MATH. SCAND. 46 (1980), 216–222

SMOOTHING H-SPACES II

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0. Introduction.

In [8] we showed that certain H-spaces obtained by homotopy mixing are homotopy equivalent to smooth, stably parallelizable manifolds. Unfortunately (as was added in proof) we needed the restriction on the fundamental group π, that $D(\mathbf{Z}\pi)=0$. It is the purpose of this sequel to [8] to remove this restriction, and also to generalize the main theorem considerably. I want to thank I. Hambleton for pointing out the error in [8].

1. Notation. Statement of results.

Throughout the paper space will mean topological space of the homotopy type of a connected C.W. complex. For a set of primes l and a nilpotent space X, X_l denotes the localization of X at l in the sense of [5]. A space X is called quasifinite if $H_*(X)=\oplus H_i(X;\mathbf{Z})$ is a finitely generated abelian group. If $H_*(X)$ is a $\mathbf{Z}_l$-module X is called l-locally quasifinite if $H_*(X)$ is finitely generated as a $\mathbf{Z}_l$-module.

To state our main theorem we need a couple of definitions. Let S^i denote the i-sphere

DEFINITION 1.1. A nilpotent space X admits a special 1-torus if, up to homotopy, there is a diagram of orientable fibrations

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^3 & \longrightarrow & S^2 \\ \downarrow & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & X & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & A & \longrightarrow & A \end{array}$$

such that

(a) A is quasifinite, B is stably reducible.

(b) Localized at 0 the diagram is homotopy equivalent to

Received May 22, 1979; in revised form January 2, 1980.

$$\begin{array}{ccccc} S^1_0 & \longrightarrow & S^3_0 & \longrightarrow & S^2_0 \\ \downarrow & & \downarrow & & \downarrow \\ S^1_0 & \longrightarrow & A_0 \times S^3_0 & \longrightarrow & A_0 \times S^2_0 \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ * & \longrightarrow & A_0 & \longrightarrow & A_0 \end{array}$$

We remark that since X is nilpotent and the fibrations are orientable A and B are nilpotent so localization makes sense.

EXAMPLE 1.2. Given a bundle $S^3 \to X \to A$ with A quasifinite stably reducible and the bundle stably trivial then X admits a special 1-torus by dividing out by the subgroup $S^1 \subset S^3$ (see [8] lemma 3.6). All compact Lie groups other than $\mathrm{SO}(3)^k \times T^l$ have subgroups isomorphic to S^3 as is seen by classification and all these Lie groups admit special 1-tori. This will be further discussed in section 5.

We need a p-local version of definition 1.1. Let X be a nilpotent space, p a prime.

DEFINITION 1.3. X admits a p-local special 1-torus if, up to homotopy, there is a diagram of orientable fibrations

$$\begin{array}{ccccc} S^1_p & \longrightarrow & S^3_p & \longrightarrow & S^2_p \\ \downarrow & & \downarrow & & \downarrow \\ S^1_p & \longrightarrow & X & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & A & \longrightarrow & A \end{array}$$

such that

(a) A is p-locally quasifinite and B is p-locally stably reducible, i.e. there are integers n and i and a map $S_p^{n+i} \to \Sigma^i B$ inducing isomorphism in homology in dimensions $\geqq n+i$.

(b) as in definition 1.1.

It is clear that if X admits a special 1-torus then X_p admits a p-local special 1-torus.

We prove the following

THEOREM 1.4. *Let X be a quasifinite H-space. Assume for every prime p that X_p is homotopy equivalent to a product $C(p) \times D(p)$ and $C(p)$ admits a p-local special 1-torus. Then X is homotopy equivalent to a smooth, stably parallellizable manifold.*

REMARK 1.5. If $H_3(X) \supset \mathbf{Z}$ the condition of the theorem is trivially satisfied for all but finitely many primes. This is because for all but finitely many primes X_p is homotopy equivalent to a product of localized spheres which must include a 3-sphere. We also note that in view of example 1.2 (see section 5) this theorem is stronger than theorem 1.1 of [8].

2. Surgery.

We use special 1-tori for the following

PROPOSITION 2.1. *Let X be a quasifinite, nilpotent Poincaré complex admitting a special* 1*-torus. Then X is homotopy equivalent to a stably parallellizable smooth manifold.*

REMARK. This result only needs condition (a) of definition 1.1. Condition (b) is needed to ensure that the property of having a special 1-torus is a generic property.

