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DERIVATIONS OF JORDAN C*-ALGEBRAS

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0. Introduction.

The classical results of Kadison [13, 14] reveal a close connection between the geometric structure (group of isometries, state space) of a C*-algebra and its Jordan algebraic (quantum mechanical) structure. This feature is also typical of the more general class of Jordan C*-algebras (JB*-algebras) introduced by Kaplansky [30]. By complexification, JB*-algebras correspond exactly to the JB-algebras investigated by Alfsen, Shultz and Størmer [1]. JB-algebras are of interest in functional analysis, spectral theory and algebra (formally real Jordan algebras). A promising application of JB-algebras is to be found in complex analysis, based on the $1-1$ correspondence between JB*-algebras and bounded symmetric domains in complex Banach spaces with tube realization [18,7].

Derivations of JB-algebras, which have been thoroughly studied in special cases (for C*-algebras and for finite dimensional JB-algebras), are of particular importance to the holomorphic automorphism group G of a bounded symmetric domain Δ of tube type and to its Lie algebra $\mathfrak{g}=\mathrm{aut}\,(\Delta)$ consisting of all complete holomorphic vector fields on Δ. More precisely, $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ into the subalgebra $\mathfrak{k}$ of all infinitesimal isometries and the subspace $\mathfrak{p}$ of all vector fields $(\alpha-\{z\alpha^*z\})\partial/\partial z$, where $\{z\alpha^*z\}$ denotes the triple product of the JB*-algebra Z associated with Δ [16, 17]. Further, the self-adjoint part X of Z induces a decomposition $\mathfrak{k}=i\mathfrak{m}\oplus\mathrm{aut}\,(X)$, where $\mathfrak{m}$ consists of all Jordan multiplications by elements of X and aut (X) is the Lie algebra of all derivations of X. The summands $\mathfrak{p}$ and $i\mathfrak{m}$ of $\mathfrak{g}$ are well-known; $\mathfrak{p}$ and Z are isomorphic as real Banach spaces and $\mathfrak{m}$ is related to derivations of self-dual Hilbert cones [8,3]. However, for JB-algebras X in general, little seems to be known about the structure of aut (X). Therefore the purpose of this paper is to study derivations of JB-algebras and to indicate their applications to complex analysis.

Our first objective is to describe derivations of Jordan operator algebras (JC-algebras) in terms of derivations of C*-algebras which are well-understood. The restriction to JC-algebras is justified by the structure theory

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developed in [1], as indicated in section 1. The principal result of section 2 shows that for reversible JC-algebras X each derivation of X can be extended to a *-derivation of the C*-algebra generated by X, and is therefore implemented by a Hilbert space operator. This is a generalization of a theorem of Sinclair [24] on Jordan derivations of C*-algebras. For infinite dimensional spin factors the above extension property is not valid.

Our second problem is to clarify the relationship between aut (X) and its ideal int (X) of inner derivations. This problem is of relevance to complex analysis since int (X) determines the subalgebra $[\mathfrak{p}, \mathfrak{p}]$ of $\mathfrak{k}$ generated by Lie products of vector fields in $\mathfrak{p}$. If dim $(X) < \infty$, then it is well-known that aut (X) $=$ int (X) and therefore $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$. In section 3 we characterize those JBW-algebras (i.e. JB-algebras with predual) X satisfying aut $(X) =$ int (X). It is shown in particular that purely exceptional JBW-algebras and reversible JW-algebras have only inner derivations. If X is a JB-algebra, then int (X) need not be uniformly dense in aut (X), but it is shown in section 4 that aut (X) is the strong operator closure of int (X). Consequently each infinitesimal isometry of a bounded symmetric domain of tube type can be approximated pointwise by vector fields in $[\mathfrak{p}, \mathfrak{p}]$.

1. Preliminaries.

1.1. Definition. A JB-*algebra* is a real Banach Jordan algebra X with unit e and product $x \circ y$ such that $\|x \circ y\| \leqq \|x\| \|y\|$ and $\|x\|^2 \leqq \|x^2 + y^2\|$ whenever $x, y \in X$. A *derivation* of X is a linear map $D: X \to X$ satisfying $D(x \circ y) = Dx \circ y + x \circ Dy$ fot all $x, y \in X$. Let aut (X) denote the Lie algebra of all derivations of X.

