_S P^S_R$$

$2 \Rightarrow 4$. In general, from the spectral sequence

$$\mathrm{Tor}^S_p(M,k) \otimes \mathrm{Tor}^R_q(S,k) \Rightarrow \mathrm{Tor}^R(M,k)$$

one obtains only an inequality of Poincaré series

$$P^M_S P^S_R \leqq P^M_R \,.$$

This is an equality if and only if the spectral sequence degenerates. In particular, if equality holds, all $d^r_{p,0}=0$ for $r \geqq 2$ so the edge homomorphism

$$\mathrm{Tor}^R_p(M,k) \to \mathrm{Tor}^S_p(M,k) \cong E^2_{p,0}$$

is surjective.

$4 \Rightarrow 1$ is obvious.

$5 \Leftrightarrow 1$. From the diagram

$$\begin{array}{ccc} \mathrm{Tor}^R(S,k) & \to & \mathrm{Tor}^S(S,k) \cong k \\ \downarrow & & \downarrow \\ \mathrm{Tor}^R(k,k) & \longrightarrow & \mathrm{Tor}^S(k,k) \end{array}$$

we see that the composite

$$\mathrm{Tor}_p^R(S,k) \xrightarrow{p_*} \mathrm{Tor}_p^R(k,k) \xrightarrow{f_*} \mathrm{Tor}_p^S(k,k)$$

is zero for $p>0$. We check the exactness by showing that the dual sequence of graded co-algebras is exact. Via the Yoneda product, $\mathrm{Ext}_R(k,k)$ and $\mathrm{Ext}_S(k,k)$ are algebras and $\mathrm{Ext}_R(S,k)$ is a left $\mathrm{Ext}_R(k,k)$ module. There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}_R(k,k)\otimes I\mathrm{Ext}_R(S,k) & \longrightarrow & I\mathrm{Ext}_R(S,k) \\ \uparrow & & \uparrow p^* \\ \mathrm{Ext}_R(k,k)\otimes I\mathrm{Ext}_R(k,k) & \xrightarrow{\varphi} & I\mathrm{Ext}_R(k,k) \\ \uparrow 1\otimes f^* & & \\ \mathrm{Ext}_R(k,k)\otimes I\mathrm{Ext}_S(k,k) & & \end{array}$$

showing that $p^*\varphi(1\otimes f^*)=0$. One then has a surjection

$$\mathrm{Ext}_R(k,k)\otimes_{\mathrm{Ext}_S(k,k)} k \to \mathrm{Ext}_R(S,k)\,.$$

Now, however, $\mathrm{Ext}_S(k,k)$ and $\mathrm{Ext}_R(k,k)$ are Hopf algebras so the left hand side has Poincaré series $P_R^k/P_S^k = P_R^S$ by hypothesis (using $1 \Rightarrow 2$) so the surjection above is, in fact, an isomorphism.

## 2.

The following theorem is useful in finding large homomorphism.

THEOREM 2.1. *If $(R,m,k)$ is a local ring and $x \in m \sim m^2$ and the homomorphism $R \to R/\mathrm{Ann}(x)$ is large, then so is the homomorphism $R \to R/(x)$.*

PROOF. The diagram

$$\begin{array}{ccc} R/\mathrm{Ann}(x) & \xrightarrow{p} & k \\ \cong\downarrow & & \downarrow\cong \\ (x) & \xrightarrow{q} & (x)\otimes_R k \end{array}$$

gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Tor}^R(R/\mathrm{Ann}(x),k) & \xrightarrow{p_*} & \mathrm{Tor}_R(k,k) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Tor}^R((x),k) & \xrightarrow{q_*} & \mathrm{Tor}^R((x)\otimes_R k,k)\,. \end{array}$$

Since $R \to R/\mathrm{Ann}(x)$ is large, $p_*$ is injective so $q_*$ is also injective. Then the diagram

$$\begin{array}{ccc} (x) & \longrightarrow & (x)\otimes_R k \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ m & \longrightarrow & m\otimes_R k \end{array}$$

yields

$$\begin{array}{ccc} \mathrm{Tor}^R((x),k) & \xrightarrow{q_*} & \mathrm{Tor}^R((x)\otimes_R k,k) \\ {\scriptstyle u_*}\downarrow & & \downarrow{\scriptstyle v_*} \\ \mathrm{Tor}^R(m,k) & \longrightarrow & \mathrm{Tor}^R(m\otimes_R k,k)\,. \end{array}$$

Since $x \notin m^2$, $(x)\otimes_R k \to m\otimes_R k$ is injective, but then $v_*$ is injective and hence also $u_*$.

Finally, the diagram

$$\begin{array}{ccccccccc} 0 & \to & (x) & \to & R & \to & R/(x) & \to & 0 \\ & & \downarrow{\scriptstyle u} & & \| & & \downarrow{\scriptstyle w} & & \\ 0 & \to & m & \to & R & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

gives

$$\begin{array}{ccc} \mathrm{Tor}_i^R(R/(x),k) & \xrightarrow{\cong} & \mathrm{Tor}_{i-1}^R((x),k) \\ {\scriptstyle w_*}\downarrow & & \downarrow{\scriptstyle u_*} \\ \mathrm{Tor}_i^R(k,k) & \xrightarrow{\cong} & \mathrm{Tor}_{i-1}^R(m,k)\,. \end{array}$$

Since $u_*$ is injective, so is $w_*$ and then $R \to R/(x)$ is large by Theorem 1.1.