PROOF OF PROPOSITION 2.1. In the diagram of orientable fibrations

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^3 & \longrightarrow & S^2 \\ \downarrow & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & X & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & A & \longrightarrow & A \end{array}$$

A is nilpotent and quasifinite, hence by [7] A is finitely dominated. It follows that X and B are finitely dominated [6]. Also A and B are Poincaré Duality spaces since X is [3]. It follows from [10] that X has 0 finiteness obstruction. Considering (B, X) a Poincaré Duality pair, we may use the stable reduction of B and a standard transversality procedure to produce a surgery problem

$$(M, \partial M) \xrightarrow{\varphi} (B, X) \qquad \hat{\varphi}: \nu_M \to \varepsilon$$

where ε is the trivial bundle. Let $\sigma(B)$ be the finiteness obstruction of B. Consider the exact sequence

$$\ldots \to H^{n+1}(\mathbf{Z}_2; \tilde{K}_0(\mathbf{Z}\pi)) \to L_n^h(\pi) \xrightarrow{\delta} L_n^p(\pi) \to$$

where $\pi = \pi_1(B) = \pi_1(X)$. The class of $\sigma(B)$, $\{\sigma(B)\}$, is an element of $H^{n+1}(\mathbf{Z}_2, \tilde{K}_0(\mathbf{Z}\pi))$. It follows from [9], that the surgery obstruction of $\partial M \to X$ is $\delta\{\sigma(B)\}$. However, since A is a P.D. space of dimension $n-3$ we have $\sigma(A) = (-1)^{n-3}\sigma(A)^*$ and by [10], $\sigma(B) = 2\sigma(A) = \sigma(A) + (-1)^{n+1}\sigma(A)^*$ and hence $\{\sigma(B)\} = 0$ in $H^{n+1}(\mathbf{Z}_2; \tilde{K}_0(\mathbf{Z}\pi))$ and we are done.

3. Reducibility of p-local H-spaces.

Browder and Spanier have shown that a finite H-space is stably reducible [2]. This is one of the steps in the attempt to prove X is a manifold, since it implies that the Spivak normal fibre space is trivial. We need to generalize the results of Browder and Spanier to a p-local situation. This is mostly straightforward. We shall nevertheless indicate the line of argument in this section. The aim of the section is to prove:

THEOREM 3.1. *Let D be a p-locally quasi finite* H-*space. Then D is p-locally stably reducible.*

We need a p-local edition of S-duality.

PROPOSITION 3.2. *Let X be a simply connected p-locally quasifinite space. Then X admits a finite p-local* C.W. *structure, i.e. X is homotopy equivalent to a space Y with a filtration $* = K_0 \subset K_1 \subset \ldots \subset K_n = Y$ such that K_i is the mapping cone of some map f_i: $S_p^{n(i)} \to K_{i-1}$, $n(i)$ a nondecreasing function of i.*

PROOF. By the Hurewicz theorem we may find a finite wedge of local spheres and a map f: $\vee\ S_p^k \to X$ such that $H_*(f)$ is onto in dimensions $\leqq k$, $k \geqq 2$. Using the relative Hurewicz theorem we inductively attach local cells to make $H_*(f)$ an isomorphism in higher dimensions. Since X is p-locally quasifinite and finitely generated $\mathbf{Z}_p$-module have free resolutions of length one, we eventually obtain a homotopy equivalence.

Let X and Y be p-locally quasifinite spaces. A p-local S-duality map is a map $X \wedge Y \to S_p^n$ so that slant product

$$f^*(i)/-: \tilde{H}_*(X) \to \tilde{H}^{n-*}(Y)$$

is an isomorphism. Here i is the generator of $H^n(S_p^n)$.

Given a p-locally finite space X we note that the suspension ΣX is a simply connected p-locally quasifinite space and thus admits a p-local C.W. structure by Proposition 3.2. We may now go through exercises F1-7 page 463 in Spanier [11] to prove existence and stable uniqueness of a p-local S-dual with the usual functoriel properties. We need the concept to complete the

PROOF OF THEOREM 3.1. $H^*(D; \mathbf{Q})$ and $H^*(D; \mathbf{Z}/p\mathbf{Z})$ are Hopf algebras and we may argue as in the finite case [1] that D satisfies Poincaré Duality with $\mathbf{Z}_p$ coefficients. We now only need to produce a map $D \to S_p^n$ inducing isomorphism in dimensions $\geqq n$. Then we may use Hopf algebra arguments (as in the finite case [2]) to prove that the composite $D^+ \wedge D^+ = (D \times D)^+ \to D$

$\to S_p^n$ is a p-local S-duality map, so D^+ is selfdual and the dual of $D^+ \to S_p^n$ will be a stable reduction.

Localized at 0 D is a product of odd dimensional spheres so if we let l be the set of primes different from p and form the homotopy pullback

$$\begin{array}{ccc} Y & \longrightarrow & \pi S_l^{2n_i+1} \\ \downarrow & & \downarrow \\ D & \longrightarrow & \pi S_0^{2n_i+1} \end{array}$$

then Y is quasifinite, nilpotent and satisfies Poincaré Duality at all primes hence [5] and [7] is a finitely dominated Poincaré Duality space. By Wall [12] Y has the homotopy type of $K \cup e^n$ where K is $n-1$ dimensional and we may thus produce a map $Y \to K \cup e^n \to S^n$ by collapsing K to a point. Localizing at p we obtain $D \cong K_p \to S_p^n$ with the required property and we are done.

4. Proof of main Theorem.

The proof will consist of two lemmas.