The set $\mathscr{C}(S, X)$ of all continuous maps from a compact space S to a JB-algebra X is a JB-algebra with pointwise algebraic operations and supremum norm. Each $x \in X$ generates a JB-subalgebra isomorphic to $\mathscr{C}(T, \mathbf{R})$, where T is a compact space. Therefore the proof for C*-algebras [21; Lemma 4.1.3] can be modified to show that all derivations of a JB-algebra X are bounded. If aut (X) is equipped with the operator norm, there is an isomorphism of Lie algebras

$$\text{aut}\,\mathscr{C}(S, X) \approx \mathscr{C}(S, \text{aut}\, X) \tag{1.2}$$

(clear, if dim $(X) < \infty$; in the general case, (1.2) follows from [29; Cor. 1.10]).

1.3. Definition. Let H be a complex Hilbert space and let $\mathscr{H}(H)$ denote the JB-algebra of all bounded hermitian operators on H with the product $x \circ y$: $= (xy + yx)/2$. A uniformly (weakly) closed unital subalgebra of $\mathscr{H}(H)$ is called a JC-*algebra* (JW-*algebra*) on H, respectively. The exceptional Jordan algebra of all self-adjoint 3×3-matrices with octonion entries is denoted by $\mathscr{H}_3(\mathrm{O})$.

By the deep results of Alfsen, Shultz and Størmer [1, 23], the study of JB-derivations can frequently be reduced to the case of JC-algebras: The second dual X'' of a JB-algebra X is a JB-algebra with the Arens product and for each $D \in \operatorname{aut}(X)$ the second transpose D'' is a derivation of X''. Further, each JB-algebra X with predual has a decomposition

$$(1.4) \qquad X = X_{sp} \oplus X_{ex} ,$$

such that X_{sp} can be realized as a JW-algebra and $X_{ex} \approx \mathscr{C}(S, \mathscr{H}_3(\mathrm{O}))$, where S is a compact space [23; Th. 3.9]. Since the center of X vanishes under $\operatorname{aut}(X)$, it follows from (1.2) and (1.4) that

$$(1.5) \qquad \operatorname{aut}(X) \approx \operatorname{aut}(X_{sp}) \oplus \mathscr{C}(S, \operatorname{aut} \mathscr{H}_3(\mathrm{O})) .$$

Finally, $\operatorname{aut} \mathscr{H}_3(\mathrm{O})$ is a well-known classical Lie algebra (a real form of F_4 [11, p. 411]).

2. Extension of derivations of JC-algebras.

The most important examples of JC-algebras are provided by C*-algebras. Henceforth, derivations of *associative* *-algebras commuting with the involution are called **-derivations*. C*-algebras Z have the fundamental property, that each *-derivation D of Z is implemented by a Hilbert space operator w lying in the weak closure of Z [21; Cor. 4.1.7]:

$$(2.1) \qquad Dz = [w, z] := wz - zw \quad \text{for all } z \in Z .$$

Sinclair [24] has shown that each Jordan derivation of the self-adjoint part X of Z is actually induced by a *-derivation of Z. In this section this result is extended to a large class of JC-algebras X, if Z is replaced by the C*-algebra generated by X. As a byproduct we obtain a new proof of the theorem of Sinclair.

2.2. DEFINITION. A JC-algebra X on a complex Hilbert space H is said to have the *extension property*, if each derivation of X can be extended to a *-derivation of the C*-algebra generated by X on H.

Not all JC-algebras have the extension property, as the counterexample of infinite dimensional spin factors shows:

2.3. EXAMPLE. If X is a real Hilbert space of dimension >2 and $e \in X$ is a unit vector with orthogonal complement Y, then $X = \mathbb{R}e \oplus Y$ with the product

$$(r_1e+y_1)\circ(r_2e+y_2) := (r_1r_2+(y_1\,|\,y_2))e+r_1y_2+r_2y_1$$

is a JB-algebra called a *spin factor*. The derivations of X are exactly the skew-adjoint operators on Y. Now suppose that $\dim(X)=\infty$ and consider a (faithful) JC-representation $X\subset\mathscr{H}(H)$. By the universal property of the Clifford representation of X, the C*-algebra Z generated by X on H can be identified with the Clifford C*-algebra of X [22]. Each derivation of the simple C*-algebra Z [22; Prop. 1] is inner [21; Th. 4.1.11]. Hence it follows from [2; Th. 4] that $D\in\operatorname{aut}(X)$ can be extended to a *-derivation of Z if and only if D is a trace-class operator.

