MATH. SCAND. 46 (1980), 223–235

COHOMOLOGY MOD 3 OF THE CLASSIFYING SPACE OF THE LIE GROUP E_6

AKIRA KONO and MAMORU MIMURA

1. Introduction.

Let E_6 be the compact, 1-connected, simple Lie group of type E_6. $H_*(E_6; Z)$ is p-torsion free for any prime $p>3$ ([7]) and has p-torsion for $p=2$ and 3 ([1], [2], [5]). The cohomology rings $H^*(E_6; Z_p)$ are known:

(1.1) $$H^*(E_6; Z_2) \cong Z_2[e_3]/(e_3^4) \otimes \Lambda(e_5, e_9, e_{15}, e_{17}, e_{23}) ,$$

(1.2) $$H^*(E_6; Z_3) \cong Z_3[e_8]/(e_8^3) \otimes \Lambda(e_3, e_7, e_9, e_{11}, e_{15}, e_{17}) ,$$

where $e_7 = \mathscr{P}^1 e_3$, $e_8 = \beta e_7$ and $e_{15} = \mathscr{P}^1 e_{11}$,

(1.3) $$H^*(E_6; Z_p) \cong \Lambda(e_3, e_9, e_{11}, e_{15}, e_{17}, e_{23}) \text{ for any prime } p>3 ,$$

where $\deg e_i = i$.

It is known that for any prime $p>3$, $H^*(E_6; Z_p)$ is universally transgressively (and hence primitively) generated and so by the Borel theorem

$$H^*(BE_6; Z_p) \cong Z_p[x_4, x_{10}, x_{12}, x_{16}, x_{18}, x_{24}] \quad \text{with } \deg x_i = i .$$

The purpose of this paper is firstly to determine the Hopf algebra structure of $H^*(E_6; Z_3)$ and then to determine the module structure of $H^*(BE_6; Z_3)$ by making use of the Eilenberg–Moore spectral sequence $\{E_r, d_r\}$ such that

$$E_2 = \mathrm{Cotor}^A(Z_3, Z_3) \quad \text{with } A = H^*(E_6; Z_3) ,$$

$$E_\infty = \mathscr{G}r\, H^*(BE_6; Z_3) .$$

(The result with Z_2-coefficient is already determined in [10]).

According to Browder [4] $H^*(E_6; Z_3)$ is not primitively generated. In fact, Araki showed in [1] that $\bar\varphi(e_{11}) = e_8 \otimes e_3$ and $\bar\varphi(e_{15}) = e_8 \otimes e_7$, where

$$\bar\varphi\colon \tilde H^*(E_6; Z_3) \to \tilde H^*(E_6; Z_3) \otimes H^*(E_6; Z_3)$$

is reduced diagonal map induced from the multiplication on E_6 (the same notation is used for F_4). Thus it seems hard to calculate the Hopf algebra $H^*(E_6; Z_3)$.

Received June 12, 1979.

The paper is organized as follows:

In section 2 we determine the Hopf algebra structure of $H^*(E_6; Z_3)$. The result is

THEOREM 2.9. *With suitably chosen e_{11}, e_{15} and e_{17} in* (1.2)

(1) *e_3, $e_7=\mathcal{P}^1e_3$, $e_8=\beta e_7$ and e_9 are universally transgressive and hence primitive.*

(2) *e_{11}, e_{15} and e_{17} can not be chosen to be primitive and $\bar{\varphi}(e_j)=e_8\otimes e_{j-8}$ for $j=11, 15, 17$.*

In section 3 we recall the results on $\mathrm{Cotor}^A(Z_3, Z_3)$ and $\mathrm{Cotor}^B(Z_3, Z_3)$ with $A=H^*(E_6; Z_3)$ and $B=H^*(F_4; Z_3)$, where F_4 is the compact, 1-connected Lie group of type F_4. It is known ([11]) that the Eilenberg–Moore spectral sequence for F_4 with Z_3-coefficient collapses.

In sections 4 and 5 we prove that the Eilenberg–Moore spectral sequence for E_6 also collapses with Z_3-coefficient by using the naturality for the inclusion $F_4 \hookrightarrow E_6$ and for dimensional reasons. Thus

THEOREM 5.6. *As modules*

$$H^*(BE_6; Z_3) \cong \mathrm{Cotor}^A(Z_3, Z_3) \quad \text{with } A=H^*(E_6; Z_3)\,.$$

However, it seems difficult to determine the algebra structure of $H^*(BE_6; Z_3)$, although we give a partial result in section 6.

To determine the algebra structure we need information of the invariant subalgebra of $H^*(BT^6; Z_3)$ under the Weyl group of E_6, where T^6 is the maximal torus of E_6 (cf. [17], [18]).

Throughout the paper $H^*(X)$ means $H^*(X; Z_3)$ unless otherwise stated.

The paper is a revised version of [9] and the result was announced in [18].

ACKNOWLEDGEMENT. The early draft of the paper was written while the second author was at Aarhus Universitet. He is most grateful to Professor Leif Kristensen for the kind invitation to Matematisk Institut at Aarhus.