THEOREM 2.2. *If $(R,m,k)$ is a local ring, $x \in m \sim m^2$ and either*

i) *$x$ is a non zero-divisor*

or ii) *$x \in (0\colon m)$,*

*then $R \to R/(x)$ is large.*

PROOF. Note that the identity map $R \to R = R/(0)$ is obviously large. But if $x$ is a non zero-divisor, $\mathrm{Ann}\,(x) = 0$ so $R \to R/(x)$ is large by Theorem 2.1. Similarly, the canonical map $R \to k = R/m$ is large so, again by Theorem 2.1, $R \to R/(x)$ is large.

Examples of large homomorphisms have arisen recently in the work of Herzog [4] and Schoeller [5]. We obtain these results as a consequence of Theorem 1.1.

DEFINITION. A local homomorphism of local rings $g\colon S \to R$ is called an algebra retract if there exists a local homomorphism $f\colon R \to S$ such that $fg = 1_S$.

THEOREM 2.3. (Herzog) *If $g: S \to R$ is an algebra retract with inverse $f: R \to S$, then for any finitely generated S-module M, regarded as an R-module via f,*

$$P_R^M = P_S^M P_R^S.$$

PROOF. Since $fg = 1_S$,

$$f_* g_*: \operatorname{Tor}^S(k,k) \to \operatorname{Tor}^R(k,k) \to \operatorname{Tor}^S(k,k)$$

is the identity on $\operatorname{Tor}^S(k,k)$ so $f_*$ is surjective, $f$ is large and Theorem 1.1 applies.

THEOREM 2.4. (Schoeller) *If $(R, m, k)$ is a local ring and $x \in m \sim m^2$ such that $R/(x)$ is a complete intersection, then*

i) $\operatorname{Tor}^R(k,k) \to \operatorname{Tor}^{R/(x)}(k,k)$ *is surjective.*

ii) $\operatorname{Tor}^R(R/(x),k) \to \operatorname{Tor}^R(k,k)$ *is injective.*

iii) *For any finitely generated $R/(x)$-module M regarded as an R-module via f: $R \to R/(x)$, $P_R^M = P_S^M P_R^S$.*

iv) *The acyclic closure* ([3]) *of the augmented algebra $R \to R/(x)$ is a minimal resolution.*

PROOF. By Tate's theorem, since $R/(x)$ is a complete intersection, $\operatorname{Tor}^{R/(x)}(k,k)$ is generated as an algebra with divided powers by its elements of degree 1 and 2. Hence it suffices to check that

$$f_{*,i}: \operatorname{Tor}_i^R(k,k) \to \operatorname{Tor}_i^{R/(x)}(k,k)$$

is surjective for $i = 1, 2$. Referring back to the spectral sequence

$$\operatorname{Tor}_i^{R/(x)}(k,k) \otimes \operatorname{Tor}^R(R/(x),k) \Rightarrow \operatorname{Tor}_{i+j}^R(k,k)$$

used in the proof of Theorem 1.1, it is clear that $E_{1,0}^2 = E_{1,0}^\infty$ so $f_{*,1}$ is surjective. Then $f_{*,2}$ is surjective if and only if $d_{2,0}^2 = 0$ which is true if and only if $\operatorname{Tor}_1^R(R/(x),k) \to \operatorname{Tor}_1^R(k,k)$ is injective. However this is equivalent to having $x \in m \sim m^2$. This proves i), and ii) and iii) follow from Theorem 1.1. Part iv) follows from the following more general result.

The following result was proved by H. Rahbar-Rochandel and independently by L. Avramov. Here is Avramov's proof.

THEOREM 2.5. *If $R \to S$ is a surjective, large homomorphism of local rings, then the acyclic closure of the augmented algebra $R \to S$ is a minimal resolution.*

PROOF. By Theorem 1.1 above and [1, Corollary 1.3(b), p. 408],

$$\operatorname{Tor}^R(S,k) \cong \operatorname{Tor}^R(k,k) \searrow f_*$$

is a free $\Gamma$-algebra. Let $P$ be the acyclic closure of $R \to S$ and suppose that $i$ is the least integer such that $d(P_{i+1}) \not\subset mP_i$. Since $P$ is the acyclic closure, $d(P_{i+1}) \subset C_i(P) + mP_i$, where $C_i(P)$ denotes the elements of $P_i$ generated by products of lower positive degree and divided powers of elements of even degree. By the choice of $i$, there is an isomorphism $P_j \otimes_R k \xrightarrow{\cong} \mathrm{Tor}_j^R(S,k)$ for $j<i$. The above information gives a relation in $\mathrm{Tor}_i^R(S,k)$ but $\mathrm{Tor}^R(S,k)$ is supposed to be a free $\Gamma$-algebra.

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