Reversible JC-algebras X defined by the property

$$x_1,\dots,x_n\in X \Rightarrow x_1\cdot\ldots\cdot x_n+x_n\cdot\ldots\cdot x_1\in X$$

and thoroughly studied by Størmer [26, 27] are complementary to spin factors in the following sense: By [26; Th. 6.4 and Th. 6.6], each JW-algebra X has a decomposition

$$X = X_{\mathrm{rev}}\oplus X_2\,, \tag{2.4}$$

such that X_{rev} is reversible and X_2 is a direct sum of JW-algebras isomorphic to $L^\infty(S,U)$, where S is a measure space and U is a spin factor [33; Th. 2]. Further, a JW-factor of dimension $\neq 3,4,6$ is a spin factor if and only if it is not reversible [26; Th. 7.1].

The following theorem is the main result of this section.

2.5. Extension Theorem. *Every reversible* JC-*algebra X on a complex Hilbert space H has the extension property, i.e. each derivation D of X can be extended to a *-derivation of the* C*-*algebra Z generated by X on H.*

For the proof we need some facts on derivations of JW-factors. Throughout, let K be one of the skew fields R, C and H of real, complex and quaternion numbers, respectively. If E and F are K-Hilbert spaces (with scalar multiplication on the right), $\mathscr{L}(E,F)$ denotes the Banach space of K-linear continuous maps from E to F. Let $z^*\in\mathscr{L}(F,E)$ be the adjoint of $z\in\mathscr{L}(E,F)$. Then $u^*v\in\mathsf{K}$ is the inner product of $u,v\in E=\mathscr{L}(\mathsf{K},E)$. Put

$$\mathscr{L}(E) := \mathscr{L}(E,E), \qquad \mathscr{H}(E) := \{x\in\mathscr{L}(E) :\ x^*=x\}$$

and

$$\mathscr{S}(E) := \{w\in\mathscr{L}(E) :\ w^*=-w\}\,.$$

If r is a positive integer, define $\mathscr{H}_r(\mathsf{K}) := \mathscr{H}(\mathsf{K}^r)$.

2.6. LEMMA. *Let E be a* K*-Hilbert space. Then each derivation D of $\mathscr{H}(E)$ has the form $Dx=[w,x]$ for all $x \in \mathscr{H}(E)$, where $w \in \mathscr{S}(E)$ satisfies* $4\|w\| \leqq 5\|D\|$.

PROOF OF 2.6 (included for the sake of completeness). Put $r:=\dim_{\mathsf{K}}(E)$. If $r=2$, then $\mathscr{H}(E)$ is a spin factor. Choose a spin system $e_1,\ldots,e_n$ [26; p. 181] and define

$$8w := \sum_{\nu=1}^{n} [De_\nu, e_\nu] . \tag{2.7}$$

If $r \geqq 3$, then D is induced by a *-derivation, again denoted by D, of $\mathscr{L}(E)$ [11; p. 143, Cor. 3]. As in [15; Lemma 2] choose an orthogonal decomposition $E = \mathsf{K} \oplus F$ and write the operators $z \in \mathscr{L}(E)$ as matrices

$$z = \begin{pmatrix} a & f \\ v & A \end{pmatrix}, \text{ where } a \in \mathsf{K},\ v \in F,\ f \in \mathscr{L}(F,\mathsf{K}) \text{ and } A \in \mathscr{L}(F) .$$

Define

$$(a) := \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix},\ (v) := \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \text{ and } (A) := \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} .$$

Applying D to the identities $(1)^2=(1)$, $(a)(1)=(a)=(1)(a)$ and $(v)(1)=(v)$, $(1)(v) =0$, we get for some $u \in F$:

$$D(1) = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix},\ D(a) = \begin{pmatrix} \tilde{a} & au^* \\ ua & 0 \end{pmatrix} \text{ and } D(v) = \begin{pmatrix} -u^*v & 0 \\ \tilde{v} & vu^* \end{pmatrix},$$

where $a \mapsto \tilde{a}$ is a *-derivation of K and $\|\tilde{v}\| \leqq \|D\|\,\|v\|$. Hence, $\tilde{a}=[s,a]$ for all $a \in \mathsf{K}$, where $s \in \mathsf{K}$ is skew-adjoint. From $(v)(a)=(va)$ and $(v_1)^*(v_2)=(v_1^*v_2)$ it follows that $Sv:=\tilde{v}+vs$ defines a bounded operator $S \in \mathscr{S}(F)$. Finally, $(1)(A) =0=(A)(1)$ and $(A)(v)=(Av)$ imply that

$$D(A) = \begin{pmatrix} 0 & -u^*A \\ -Au & [S,A] \end{pmatrix} .$$

Hence,

$$w := \begin{pmatrix} s & -u^* \\ u & S \end{pmatrix} \in \mathscr{S}(E)$$

satisfies $Dx=[w,x]$ for all $x \in \mathscr{H}(E)$.