2. Hopf algebra structure of $H^*(E_6)$.

In this section we will determine the Hopf algebra structure of $H^*(E_6)$. The following is due to Araki [1]:

THEOREM 2.1. (1) $H^*(E_6)\cong Z_3[e_8]/(e_8^3)\otimes\Lambda(e_3, e_7, e_9, e_{11}, e_{15}, e_{17})$,

(2) $H^*(F_4)\cong Z_3[e_8]/(e_8^3)\otimes\Lambda(e_3, e_7, e_{11}, e_{15})$,

where $\deg e_i=i$, $e_7=\mathcal{P}^1e_3$, $e_8=\beta e_7$ *and* $e_{15}=\mathcal{P}^1e_{11}$,

(3) F_4 *is totally non-homologous to zero* mod 3 *in* E_6,
(4) $H^*(E_6/F_4; Z) \cong \Lambda(\bar{e}_9, \bar{e}_{17})$ *with* $\deg \bar{e}_i = i$,
(5) $\bar{\varphi}(e_j) = e_8 \otimes e_{j-8}$ *for* $j = 11, 15$ *and* e_i *is universally transgressive and hence primitive for* $i = 3, 7, 8, 9$.

Thus it suffices to determine $\bar{\varphi}(e_{17})$ in order to know the Hopf algebra structure of $H^*(E_6)$. (The result is stated in Theorem 2.13).

Consider the two fiberings:

$$F_4 \to E_6 \xrightarrow{\varrho} E_6/F_4 \, , \tag{2.2}$$

$$E_6/F_4 \to BF_4 \to BE_6 \, . \tag{2.2)'$$

NOTATION. By abuse of notation we denote by $\bar{e}_9$ and $\bar{e}_{17}$ the mod 3 reduction of $\bar{e}_9$ and $\bar{e}_{17}$ respectively.

The following is easy to see.

PROPOSITION 2.3. (1) $e_9 = \varrho^*(\bar{e}_9)$ (*up to sign*).
(2) *One of the following is true*:

(2.4) $\bar{e}_{17}$ *is transgressive with respect to* (2.2)′.
(2.5) *there is an element* $f_9 \neq 0$ *with* $\deg f_9 = 9$ *such that*

$$d_9(1 \otimes \bar{e}_{17}) = f_9 \otimes \bar{e}_9 \quad \text{and} \quad f_9 \tau(\bar{e}_9) = 0 \, ,$$

where d_9 *is the differential and* τ *is the transgression in the Serre spectral sequence associated with* (2.2)′.

(3) (2.4) *and* (2.5) *are equivalent to the following* (2.4)′ *and* (2.5)′ *respectively*:

(2.4)′ $\varrho^*(\bar{e}_{17}) = e_{17}$ *is universally transgressive*,

(2.5)′ e_{17} *can not be chosen to be primitive.*

PROOF. (1) By (3) of Theorem 2.1 $0 \neq \varrho^*(\bar{e}_9) \in H^9(E_6) \cong Z_3$.
(2) is obvious.
(3) Consider the commutative diagram:

$$\begin{array}{ccccc} E_6 & \longrightarrow & * & \longrightarrow & BE_6 \\ \downarrow & & \downarrow & & \| \\ E_6/F_4 & \to & BF_4 & \to & BE_6 \end{array}$$

where the upper sequence is the universal E_6-bundle. It follows from this diagram that (2.4) implies (2.4)′ or equivalently that (2.5)′ implies (2.5). Since $e_{17} = \varrho^*(\bar{e}_{17})$ and since $H^9(BE_6) \cong Z_3$ generated by $\tau(e_8) = \bar{x}_9$, we may take f_9

$=\bar{x}_9$, that is, $\bar{\varphi}(e_{17})=e_8\otimes e_9$ in (2.5). Note that $\bar{\varphi}(e_8e_9)=e_8\otimes e_9+e_9\otimes e_8$ and that e_8e_9 is the only decomposable element in $H^{17}(\mathrm{E}_6)$ the image of which has the form $e_8\otimes e_9$. Therefore $\bar{\varphi}(e_{17}+\text{decomp.})\neq 0$. This shows that (2.4)′ implies (2.4) or equivalently that (2.5) does (2.5)′.

The following theorem is proved in [11].

THEOREM 2.6. (Kono–Mimura–Shimada)

(1) *The Eilenberg–Moore spectral sequence for* F_4 *with* $\mathbf{Z}_3$*-coefficient collapses,*

(2) $H^*(B\mathrm{F}_4)\cong \mathbf{Z}_3[\bar{x}_4,\bar{x}_8,\bar{x}_9,\bar{x}_{20},\bar{x}_{21},\bar{x}_{25},\bar{x}_{26}]/S$ *for* $*\leqq 35$ (see section 3 for the ideal S),

(3) $\bar{x}_8=\mathscr{P}^1\bar{x}_4$, $\bar{x}_9=\beta\bar{x}_8$, $\bar{x}_{20}=\mathscr{P}^3\bar{x}_8$, $\bar{x}_{21}=\beta\bar{x}_{20}$, $\bar{x}_{25}=\mathscr{P}^1\bar{x}_{21}$, $\bar{x}_{26}=\beta\bar{x}_{25}$.

(4) *all the relations for the degrees* $\leqq 35$ *are obtained from* $\bar{x}_4\bar{x}_9=0$ *over* $\mathcal{A}_3$.

NOTATION. $A=H^*(\mathrm{E}_6)$ and $B=H^*(\mathrm{F}_4)$.