LEMMA 4.1. *If X is a quasifinite* H-*space and $X_p = C(p) \times D(p)$ where $C(p)$ admits a p-local special* 1-*torus, then X_p does.*

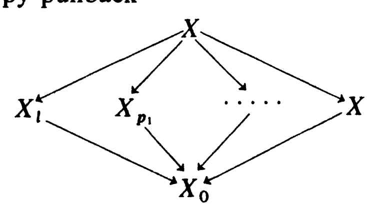
PROOF. Crossing the special 1-torus diagram

$$\begin{array}{ccc} C(p) & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \end{array}$$

with $D(p)$ reduces the lemma to showing $B \times D(p)$ is p-locally stably reducible. Now $D(p)$ is a retract of a p-locally quasifinite H-space and is thus itself a p-locally quasifinite H-space and thus p-locally stably reducible by theorem 3.1.

LEMMA 4.2. *If X is a quasifinite* H-*space such that each X_p admits a p-local special* 1-*torus, then X admits a special* 1-*torus.*

PROOF. At all but finitely many primes X is a product of spheres, so we may consider X a homotopy pullback



where X_l is a product of odd dimensional spheres and X_{p_i} also admit special 1-tori. Mixing the special 1-tori in the obvious way we obtain $\bar{X}$ which admits a special 1-torus and such that $\bar{X}_l \cong X_l$ and $\bar{X}_{p_i} \cong X_{p_i}$; in other words $\bar{X}$ is in the genus of X. We now argue as in [8] Proposition 3.2 to show that admitting a special 1-torus is a generic property for an H-space. The key step is the result of Zabrodsky that one obtains the whole genus of an H-space by mixings defined by diagonal matrices and the observation in [8] that one of these diagonal entries may be assumed to be 1.

Combining these two lemmas with Proposition 2.1 proves theorem 1.4.

5. Examples.

In this section we show that compact Lie groups other than $SO(3)^k \times T^l$ do admit special 1-tori. This implies that our theorem 1.4 is indeed stronger than theorem 1.1 of [8].

PROPOSITION 5.1. *Let G be a compact connected Lie group which is not isomorphic to* $SO(3)^k \times T^l$. *Then G has a subgroup isomorphic to S^3.*

PROOF. We use classification of compact Lie groups. Any compact connected Lie group is a quotient of $H \times T^l$ by a discrete central subgroup A. Here H is a simply connected compact Lie group. Furthermore H is a product of groups in a list, see [4, p. 346]. If we can find an S^3 subgroup of H that intersects A trivially we are done. There are two cases. First assume $H=(S^3)^k$. Then A cannot contain the center of $(S^3)^k$ since if it does G will be isomorphic to $SO(3)^k \times T^l$. This being the case it is easy to find a subgroup isomorphic to S^3 not intersecting A. If H is not a product of S^3's it has a simple factor different from S^3 and we will be done if we can find a subgroup isomorphic to S^3 in this factor, intersecting the center trivially. We do this by checking the list. We have $S^3 = SU(2) \subset SU(n)$ $(n \geqq 3)$ intersecting the center trivially since the central elements of $SU(n)$ are diagonal matrices with the same n'th root of unity as entry. Similarly, $S^3 = SP(1) \subset Sp(n)$ $(n \geqq 2)$ and $S^3 = SU(2) \subset SO(4) \subset SO(n)$, $n \geqq 5$ do not contain $-I$ which is the only central element $\neq I$. Furthermore $E_8 \supset E_7 \supset E_6 \supset SU(6) \supset S^3$ and since the center of E_6 has order 3, S^3 must intersect it trivially and E_6 must intersect the center of E_7 (cyclic of order 2) trivially. Finally $F_4 \supset SP(3)$ and $G_2 \supset SU(2)$ and these groups have trivial center. We are done.

REMARK 5.2. It would be nice to have a conceptual proof of Proposition 5.1. Working in the Lie algebra it is not hard to find a subgroup isomorphic to $SO(3)$ or S^3 but it is crucial for us to be in the latter case.

PROPOSITION 5.3. *Let G be a compact Lie group with S^3 as a subgroup $G \supset S^3$. Then*

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^3 & \longrightarrow & S^2 \\ \downarrow & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & G & \longrightarrow & G/S^1 \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & G/S^3 & \longrightarrow & G/S^3 \end{array}$$

is a special 1-*torus in G.*

PROOF. Lemma 3.4 of [8] shows that G/S^1 is stably parallellizable. It follows from lemma 3.3 of [8] that $H^3(G; \mathbf{Q}) \to H^3(S^3; \mathbf{Q})$ is onto. Let $G_0 \to K(\mathbf{Q}, 3) = S_0^3$ represent an element in $H^3(G; \mathbf{Q})$ hitting the generator of $H^3(S^3; \mathbf{Q})$ then one sees by a spectral sequence argument that $G_0 \to (G/S^3)_0 \times S_0^3$ is a homology equivalence hence a homotopy equivalence and we are done.

FINAL REMARKS. In case $D(\mathbf{Z}\pi) = 0$ we could replace the concept special 1-torus by the concept 1-torus (see [8]). Since admitting a 1-torus is a weaker condition than admitting a special 1-torus, it is not entirely a loss to have both concepts.

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