In the complex case it is possible to choose $\|w\| \leqq \|D\|$ by [25; Th. 4]. If $\mathsf{K} \neq \mathsf{C}$, we may assume that $r=2$ by considering a 2-dimensional subspace containing $h \in E$ and wh. From (2.7) it follows that $2\|w\| \leqq \|D\|$ if $\mathsf{K}=\mathsf{R}$ and $4\|w\| \leqq 5\|D\|$ if $\mathsf{K}=\mathsf{H}$.

PROOF OF 2.5. For each pure state f of Z, let $\pi_f\colon Z'' \to \mathscr{L}(K_f)$ be the canonical W*-representation associated with the normal state f of the second dual Z'' [21; p. 41]. Let σ be the weak operator topology on $\mathscr{L}(K_f)$. Then

$$\pi_f(Z'') = \overline{\pi_f Z}^{\sigma} = \mathscr{L}(K_f)\,. \tag{2.8}$$

If X'' is embedded in Z'', it can be proved as in [21; Prop. 1.16.2] that the unit ball of $\pi_f(X'')$ is σ-compact. Hence it follows from the Kaplansky density theorem for Jordan algebras that

$$Y := \pi_f(X'') = \overline{\pi_f X}^{\sigma} \tag{2.9}$$

is reversible. By (2.9) and the proof of [21; Lemma 4.1.4], there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{D} & X \\ {\scriptstyle \pi_f}\downarrow & & \downarrow{\scriptstyle \pi_f} \\ Y & \xrightarrow{\delta} & Y \end{array}, \tag{2.10}$$

where $\delta \in \operatorname{aut}(Y)$ satisfies $\|\delta\| \leqq \|D\|$. Since the complex algebra $\mathbf{C}[X]$ generated by X is uniformly dense in Z, it follows from (2.8) that $\mathbf{C}[\pi_f X]$ is σ-dense in $\mathscr{L}(K_f)$. Thus Y is a JW-factor acting irreducibly on K_f. Hence Y is of type I by [27; Th. 4.1]. Let r be the degree of Y. If $r=2$, then Y is a spin factor of dimension $\leqq 6$ by [26; Th. 7.1]. If $r \geqq 3$, it follows from [26; Th. 3.9] that there exists a K-Hilbert space structure E on K_f such that $Y = \mathscr{H}(E)$. From (2.7) and 2.6 we can deduce that

$$\delta y = [w_f, y] \quad \text{for all } y \in Y, \tag{2.11}$$

where $w_f \in \mathscr{S}(K_f)$ satisfies $4\|w_f\| \leqq 5\|D\|$. Denote by K, π and w the direct sum (over all pure states f of Z) of K_f, π_f and w_f, respectively. Then (2.10) and (2.11) imply that $\pi(Dx) = [w, \pi x]$ for all $x \in X$. Hence $[w, \pi z] \in \pi Z$ for all $z \in Z$ and thus $\pi(Dz) := [w, \pi z]$ defines a *-derivation of Z extending D, since $\pi\colon Z \to \mathscr{L}(K)$ is faithful.

2.12. COROLLARY (Sinclair [24]). *Each Jordan derivation of a* C**-algebra is already a derivation of the associative product.*

3. Inner derivations of JBW-algebras.

If B and C are subsets of a Lie algebra with bracket $[\cdot,\cdot]$, let $[B, C]$ denote the set of all finite sums of elements $[b, c]$, where $b \in B$ and $c \in C$.

3.1. DEFINITION. Let X be a JB-algebra and put $\mathfrak{m} := \{xM : x \in X\}$, where xM denotes the multiplication operator defined by $(xM)y := x \circ y$ for all

$x, y \in X$. The elements of the ideal $\mathrm{int}\,(X) := [\mathfrak{m}, \mathfrak{m}]$ of $\mathrm{aut}\,(X)$ are called *inner derivations* of X.