It is quite easy to obtain

LEMMA 2.7. *If* (2.4) *were true, then*

$$\mathrm{Cotor}^A(\mathbf{Z}_3,\mathbf{Z}_3)\cong \mathrm{Cotor}^B(\mathbf{Z}_3,\mathbf{Z}_3)\otimes \mathbf{Z}_3[y_{10},y_{18}]\,,$$

(cf. section 3 of [11]).

NOTATION. When an element $\alpha\in\mathrm{Cotor}^A(\mathbf{Z}_3,\mathbf{Z}_3)$ or $\mathrm{Cotor}^B(\mathbf{Z}_3,\mathbf{Z}_3)$ is a permanent cycle, we denote by $\bar{\alpha}$ the element of $H^*(BG)$, $G=\mathrm{E}_6,\mathrm{F}_4$, represented by α. Further, by abuse of notation, we denote by the same symbol the corresponding elements of $H^*(B\mathrm{E}_6)$ and $H^*(B\mathrm{F}_4)$ under $i^*\colon H^*(B\mathrm{E}_6)\to H^*(B\mathrm{F}_4)$, where $i\colon \mathrm{F}_4\to\mathrm{E}_6$ is the inclusion.

LEMMA 2.8. *If* (2.4) *were true, then the Eilenberg–Moore spectral sequence for* E_6 *with* $\mathbf{Z}_3$*-coefficient would collapse for degrees* $\leqq 35$.

PROOF. The elements y_{10} and y_{18} are permanent cycles. In fact, the elements $\bar{y}_{10}$ and $\bar{y}_{18}$ represented by them are the transgression images of e_9 and e_{17} respectively by (2.4)′. Meanwhile it is clear (without assuming (2.4)) that

$$i^*\colon H^*(B\mathrm{E}_6)\cong H^*(B\mathrm{F}_4)\cong \mathbf{Z}_3[\bar{x}_4,\bar{x}_8,\bar{x}_9]\quad \text{for } *\leqq 9\,,$$

where $\bar{x}_8=\mathscr{P}^1\bar{x}_4$ and $\bar{x}_9=\beta\mathscr{P}^1\bar{x}_4$.

The elements $\mathscr{P}^3\bar{x}_8$, $\beta\mathscr{P}^3\bar{x}_8$, $\mathscr{P}^1\beta\mathscr{P}^3\bar{x}_8$ and $\beta\mathscr{P}^1\beta\mathscr{P}^3\bar{x}_8$ are not decomposable in $H^*(B\mathrm{E}_6)$, since they are not decomposable in $H^*(B\mathrm{F}_4)$. On the other hand there is only one non-decomposable element x_{20}, x_{21}, x_{25} and x_{26} of $\mathrm{Cotor}^A(\mathbf{Z}_3, \mathbf{Z}_3)$, $A = H^*(E_6)$, in each degree 20, 21, 25 and 26 respectively. Thus x_i are permanent cycles for $i = 20, 21, 25, 26$.

REMARK 2.9. By the Adem relation we have

$$\begin{aligned} \beta\bar{x}_9 &= 0\,, \\ \mathscr{P}^1\bar{x}_9 &= \mathscr{P}^1\beta\mathscr{P}^1\bar{x}_4 = \beta\mathscr{P}^2\bar{x}_4 + \mathscr{P}^2\beta\bar{x}_4 = 0\,, \\ \mathscr{P}^2\bar{x}_9 &= \mathscr{P}^2\beta\mathscr{P}^1\bar{x}_4 = \beta\mathscr{P}^3\bar{x}_4 - \mathscr{P}^3\beta\bar{x}_4 = 0\,, \\ \mathscr{P}^3\bar{x}_9 &= \mathscr{P}^3\beta\mathscr{P}^1\bar{x}_4 = \beta\mathscr{P}^3\mathscr{P}^1\bar{x}_4 = \bar{x}_{21}\,. \end{aligned}$$

Then all the relations in degrees $\leqq 30$ can be obtained by applying the operations to the relation $\bar{x}_4\bar{x}_9 = 0$. In fact,

$$\begin{array}{ccccc} \bar{x}_4\bar{x}_9 = 0 & \xrightarrow{\mathscr{P}^1} & \bar{x}_8\bar{x}_9 = 0 & \xrightarrow{\beta} & \bar{x}_9^2 = 0 \\ \downarrow{\scriptstyle \mathscr{P}^3} & & \downarrow{\scriptstyle \mathscr{P}^3} & & \\ \bar{x}_{21}\bar{x}_4 = 0 & & \bar{x}_{20}\bar{x}_9 + \bar{x}_8\bar{x}_{21} = 0 & & \\ \downarrow{\scriptstyle \mathscr{P}^1} & \text{sum} & & & \\ \bar{x}_{25}\bar{x}_4 + \bar{x}_{21}\bar{x}_8 = 0 & & \bar{x}_{25}\bar{x}_4 + \bar{x}_{20}\bar{x}_9 = 0 & \xrightarrow{\beta} & \bar{x}_{26}\bar{x}_4 + \bar{x}_{21}\bar{x}_9 = 0\,. \end{array}$$

Thus we have shown

LEMMA 2.10. *If* (2.4) *were true, we would have an isomorphism as algebras*

$$H^*(BE_6) \cong \mathbf{Z}_3[\bar{y}_{10}, \bar{y}_{18}] \otimes H^*(BF_4) \quad \text{for } * \leqq 30\,.$$

