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HYPERFINITE STOCHASTIC INTEGRATION I: THE NONSTANDARD THEORY

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Introduction.

For a number of years nonstandard measure theory was in a somewhat problematic state. Although many interesting papers appeared (e.g. Bernstein and Wattenberg [5], Loeb [14], [15] and Bernstein and Loeb [4]), the theory suffered from the fact that nonstandard measures refuse to be countably additive. This made application to and comparison with standard measure- and probability-theory difficult.

Many of these difficulties were resolved when Peter A. Loeb in [16] showed how to turn *-measures into real-valued, countably additive measures. This opened the way for the use of nonstandard constructions in probability theory, and Loeb himself gave the first example by constructing a Poisson process. Shortly afterwards R. M. Anderson [1] used Loeb's technique to make a nonstandard construction of a Brownian motion in a very simple and attractive way. Anderson also gave a nonstandard treatment of stochastic integration with respect to this Brownian motion process, and this theory was used by H. J. Keisler in [10] to obtain new results on stochastic differential equations.

Keisler [9] (see also Hoover [7]) based his hyperfinite model theory on the Loeb-measure, and A. E. Hurd [8] used it in analyzing equilibrium states in statistical mechanics. In his thesis [2], Anderson further developed the theory of Loeb-spaces, found a representation of Radon-spaces as inverse images of hyperfinite Loeb-spaces, and gave applications to economics. This list does by no means exhaust the applications of the Loeb-measure, but it should be long and varied enough to indicate the fruitfulness of the idea.

In this paper we shall carry on the work on nonstandard stochastic integration begun by Anderson in [1]. Let us first give a short sketch of standard stochastic integration. The problem is simply to give a precise meaning to the heuristic formula $\int X\,dM$ where $X, M\colon \mathbf{R}_+ \times \Omega \to \mathbf{R}$ are suitable processes on a proabability space Ω. This is nontrivial since "typical" examples of M (e.g. Brownian motions) have paths of unbounded variation, and the

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integral $\int X\,dM$ thus can *not* be defined by pathwise Stieltjes integration. However, it was realized by N. Wiener and K. Itô that in the Brownian motion case the integral $\int X\,dM$ can be defined as an L^2-limit, and this has later been extended to more general classes of processes M (so called local L^2-martingales; see Meyer [20] or Métivier [18] for expositions.)

We now turn to Anderson's work: Let $\eta \in {}^*\mathbf{N} \setminus \mathbf{N}$, and let $T = \{k/\eta : k \in {}^*\mathbf{N},\ k < \eta^2\}$. Anderson constructed a hyperfinite *-probability space Ω and an internal process $\chi: T \times \Omega \to {}^*\mathbf{R}$ — which in fact was nothing but a hyperfinite random walk — such that the process $\beta: \mathbf{R}_+ \times \Omega \to \mathbf{R}$ defined by $\beta(t,\omega) = {}^\circ\chi(\bar{t},\omega)$, was a Brownian motion (here $\bar{t}$ is the least element of T larger than t. For a definition of Anderson's process, see Example 1 below). If $Y: T \times \Omega \to {}^*\mathbf{R}$ is an internal process, he defined the stochastic integral $\int Y\,d\chi$ to be the process

$$(k/\eta,\omega) \to \sum_{i=0}^{k-1} Y(i/\eta,\omega)(\chi(i+1/\eta,\omega) - \chi(i/\eta,\omega));$$

i.e. as a pathwise "Stieltjes integral". Anderson was now able to show that given a process X such that $\int X\,d\beta$ was defined by the standard theory, he could find an internal process Y (called a 2-lifting of X) such that ${}^\circ\int_0^{\bar{t}} Y\,d\chi = \int_0^t X\,d\beta$. Thus he could do by nonstandard methods what had proved impossible by standard ones; to write the stochastic integral as a pathwise Stieltjes integral. This intuitive definition of the stochastic integral enabled him to give a simple proof of Itô's Lemma. The work by Keisler [10] on stochastic differential equations has later confirmed the importance of this approach to stochastic integration.

The purpose of this paper is to generalize Anderson's theory about χ to a larger class of hyperfinite processes (called local λ^2-martingales), and to prove theorems on the structure of these processes. We shall prove that the paths are finite and have S-right- and S-left-limits a.e.; and characterize those processes which have S-continuous paths a.e. We then define stochastic integrals $\int X\,dM$ with respect to such martingales as sums $\sum X(t_i)(M(t_{i+1}) - M(t_i))$, and prove that if M is S-continuous and X is reasonably nice, then $\int X\,dM$ is S-continuous. Nonstandard versions of the Transformationformula and Itô's formula are proved; and we finally obtain a Doob–Meyer decomposition of hyperfinite processes.