If Δ is the bounded symmetric domain associated with a JB-algebra X (i.e. the open unit ball of the JB*-algebra $Z := X \oplus iX$) and if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g} := \mathrm{aut}\,(\Delta)$, then the map $\lambda \mapsto ((\lambda e)M, \lambda - (\lambda e)M)$ induces decompositions

$$(3.2) \qquad [\mathfrak{p}, \mathfrak{p}] = i\mathfrak{m} \oplus \mathrm{int}\,(X) \subset \mathfrak{k} = i\mathfrak{m} \oplus \mathrm{aut}\,(X)\,.$$

Since the infinitesimal isometries in $[\mathfrak{p}, \mathfrak{p}]$ are explicitly known, the problem arises, to what extent $\mathfrak{k}$ or $\mathrm{aut}\,(X)$ is determined by $[\mathfrak{p}, \mathfrak{p}]$ or $\mathrm{int}\,(X)$, respectively. If $\dim\,(X) < \infty$, then the answer is well-known [6; p. 281, Satz 3.1]:

$$(3.3) \qquad \mathrm{aut}\,(X) = \mathrm{int}\,(X) \text{ and therefore } \mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]\,.$$

For the self-adjoint part X of a C*-algebra, $\mathrm{aut}\,(X)$ does not agree with $\mathrm{int}\,(X)$ in general (see (4.1)). Therefore we consider in this section JB-algebras with predual (JBW-algebras) and determine completely those JBW-algebras which have only inner derivations. Recall that by (1.4) and (2.4), each JBW-algebra X has a canonical decomposition

$$(3.4) \qquad X = X_{\mathrm{ex}} \oplus X_2 \oplus X_{\mathrm{rev}}\,,$$

such that

(i) X_{ex} is a purely exceptional JBW-algebra,
(ii) X_2 is a JW-algebra of type I_2 isomorphic to $\oplus_{j \in J} L^\infty(S_j, U_j)$, where $S_j \neq \emptyset$ is a measure space and U_j is a spin factor for each $j \in J$,
(iii) X_{rev} is a reversible JW-algebra.

3.5. Theorem. *Suppose X is a* JBW-*algebra with canonical decomposition* (3.4). *Then* $\mathrm{aut}\,(X) = \mathrm{int}\,(X)$ *if and only if* $\sup_{j \in J} \dim U_j < \infty$.

The proof of 3.5 consists of several steps giving more precise information in special cases.

3.6. Proposition. *Each purely exceptional* JBW-*algebra X has only inner derivations.*

Proof. Since $X \approx \mathscr{C}(S, \mathscr{H}_3(\mathrm{O}))$ for some compact space S, the assertion follows easily from (1.2).

3.7. Example. Let $X = \mathbf{R}e \oplus Y$ be a spin factor. Then int (X) consists of finite rank operators since $[xM, yM]$ has rank $\leqq 2$ for all $x, y \in X$. Conversely, if $D \in \operatorname{aut}(X)$ has finite rank and if $e_1, \ldots, e_n$ is a Hilbert basis of DX, then

$$2D = \sum_{\nu=1}^{n} [(De_\nu)^* M, (e_\nu) M] \in \operatorname{int}(X).$$

If a JB-algebra X is given as in (3.4.ii), these remarks imply that aut (X) $=$ int (X) if and only if $\sup_{j \in J} \dim U_j < \infty$.

Each JW-algebra X has a type decomposition of the form

$$\mathrm{I}_{\mathrm{fin}} \oplus \mathrm{I}_\infty \oplus \mathrm{II}_1 \oplus \mathrm{II}_\infty \oplus \mathrm{III}$$

(cf. [28; Th. 13]). X is called *properly non-modular*, if its modular part $\mathrm{I}_{\mathrm{fin}} \oplus \mathrm{II}_1$ vanishes. A *real* W*-*algebra* on a complex Hilbert space H is a weakly closed self-adjoint real subalgebra W of $\mathscr{L}(H)$. If d is a cardinal number, let $\mathscr{M}_d(W)$ be the set of all bounded operators on the Hilbert sum of d copies of H which are matrices with entries in W.

3.8. Theorem. *Let X be a properly non-modular* JW-*algebra. Then each* $D \in \operatorname{aut}(X)$ *is the sum of* 6 *commutators* $[xM, yM]$, *where* $x, y \in X$. *In particular,* aut $(X) =$ int (X).