PROPOSITION 2.11. *The statement* (2.3) *is false.*

PROOF. By the Adem relation $\bar{y}_{10}^3 = \mathscr{P}^5\bar{y}_{10} = \mathscr{P}^2\mathscr{P}^3\bar{y}_{10}$. Here $\mathscr{P}^3\bar{y}_{10} \in (\bar{x}_4, \bar{x}_8)$, the ideal generated by $\bar{x}_4$ and $\bar{x}_8$, since $\deg\mathscr{P}^3\bar{y}_{10} = 22$. Meanwhile the ideal $(\bar{x}_4, \bar{x}_8)$ is closed under the operation $\mathscr{P}^1$ and $\mathscr{P}^2$. Thus $\bar{y}_{10}^3 \in (\bar{x}_4, \bar{x}_8)$, which contradicts to the fact that $\bar{y}_{10} \notin (\bar{x}_4, \bar{x}_8)$.

COROLLARY 2.12. *There are no primitive elements in* $H^{17}(E_6)$.

Putting $e_{17} = \varrho^*(\bar{e}_{17})$ we obtain

THEOREM 2.13. *In* (1.2), *the elements* e_i *are primitive as well as universally transgressive for* $i = 3, 7, 8, 9$; *and* $\bar{\varphi}(e_j) = e_8 \otimes e_{j-8}$ *for* $j = 11, 15, 17$.

REMARK 2.14. (1) This result was independently proved by Toda in [18], in which he calculated the invariant submodule $H^*(BT^6)^{W(E_6)}$ under the Weyl group $W(E_6)$ and studied the Serre spectral sequence associated with the fibering EIII $\to BE_6 \to B(D_5 \cdot T^1)$. There is also given a proof in [8].

(2) Lemma 2.10 can be obtained by the comparison theorem on the Serre spectral sequence for the fibering (2.2)'.

3. $\mathrm{Cotor}^A(Z_3, Z_3)$ and $\mathrm{Cotor}^B(Z_3, Z_3)$.

Recall the following

PROPOSITION 3.1. (Theorem 5.20 of [12])

$$\mathrm{Cotor}^A(Z_3, Z_3) \cong Z_3[x_4, x_8, x_9, x_{20}, x_{21}, x_{25}, x_{26}, x_{36}, x_{48}] \otimes Z_3[y_{10}, y_{22}, y_{26}, y_{27}, y_{54}, y_{58}, y_{60}, y_{64}, y_{76}]/R$$

where R is the ideal generated by

(3.2) $x_9^2, x_{21}^2, x_{25}^2, y_{27}^2$,

(3.3) $x_9x_{21}+x_{26}x_4,\quad x_9x_{25}+x_{26}x_8,\quad x_9y_{27}+x_{26}y_{10}$,

$x_{21}x_{25}+x_{26}x_{20}, x_{21}y_{27}+x_{26}y_{22}, x_{25}y_{27}-x_{26}y_{26}$,

(3.4) $x_9\partial^2Q, x_{21}\partial^2Q, x_{25}\partial^2Q, y_{27}\partial^2Q, x_{26}\partial^2Q$,

(3.5) $x_9x_4, x_9x_8, x_9y_{10}, x_{21}x_4, x_{25}x_8, y_{27}y_{10}$,

(3.6) $x_{21}x_8+x_9x_{20} = x_{25}x_4-x_9x_{20}$,

$x_{21}y_{10}+x_9y_{22} = y_{27}x_4-x_9y_{22}$,

$x_{25}y_{10}-x_9y_{26} = y_{27}x_8+x_9y_{26}$,

(3.5)' $x_{21}x_{20}, x_{25}x_{20}, x_{21}y_{22}, y_{27}y_{22}, x_{25}y_{26}, y_{27}y_{26}$,

(3.6)' $y_{27}x_{20}+x_{25}y_{22} = x_{21}y_{26}-x_{25}y_{22} = -y_{27}x_{20}-x_{21}y_{26}$.

(For ∂^2Q see [12].)

For sake of our convenience we use here different notations for the generators from those in [12]. The correspondance is as follows:

$\mathrm{Cotor}^A(Z_3, Z_3)$ x_i

gen. in [12] a_i $(i=4,8,9)$, y_i $(i=20,21,25)$, x_i $(i=26,36,48)$

$\mathrm{Cotor}^A(Z_3, Z_3)$ y_j

gen. in [12] a_j ($j=10$), $-y_j$ ($j=22$), x_j ($j=54$),

y_j ($j=26, 27, 58, 60, 64, 76$)

Recall also the following

PROPOSITION 3.7. (Theorem 3.9 of [11])

$$\mathrm{Cotor}^B(\mathbf{Z}_3, \mathbf{Z}_3) = \mathbf{Z}_3[x_4, x_8, x_9, x_{20}, x_{21}, x_{25}, x_{26}, x_{36}, x_{48}]/S,$$

where S is the ideal generated by x_4x_9, x_8x_9, x_9^2, x_4x_{21}, x_8x_{25}, $x_4x_{25}, +x_8x_{21}$, $x_{20}x_{21}$, $x_{20}x_{25}$, x_{21}^2, x_{25}^2, $x_9x_{20}-x_4x_{25}+x_8x_{21}$, $x_{20}^3-x_4^3x_{48}+x_8^3x_{36}$, $x_{26}x_4+x_{21}x_9$, $x_{26}x_8+x_{25}x_9$, $x_{26}x_{20}-x_{21}x_{25}$.