In another paper [12], we shall show that from this nonstandard theory for stochastic integration with respect to such hyperfinite martingales, we may obtain the standard theory for stochastic integration with respect to "the standard parts" of the martingales. This extends the results of Anderson [1].

In a third paper [13], we shall show that all standard local L^2-martingales can — in a suitable sense — be represented as "the standard parts" of local λ^2-

martingales; and we shall argue that this combined with the results in [12] shows that anything that can be done by the standard theory for stochastic integration, can be done by the nonstandard theory. The hope is — of course — that the nonstandard theory will be simpler and more intuitive to work with.

This and the subsequent papers are based on a common previous version, the [11] of the references. That version is at times a little more comprehensive and contains some additional material.

Our main references will be Métivier [18] on stochastic integration, and Stroyan and Luxemburg [23] on nonstandard analysis.

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1. Setting.

We shall work with *polysaturated models* for nonstandard analysis. That is we begin with a set K large enough to include all the standard entities we shall need (e.g. the real numbers $\mathbf{R}$, the relevant measure spaces), and we construct the superstructure $V(K)$ over K. We then take ${}^*V(K)$ to be a card $(V(K))$-saturated nonstandard model of $V(K)$. This will be our polysaturated model. For a proof of the existence of such models and a survey of their properties, see Stroyan and Luxemburg [23], sections 7.4–7.6.

Let $\langle \Omega, \mathcal{G}, P \rangle$ be a *-measure space. We may define a finitely additive measure ${}^\circ P$ on the algebra $\mathcal{G}$ by ${}^\circ P(A) = \mathrm{st}\,(P(A))$ for all $A \in \mathcal{G}$. Peter A. Loeb proved in [16] that ${}^\circ P$ satisfies the conditions of Carathéodory's Extension Theorem (Royden [22, page 257]), and that ${}^\circ P$ thus may be extended to a measure on the σ-algebra $\sigma(\mathcal{G})$ generated by $\mathcal{G}$. The completion of this measure we denote by $L(P)$ and call the *Loeb-measure* of P. The σ-algebra $\sigma(\mathcal{G})$ is called the *Borel-algebra* of $\mathcal{G}$ and its completion $L(\mathcal{G})$ with respect to $L(P)$ is called the *Loeb-algebra* of $\mathcal{G}$. The measure space $\langle \Omega, L(\mathcal{G}), L(P) \rangle$ is called the *Loeb-space* of $\langle \Omega, \mathcal{G}, P \rangle$. When ${}^\circ P(\Omega)$ is finite, it follows from the uniqueness part of Carathéodory's Theorem that $L(P)$ is uniquely determined by P; C. Ward Henson has proved that this still holds when ${}^\circ P(\Omega) = \infty$ ([24]). If Ω is *-finite and $P(\Omega) = 1$, $\langle \Omega, \mathcal{G}, P \rangle$ is called a *hyperfinite probability space.*

The measure- and integration-theory of Loeb-spaces were developed in Loeb [16] and Anderson [1]. Expositions are given in Loeb [17] and (without proofs) in Keisler [10].

2. Hyperfinite processes.

We now define the basic concepts in the theory of hyperfinite processes:

A *hyperfinite time-line* is a hyperfinite subset of T of $^*\mathbf{R}_+$ such that $0 \in T$ and for each $a \in \mathbf{R}_+$ there is a $t \in T$ such that $t \approx a$. If S and T are hyperfinite time-lines with $S \subset T$, we call S a *subline* of T.

By a *hyperfinite stochastic process* X we mean an internal mapping $X: T \times \Omega \to {}^*\mathbf{R}$ where T is a hyperfinite time-line and $\langle \Omega, \mathcal{G}, P \rangle$ is a hyperfinite probability space such that the mappings $\omega \to X(t, \omega)$ are $\mathcal{G}$-measurable for all $t \in T$.