Proof. X is reversible by [26; Th. 6.4 and Th. 6.6]. Applying [26; Lemma 6.1], [27; Lemma 2.3 and Th. 2.4] and the extension theorem 2.5, we may assume that X is the self-adjoint part of a properly infinite real W*-algebra W and that $Dx = [w, x]$ for all $x \in X$, where $w \in W$ is skew-adjoint. It follows from [4; p. 103, Th. 1] that there is a spatial *-isomorphism φ: $W \to \mathscr{M}_2(W)$. Define

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} := \varphi(w) \text{ and } A := \varphi^{-1}\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}.$$

Similarly, we may identify W and $\mathscr{M}_{\aleph_0}(W)$ by a spatial *-isomorphism, such that

$$A = \begin{bmatrix} a & 0 & \cdots \\ b_1 & 0 & \cdots \\ b_2 & 0 & \cdots \\ \vdots & \vdots & \end{bmatrix}.$$

Then $A=[P,Q]$ by [20; p. 512], where

$$P := \begin{bmatrix} 0 & \cdot & \cdot & & & \\ 1 & 0 & \cdot & & \cdot & \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & 0 & \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \end{bmatrix}, \quad Q := \begin{bmatrix} b_1 & -a & 0 & \cdot & & \\ b_2 & 0 & -a & \cdot & & \\ b_3 & 0 & 0 & -a & & \\ & & & & \cdot & \\ \cdot & \cdot & \cdot & \cdot & & \\ & & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \end{bmatrix}.$$

Put $2x_1:=P+P^*$. Now it can be verified that $P-P^*=2[x_2,x_3]$, where

$$x_2 := \begin{bmatrix} 0 & & & & & 0 \\ & 1 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & \cdot \\ 0 & & & & & \cdot \end{bmatrix}, \quad 2x_3 := \begin{bmatrix} 0 & 1 & & & & 0 \\ 1 & 0 & -1 & & & \\ & -1 & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & \cdot & \\ 0 & & & & & \cdot \end{bmatrix}.$$

By the Jacobi identity, $A=[x_1,Q]+[x_2,[x_3,Q]]+[x_3,[Q,x_2]]$. Applying a similar argument to

$$B := \varphi^{-1}\begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix},$$

we obtain $w=\sum_{\nu=1}^{6}[x_\nu,w_\nu]$, where $x_\nu \in X$ and $w_\nu \in W$ satisfy $\|x_\nu\|\,\|w_\nu\| \leqq 4\|w\|$. Put $2y_\nu:=w_\nu+w_\nu^*$ for all ν. Then

$$D = 4\sum_{\nu=1}^{6}[x_\nu M, y_\nu M].$$

3.9. THEOREM. *Each derivation D of a reversible* JW*-algebra X of type* $\mathrm{I}_{\mathrm{fin}}$ *can be written as* $D=4\sum_{\nu=1}^{5}[x_\nu M,y_\nu M]$, *where* $x_\nu, y_\nu \in X$ *satisfy* $\|x_\nu\|\,\|y_\nu\| \leqq 5\|D\|$.

PROOF. Suppose that $X=\mathscr{H}_r(\mathsf{K})$, where r is a positive integer. Then it follows from 2.6 that $Dx=[w,x]$ for all $x \in X$, where w is a skew-adjoint $r\times r$-matrix over K satisfying $4\|w\| \leqq 5\|D\|$. One can assume that $\|w\| \leqq 1$.

If $\mathsf{K}=\mathsf{R}$, we have

$$\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\right] \quad \text{for all } a \in \mathsf{R}$$

which shows that $w=[x,y]$, where $x,y \in X$ satisfy $\|x\|\,\|y\| \leqq 1$. If $\mathsf{K} \neq \mathsf{R}$, one can assume that w is a diagonal matrix with diagonal entries $a_1,\ldots,a_r \in i\mathsf{R}$. If trace $(w)=0$, one can further assume that for each $k\leqq r$ the partial sums $s_k := \sum_{j=1}^{k} a_j$ have modulus $\leqq 1$. Define P similar as in the proof of 3.8 and let Q be the $r\times r$-matrix with $Q_{k,k+1} := -s_k$ for $k<r$ and with zero entries elsewhere. Then $w=[P,Q]$. Now argue as in the proof of 3.8. If $p,q\in \mathsf{H}$ are skew-adjoint and $\alpha\in \mathsf{R}$ satisfies $2\alpha^2=|p|^2+|q|^2$, it follows that