Let $i\colon \mathrm{F}_4 \to \mathrm{E}_6$ be the natural inclusion. Consider the induced homomorphism

$$i^*\colon \mathrm{Cotor}^A(\mathbf{Z}_3, \mathbf{Z}_3) \to \mathrm{Cotor}^B(\mathbf{Z}_3, \mathbf{Z}_3).$$

By the observation on the cochain level in the correspondence between $\mathrm{Cotor}^A(\mathbf{Z}_3, Z_3)$ and $\mathrm{Cotor}^B(\mathbf{Z}_3, \mathbf{Z}_3)$ we obtain

COROLLARY 3.8. $i^*(x_i)=x_i$ *for all i and* $i^*(y_j)=0$ *for all j. In particular* i^* *is an isomorphism on the subalgebra generated by all* x_i*'s.*

REMARK 3.9. Consider the spectral sequence $\{E_r(\mathrm{E}_6), d_r\}$ with $\mathbf{Z}_3$-coefficient converging to $H^*(B\mathrm{E}_6)$, where each term $E_r(\mathrm{E}_6)$ is of bidegree, the homological (or external) degree and the internal degree. The differential d_r raises the homological degree by r. Denoting by $h(z)$ the homological degree for an element $z \in \mathrm{Cotor}^A(\mathbf{Z}_3, \mathbf{Z}_3)=E_2(\mathrm{E}_6)$, we give a list of the homological degree of the generators:

$$h(z) = 1 \quad \text{for } z=x_4, x_8, x_9, y_{10},$$

$$h(z) = 2 \quad \text{for } z=x_{20}, x_{21}, y_{22}, x_{25}, y_{26}, x_{26}, y_{27},$$

$$h(z) = 3 \quad \text{for } z=x_{36}, x_{48}, y_{54},$$

$$h(z) = 5 \quad \text{for } z=y_{58}, y_{60}, y_{64},$$

$$h(z) = 6 \quad \text{for } z=y_{76}.$$

4. The module structure of $H^*(B\mathrm{E}_6)$—I ($* \leqq 35$).

NOTATION. When an element $\alpha \in \mathrm{Cotor}^A(\mathbf{Z}_3, \mathbf{Z}_3)$ is a permanent cycle, we denote by $\bar{\alpha}$ the element of $H^*(B\mathrm{E}_6)$ represented by α.

Let i: $\mathrm{F}_4 \to \mathrm{E}_6$ be the inclusion considered in [1].

PROPOSITION 4.1. *The induced homomorphism*

$$i^*: H^*(BE_6) \to H^*(BF_4)$$

is surjective for $* \leqq 35$.

PROOF. By Toda [17] or by Kono–Mimura–Shimada [11], we know that $H^*(BF_4)$ for $* \leqq 35$ is generated by the generator $\bar{x}_4 \in H^4(BF_4) \cong Z_3$ over $\mathcal{A}_3$. Meanwhile, according to Araki [1], $i^*: H^3(E_6) \cong H^3(F_4)$ and hence $i^*: H^4(BE_6) \cong H^4(BF_4)$. Thus the proposition follows from the naturality of the cohomology operations.

COROLLARY 4.2. *The elements* x_i $(i=4,8,9,20,21,25,26)$ *are permanent cycles. In fact, there hold*

$$\bar{x}_8 = \mathcal{P}^1\bar{x}_4,\ \bar{x}_9 = \beta\bar{x}_8,\ \bar{x}_{20} = \mathcal{P}^3\bar{x}_8\,,$$

$$\bar{x}_{21} = \beta\bar{x}_{20},\ \bar{x}_{25} = \mathcal{P}^1\bar{x}_{21},\ \bar{x}_{26} = \beta\bar{x}_{25}\,,$$

where $\bar{x}_4 \in H^4(BE_6)$ *is the generator.*

PROOF. $i^*(\bar{x}_i)$ is not decomposable in $H^*(BF_4)$ and hence is represented by x_i.

LEMMA 4.3. *The elements* y_j $(j=10,22,26,27)$ *are permanent cycles.*

PROOF. Put $\bar{y}_{10}=\tau(e_9)$, the transgression image of e_9. (Remark that this transgression has no indeterminacy.) Consider the Serre spectral sequence with Z_3-coefficient associated with the fibering (2.2)′, where $H^*(E_6/F_4) \cong \Lambda(\bar{e}_9, \bar{e}_{17})$. By Propositions 2.3 and 2.11 we have

$$d_9(1\otimes\bar{e}_{17}) = \pm\bar{x}_9\otimes\bar{e}_9 \quad \text{and} \quad d_{10}(1\otimes\bar{e}_9) = \bar{y}_{10}\otimes 1\,,$$

where $\bar{y}_{10}\bar{x}_9=0$. In this spectral sequence the relations $\bar{x}_9\bar{x}_4=\bar{x}_9\bar{x}_4^2=\bar{x}_9\bar{x}_8=\bar{x}_9^2=0$ imply respectively that $\bar{x}_4\otimes\bar{e}_{17}$, $\bar{x}_9^2\otimes\bar{e}_{17}$, $\bar{x}_8\otimes\bar{e}_{17}$ and $\bar{x}_9\otimes\bar{e}_{17}$ are d_{10}-cycles. Since $i^*: H^*(BE_6) \to H^*(BF_4)$ is surjective for $* \leqq 35$, there are elements $\bar{y}_k$ $(k=22,26,27)$ such that

$$d_{18}(\bar{x}_{k-18}\otimes\bar{e}_{17}) = \bar{y}_k\otimes 1 \qquad (k=22,26,27)$$

$$d_{18}(\bar{x}_4^2\otimes\bar{e}_{17}) = \bar{y}_{22}\bar{x}_4\otimes 1 \neq 0 \quad \text{and} \quad \bar{y}_{22}\bar{x}_4 \neq \bar{y}_{26}\,.$$

Of course $\bar{y}_k \in \operatorname{Ker} i^*$ $(k=22,26,27)$. Further they are not decomposable. It is then easy to see that $\bar{y}_k$ are represented by y_k $(k=22,26,27)$ respectively.

COROLLARY 4.4. *The Eilenberg–Moore spectral sequence for* E_6 *with* Z_3-*coefficient collapses for degrees* $\leqq 35$.

COROLLARY 4.5. $\mathrm{Ker}\, i^*$ *is generated by* $\bar{y}_k$ $(k=10,22,26,27)$ *for degrees* $\leqq 35$.

REMARK 4.6. Using the argument in [3] one can obtain

$$\bar{y}_{26} = \mathscr{P}^1\bar{y}_{22} \quad \text{and} \quad \bar{y}_{27} = \beta\bar{y}_{26},$$

since $\mathscr{P}^1\bar{e}_{17}=0$, $\beta\bar{e}_{17}=0$, $\mathscr{P}^1\bar{x}_4=\bar{x}_8$ and $\beta\bar{x}_8=\bar{x}_9$. Furthermore, by a similar argument to that in section 2 one can show

$$\mathscr{P}^3\bar{y}_{10} = \bar{y}_{22}.$$

5. The module structure of $H^*(BE_6)$—II $(*>35)$.

NOTATION. $N=\{x_{36}, x_{48}, y_{54}, y_{58}, y_{60}, y_{64}, y_{76}\}$.

In the below we will show that the elements in N are all permanent cycles in the Eilenberg–Moore spectral sequence $\{E_r(E_6), d_r\}$.

LEMMA 5.1. *Let* $f\in N$. *If* $d_r(f)\neq 0$, *then* $d_r(f)$ *is expressed as a sum of monomials containing* y_j *and an odd number of* x_i $(i=9,21,25)$ *or* y_{27}.

PROOF. If $d_r(f)\neq 0$, each term of $d_r(f)$ must contain y_j, since the induced homomorphism i^*: $E_r(E_6) \to E_r(F_4)$ is injective on the subalgebra generaled by all the x_i's (cf. Corollary 3.8). As every element of N is of even degree, each term of $d_r(f)$ must contain an odd number of x_i's $(i=9,21,25)$ or y_{27}.

Recall from Proposition 3.1 the following relations in $E_2(E_6)$:

(R.1) $x_4x_9 = x_8x_9 = x_4x_{21} = x_8x_{25} = x_{20}x_{21} = x_{20}x_{25} = 0$,

(R.2) $y_{10}x_9 = y_{22}x_{21} = y_{26}x_{25} = 0$,

(R.3) $y_{22}y_{27} = y_{26}y_{27} = y_{10}y_{27} = 0$,

(R.4) $x_9^2 = x_{21}^2 = x_{25}^2 = y_{27}^2 = 0$,

(R.5) $y_{22}x_{25} = y_{26}x_{21} = -x_{20}y_{27}$,

(R.6) $x_{20}x_9 = -x_8x_{21} = x_4x_{25}$, $\quad y_{22}x_9 = -y_{10}x_{21} = x_4y_{27}$,

$y_{26}x_9 = -x_8y_{27} = y_{10}x_{25}$,

(R.7) $x_{26}x_4 = -x_{21}x_9$, $\quad x_{26}x_{20} = -x_{21}x_{25}$, $x_{26}x_8 = -x_{25}x_9$,

$x_{26}y_{22} = -x_{21}y_{27}$, $x_{26}y_{10} = -y_{27}x_9$, $x_{26}y_{26} = x_{25}y_{27}$,

(R.8) *the other relations*,

$$\text{(R}'\text{)}\quad y_{10}^2x_{21} = -y_{10}y_{22}x_9 = 0 \qquad \text{(by (R.6) and (R.2))},$$

$$x_8y_{10}x_{21} = x_8y_{22}x_9 = 0 \qquad \text{(by (R.6) and (R.1))},$$

$$y_{10}^2x_{25} = y_{10}y_{26}x_9 = 0 \qquad \text{(by (R.6) and (R.2))},$$

$$x_4y_{10}x_{25} = x_4y_{26}x_9 = 0 \qquad \text{(by (R.6) and (R.1))}.$$

LEMMA 5.2. *Let $f \in N$. Then*

$$d_r(f) = y_{10}x_{21}f_1 + y_{10}x_{25}f_2 + y_{26}x_{21}f_3 + y_{27}f_4,$$

where

f_1 is a monomial containing $x_{26}, y_{26}, x_{36}, x_{48}$,

f_2 is a monomial containing $y_{22}, x_{26}, x_{36}, x_{48}$,

f_3 is a monomial containing $y_{26}, x_{26}, x_{36}, x_{48}$,

f_4 is a monomial containing x_{26}, x_{36}, x_{48}.