$$\left[\begin{pmatrix} 0 & p \\ -p & 0\end{pmatrix}, \begin{pmatrix} 0 & -q \\ q & 0\end{pmatrix}\right] = \begin{pmatrix} s & 0 \\ 0 & s\end{pmatrix} \quad \text{and}$$

$$\left[\begin{pmatrix} 0 & p & \alpha \\ -p & 0 & -q \\ \alpha & q & 0\end{pmatrix}, \begin{pmatrix} \alpha & -q & 0 \\ q & 0 & -p \\ 0 & p & -\alpha\end{pmatrix}\right] = \begin{pmatrix} s & 0 & 0 \\ 0 & 2s & 0 \\ 0 & 0 & s\end{pmatrix},$$

where $s:=[p,q]$. Applying these formulae to the remaining case of a scalar matrix w, it is easy to show that 3.9 is true if $X=\mathscr{H}_r(\mathsf{K})$.

Now suppose that X is a reversible JW-algebra of type $\mathrm{I}_{\mathrm{fin}}$. Decomposing X into homogeneous JW-algebras with finite faithful normal trace and applying [12; Satz 36], we may assume that $X=\mathscr{C}(S,U)$, where S is a Stone space and U is a reversible spin factor or $U=\mathscr{H}_r(\mathsf{K})$ and $r\geqq 3$ is finite. Each $D\in \operatorname{aut}(X)$ can be viewed as a map $D\in\mathscr{C}(S,\operatorname{aut} U)$ by (1.2). If $s\in S$, the above reasoning implies that

$$D(s) = 4\sum_{\nu=1}^{5} [x_\nu(s)M, y_\nu(s)M]\ ,$$

where the maps $x_\nu, y_\nu\colon S\to U$ satisfy

$$\|x_\nu(s)\|\,\|y_\nu(s)\| \leqq 5\|D\| \qquad \text{for all } s\in S\ .$$

By [9; Th. 2.5], one can assume that $x_\nu, y_\nu\in X$ for all ν.

3.10. THEOREM. *Let X be a reversible* JW-*algebra. Then each derivation of X is inner, i.e.* $\operatorname{aut}(X)=\operatorname{int}(X)$.

PROOF. By 3.8 and 3.9, it suffices to consider JW-algebras without type I summand. From the extension theorem 2.5 and [21; Th. 4.1.6] it follows that each $D\in\operatorname{aut}(X)$ has the form $Dx=[w,x]$ for all $x\in X$, where $w=-w^*$ lies in the complex W^*-algebra W generated by X. It has been shown in [31, 32], that each operator having central trace 0 in a finite W^*-algebra W is a sum of 10 commutators in W (the separability condition assumed in [31; Th. 4.1] is not necessary). Using this result and [20; p. 512], we may assume that $w\in[W,W]$. By [26; Lemma 6.1] and [27; Lemma 2.3 and Th. 2.4], we may further assume that X is the self-adjoint part of a real W^*-algebra V and $w\in[V,V]$. Since X is supposed to have no type I summand, V is of continuous type. Hence V is *-

isomorphic to $\mathcal{M}_2(A)$, where A is a real *-algebra [4; p. 121], Since $w^* = -w$, the following Lemma implies that $w \in [X,X]$ and therefore $D \in \operatorname{int}(X)$.

3.11. LEMMA. *Let A be an associative real *-algebra with unit* 1 *and denote by V the *-algebra of all* 2×2*-matrices over A. Put*

$$S := \{v \in V : v^* = -v\} \text{ and } X := \{v \in V : v^* = v\} .$$

Then $[S,S] \subset [X,X]$.