PROOF. (1) For f with $\deg f \leqq 54$: By (R.2) we may put

$$d_r(f) = y_{10}x_{21}f_1 + y_{10}x_{25}f_2 + y_{22}x_9f'_1 + y_{22}x_{25}f'_3 + y_{26}x_9f'_2 + y_{26}x_{21}f_3 + y_{27}f_4.$$

Then by (R.6) we may suppose $f'_i = 0$ ($i = 1, 2, 3$). The relations (R.1), (R.2), (R.7) and (R′) imply that

$$d_r(f) = y_{10}x_{21}f_+ y_{10}x_{25}f_2 + y_{26}x_{21}f_3 + y_{27}f_4, \tag{5.3}$$

where

f_1 is a monomial of $x_{26}, y_{26}, x_{36}, x_{48}, y_j \quad (j \geqq 54)$,

f_2 is a monomial of $y_{22}, x_{26}, x_{36}, x_{48}, y_j \quad (j \geqq 54)$,

f_3 is a monomial of $x_{26}, y_{26}, x_{36}, x_{48}, y_j \quad (j \geqq 54)$,

further, by (R.3) and (R.6),

f_4 is a monomial of $x_{26}, x_{36}, x_{48}, y_j \quad (j \geqq 54)$.

Since $\deg f \leqq 54$, f_i does not contain y_j ($j \geqq 54$).

(2) For f with $\deg f \geqq 54$: By the same reason as in (1), $d_r(f)$ is of the form (5.3). In $E_2(\mathrm{E}_6)$, there are no terms containing y_j ($j \geqq 54$) in degrees 55, 59, 61, 65, 77 for dimensional reasons and by the relations (R.1) and (R.4). So we can get the lemma.

PROPOSITION 5.3. (1) *x_{36} and x_{48} are permanent cycles and there holds*

$$\bar{x}_{48} = \mathscr{P}^3\bar{x}_{36}\,.$$

(2) *y_j $(j=54, 58, 64, 76)$ are permanent cycles.*

(3) *y_{60} is a permanent cycle.*

PROOF. (1) and (2): For dimensional reasons we easily see that all the f_i's are zero in $d_r(f)$ of Lemma 5.2 for $f \in N - \{x_{60}, x_{48}\}$. Namely, all the generators of $N - \{x_{60}, x_{48}\}$ are permanent cycles. Specifically we get the element $\bar{x}_{36}$ in $H^*(BE_6)$. Consider the induced homomorphism $i^*\colon H^*(BE_6) \to H^*(BF_4)$. Then $i^*\mathscr{P}^3\bar{x}_{36} = \mathscr{P}^3\bar{x}_{36}$ is not decomposable in $H^{48}(BF_4)$. So there exists an element in $H^{48}(BE_6)$ which is represented by x_{48}. That is, x_{48} is a permanent cycle and $\mathscr{P}^3\bar{x}_{36} = \bar{x}_{48}$.

(3) For dimensional reasons $d_r(y_{60}) = \alpha y_{10}x_{25}y_{26}$ with $\alpha \in \mathbf{Z}_3$, where the homological degree of both elements are $h(y_{60}) = h(y_{10}x_{25}y_{26}) = 5$ by Remark 3.9. However d_r raises the homological degree by $r \geqq 2$, $d_r(y_{60}) = 0$.

COROLLARY 5.4. *The induced homomorphism*

$$i^*\colon H^*(BE_6) \to H^*(BF_4)$$

is surjective.

By Proposition 5.3 together with Corollary 4.2 and Lemma 4.3 we have seen that all the algebra generators in $\mathrm{Cotor}^A(\mathbf{Z}_3, \mathbf{Z}_3)$ are permanent cycles in the Eilenberg–Moore spectral sequence. Thus

THEOREM 5.5. *The Eilenberg–Moore spectral sequence for* E_6 *with* $\mathbf{Z}_3$-*coefficient collapses.*

THEOREM 5.6. *As modules*

$$H^*(BE_6) \cong \mathrm{Cotor}^A(\mathbf{Z}_3, Z_3) \qquad (A = H^*(E_6))\,.$$

6. Some algebra relations.

By the Adem relation one can obtain

$$\text{(6.1)} \qquad \mathscr{P}^i\bar{x}_9 = 0 \text{ for } i=1,2, \quad \mathscr{P}^3\bar{x}_9 = \bar{x}_{21}, \; \beta\bar{x}_9 = 0\,.$$

Similar results for the other generators can be obtained.

LEMMA 6.1. *All the relations among x_i $(i<36)$ and y_j $(j<54)$ in the $E_\infty(E_6)$-term give rise to the corresponding relations in $H^*(BE_6)$.*

PROOF. In fact, as is easily checked, these relations are obtained over $\mathfrak{A}_3$ from the relations $\bar{x}_4\bar{x}_9=0$ and $\bar{y}_{10}\bar{x}_9=0$.

REMARK 6.2. (1) Let T^6 be the maximal torus of E_6 and $k\colon T^6\subset \mathrm{E}_6$ be the inclusion. The computation by Toda in [18] on $H^*(BT^6)^{W(\mathrm{E}_6)}$, the invariant subalgebra under the Weyl group, imply that the map i' in the diagram below is surjective and its kernel is generated by $\bar{x}_9, \bar{x}_{21}, \bar{x}_{25}, \bar{x}_{26}, \bar{y}_{27}$:

$$\begin{array}{ccc} H^*(B\mathrm{E}_6) & \xrightarrow{\quad k^* \quad} & H^*(BT^6) \\ & {\scriptstyle i'}\searrow \qquad \nearrow{\scriptstyle \subset} & \\ & H^*(BT^6)^{W(\mathrm{E}_6)} & \end{array}$$

(2) The following relations

$$x_4y_{26}+x_8y_{22}-y_{10}x_{20} = 0,\ x_{20}^3 = x_8^3x_{36}-x_4^3x_{48}\,,$$
$$y_{22}^3 = y_{10}^3x_{36}-x_4^3y_{54}, \qquad x_{20}^2y_{22} = x_4y_{58}+y_{10}x_8^2x_{36}\,,$$
$$y_{22}^2x_{20} = x_4y_{60}+x_8y_{10}^2x_{36},\ y_{10}x_{58}+x_8y_{60}-x_4y_{64} = 0\,,$$
$$y_{58}y_{22} = y_{76}x_4-x_8y_{10}x_{36}y_{26}\,,$$

give the forms $i'\bar{x}_{48}, i'\bar{y}_j$ $(j=26, 54, 58, 60, 64, 76)$ in $H^*(BT^6)^{W(\mathrm{E}_6)}$ modulo higher terms (cf. [18]). In fact, $i'\bar{y}_j$ $(j=26, 58, 60, 64, 76)$ become of this form, if one chooses adequately the elements in lower degrees. Meanwhile, in the form

$$i'\bar{x}_{48} = i'((-\bar{x}_{36}\bar{x}_8^3+\bar{x}_{20}^3+\bar{x}_{20}^2\bar{x}_8^2\bar{x}_4)/\bar{x}_4^3)$$

the weight of $x_{20}^2x_8^2x_4$ is higher by 1 than the others. Then all the elements are decomposable in $H^*(BT^6)^{W(\mathrm{E}_6)}[i'(\bar{x}_4)^{-1}]$.

REFERENCES

1. S. Araki, *On the non-commutativity of Pontrjagin rings* mod 3 *of some compact exceptional groups*, Nagoya Math. J. 17 (1960), 225–260.
2. S. Araki, *Cohomology* mod 2 *of the compact exceptional groups* E_6 *and* E_7, J. Math. Osaka City Univ. 12 (1961), 43–65.
3. S. Araki, *Steenrod's reduced power in the spectral sequences associated with a fibering* I, II, Mem. Fac. Sci. Kyûshû Univ., Ser. A. Math. 11 (1957), 15–64, 81–97.
4. W. Browder, *Homology ring of groups*, Amer. J. Math. 90 (1968), 318–333.
5. A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. 57 (1953), 115–207.
6. A. Borel, *Sur l'homologie et la cohomologie des groupes de Lie compacts connexes*, Amer. J. Math. 76 (1954), 273–342.
7. A. Borel, *Sous-groupes commutatifs et torsions des groupes de Lie compacts connexes*, Tôhoku Math. J. 13 (1961), 216–240.
8. A. Kono, *Hopf algebra structure of simple Lie groups*, J. Math. Kyoto Univ. 17 (1977), 259–298.

9. A. Kono and M. Mimura, *Cohomology* mod 3 *of the classifying space of the compact, 1-connected Lie group of type* E_6, Aarhus Univ. Preprint series 1974/75 No. 30.
10. A. Kono and M. Mimura, *Cohomology* mod 2 *of the classifying space of the compact connected Lie group of type* E_6, J. Pure and Applied Algebra 6 (1975), 61–81.
11. A. Kono, M. Mimura, and N. Shimada, *Cohomology of classifying space of certain associative* H-*spaces*, J. Math. Kyoto Univ. 15 (1975), 607–617.
12. M. Mimura and Y. Sambe, *On the cohomology* mod *p of the classifying spaces of the exceptional Lie groups*, I, J. Math. Kyoto Univ. 19 (1979), 553–581.
13. M. Rothenberg and N. E. Steenrod, *The cohomology of classifying spaces of* H-*spaces*, (mimeographed notes).
14. M. Rothenberg and N. E. Steenrod, *The cohomology of classifying spaces of* H-*spaces*, Bull. Amer. Math. Soc. 71 (1965), 872–875.
15. J.-P. Serre: *Homologie singulière des espaces fibrés*, Ann. of Math. 54 (1951), 425–505.
16. N. E. Steenrod and D. B. A. Epstein, *Cohomology operations* (Ann. of Math. Studies 50), Princeton Univ. Press, Princeton, 1962.
17. H. Toda, *Cohomology* mod 3 *of the classifying space* BF_4 *of the exceptional group* F_4, J. Math. Kyoto Univ. 13 (1972), 97–115.
18. H. Toda, *Cohomology of the classifying space of exceptional Lie groups*, Manifolds, pp. 265–271, Tokyo 1973.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY, JAPAN

MATEMATISK INSTITUT
AARHUS UNIVERSITET, DENMARK

AND

MATHEMATICAL INSTITUTE
YOSHIDA COLLEGE
KYOTO UNIVERSITY, JAPAN