PROOF. If

$$s = \begin{pmatrix} a & -c^* \\ c & b \end{pmatrix} \in [S,S] ,$$

then $a+b$ is a finite sum of commutators $[\alpha_1,\alpha_2]$ and $[\beta_1,\beta_2]$, where $\alpha_1,\alpha_2 \in A$ are self-adjoint and $\beta_1,\beta_2 \in A$ are skew-adjoint. Hence the assertion follows from the formulae

$$\begin{pmatrix} 0 & -c^* \\ c & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix}\right],$$

$$2\begin{pmatrix} a-b & 0 \\ 0 & b-a \end{pmatrix} = \left[\begin{pmatrix} 0 & a-b \\ b-a & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right],$$

$$\begin{pmatrix} [\alpha_1,\alpha_2] & 0 \\ 0 & [\alpha_1,\alpha_2] \end{pmatrix} = \left[\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{pmatrix}\right],$$

$$\begin{pmatrix} [\beta_1,\beta_2] & 0 \\ 0 & [\beta_1,\beta_2] \end{pmatrix} = \left[\begin{pmatrix} 0 & -\beta_2 \\ \beta_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\beta_1 \\ \beta_1 & 0 \end{pmatrix}\right].$$

4. Approximation by inner derivations.

For JB-algebras in general, the solution of the problem, how to describe $\operatorname{aut}(X)$ in terms of $\operatorname{int}(X)$, requires topological arguments. If E is a Banach space, let τ_E and σ_E denote the strong operator topology and the weak operator topology, respectively, on the algebra $\mathcal{L}(E)$ of all bounded operators on E. It is proved in this section that $\operatorname{aut}(X)$ is the closure of $\operatorname{int}(X)$ in the topology τ_X of simple convergence on X. By 3.7, $\operatorname{int}(X)$ is not uniformly dense in $\operatorname{aut}(X)$ if X is a spin factor of infinite dimension. Another example of this type is the following

4.1. EXAMPLE. Let X be the self-adjoint part of the C*-algebra

$$Z := \{c1_H + z : c \in \mathbf{C},\ z \text{ compact operator on } H\}$$

acting on a Hilbert space H of dimension $\aleph_0$. Then

$$\operatorname{aut}(X) = \{x \mapsto [w,x] : w \in \mathscr{S}(H)\} .$$

Obviously $w \in \mathscr{S}(H)$ induces a (Jordan) inner derivation if and only if

$$w \in [X,X] \bmod \mathbf{C}1_H ,$$

which by [19; Th. 1] is equivalent to $w \in Z$. Therefore int (X) is uniformly closed but different from aut (X).

Since JB-derivations are automatically bounded, the pointwise limit of a net of inner derivations lies in aut (X). Thus τ_X seems to be a natural topology on aut (X).

4.2. Approximation Theorem. *For every* JB-*algebra* X, aut (X) *is the closure of* int (X) *in the strong operator topology* τ_X *(i.e. the topology of simple convergence on* X).

Proof. Suppose that X is a JW-algebra of type I_2. Choose $D \in \operatorname{aut}(X)$ and $x_1,\ldots,x_m \in X$. We may assume that each x_μ has vanishing central trace (cf. [28; p. 40]). Put $x_{m+\mu} := Dx_\mu$ for all $\mu \leqq m$. By [33; Th. 2], X is a direct sum of JW-algebras isomorphic to $L^\infty(S,U)$, where S is a measure space and U is a spin factor. By the orthonormalization process, there exist $e_1,\ldots,e_{2m} \in X$ such that

$$x_\nu = \sum_{\mu=1}^{2m} e_\mu \circ (e_\mu \circ x_\nu)$$

and $e_\mu \circ x_\nu$ are central for all $\mu, \nu \leqq 2m$. This implies for all $\nu \leqq m$:

$$2Dx_\nu = \sum_{\mu=1}^{2m} [(De_\mu)M, (e_\mu)M]x_\nu .$$

If X is a JB-algebra with predual, the above argument together with 3.5 shows that $\operatorname{aut}(X) = \overline{\operatorname{int}(X)}^{\tau_X}$.By [1; Prop. 3.9], the unit ball of a JB-algebra X is strongly dense in the unit ball of X''. Since each $D \in \operatorname{aut}(X)$ can be extended to a derivation of X'' and multiplication is jointly strongly continuous on bounded subsets of X'' by [1; Prop. 3.7], it follows that

$$\operatorname{aut}(X) = \overline{\operatorname{int}(X)}^{\sigma_X} = \overline{\operatorname{int}(X)}^{\tau_X}$$

by [5; p. 77, Prop. 11].

4.3. Corollary. *If* Δ *is a bounded symmetric domain of tube type and if* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *is the Cartan decomposition of* $\mathfrak{g} := \operatorname{aut}(\Delta)$, *then* $\mathfrak{k}$ *is the closure of* $[\mathfrak{p},\mathfrak{p}]$ *in the topology of simple convergence.*

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