If T is a hyperfinite time-line, an *internal basis* $\langle \Omega, \{\mathcal{G}_t\}_{t \in T}, P \rangle$ indexed by T consists of a hyperfinite set Ω, an increasing *-sequence $\{\mathcal{G}_t\}_{t \in T}$ of *-algebras of subsets of Ω, and a *-probability measure P on $\mathcal{G}_{t_\infty}$, where t_∞ is the largest element in T. For convenience we shall often assume that $\mathcal{G}_{t_\infty}$ is the internal power-set of Ω such that each one-point set $\{\omega\}$ is $\mathcal{G}_{t_\infty}$-measurable.

Let us illustrate these definitions by an example:

EXAMPLE 1. In [1], Anderson constructed the following process: Choose an $\eta \in {}^*\mathbf{N} \setminus \mathbf{N}$, and let the time-line be $T = \{k/\eta : k \in {}^*\mathbf{N}, k < \eta^2\}$. The set Ω is the set $\{-1, 1\}^T$ of all internal functions from T to $\{-1, 1\}$. If $\omega \in \Omega$, we shall write ω_i for $\omega(i/\eta)$. The internal algebra $\mathcal{G}_{k/\eta}$ will be the *-algebra generated by the equivalence classes of the relation $\sim_k$, where $\omega \sim_k \omega'$ if $\omega_i = \omega'_i$ for all $i < k$. The probability measure P is the uniform measure on Ω; $P\{\omega\} = 2^{-\eta^2}$ for all $\omega \in \Omega$. Then $\langle \Omega, \{\mathcal{G}_t\}, P \rangle$ is an internal basis. We define a hyperfinite stochastic process $\chi: T \times \Omega \to {}^*\mathbf{R}$ by

$$\chi(k/\eta, \omega) = \sum_{i=0}^{k-1} \omega_i / \sqrt{\eta} \, .$$

Anderson proved that if we define $\beta: \mathbf{R}_+ \times \Omega \to \mathbf{R}$ by $\beta(t, \omega) = {}^\circ\chi(\bar{t}, \omega)$, where $\bar{t}$ is the least element of T larger than t, then β is a Brownian motion.

We see that for each $t \in T$, $\omega \to \chi(t, \omega)$ is $\mathcal{G}_t$-measurable, and this motivates our next definition:

A hyperfinite process $X: T \times \Omega \to {}^*\mathbf{R}$ is said to be *adapted* to the internal bases $\langle \Omega, \{\mathcal{G}_t\}, P \rangle$ if $\omega \to X(t, \omega)$ is $\mathcal{G}_t$-measurable for all $t \in T$.

NOTATION. We shall often write $X_t(\omega)$ or—suppressing ω—X_t for $X(t, \omega)$. Let $0 = t_0 < t_1 < \ldots < t_\xi = t_\infty$ be the elements of T in increasing order. If $X: T \times \Omega \to {}^*\mathbf{R}$ is a process, we shall write $\Delta X(t_i, \omega)$ for $X(t_{i+1}, \omega) - X(t_i, \omega)$. We use the following notation for summation: If $s = t_i$, $t = t_j$ and $i < j$, we write

$$\sum_s^t X(r, \omega) \quad \text{for} \quad \sum_{k=i}^{j-1} X(t_k, \omega) \, .$$

Notice that the term $X(t,\omega)$ is *not* included in the sum $\sum_s^t X(r,\omega)$. We use similar notation when the limits s and t of the summation depends on ω.

By the *quadratic variation* of a process $X\colon T\times\Omega \to {}^*\mathbf{R}$, we mean the process $[X]\colon T\times\Omega \to {}^*\mathbf{R}$ defined by

$$[X](t,\omega) = \sum_0^t (\Delta X(s,\omega))^2 .$$

Observe that if X is adapted to a basis $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$, then so is $[X]$. If χ is the process of Example 1, then $[\chi](t,\omega)=t$.

An *internal stopping time* adapted to the internal basis $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$ is an internal mapping $\tau\colon \Omega \to T$ such that for all $t \in T$, the set $\{\omega \in \Omega : \tau(\omega)\leqq t\}$ is in $\mathscr{G}_t$.

Definition 2. A hyperfinite process $M\colon T\times\Omega \to {}^*\mathbf{R}$ is called a *hyperfinite martingale* (*hyperfinite sub-martingale, hyperfinite super-martingale*) adapted to an internal basis $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$, if M is adapted to $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$ as a process and for all $s,t \in T$, $s<t$, and all $A \in \mathscr{G}_s$:

$$E(1_A(M_t-M_s)) = 0 \qquad (E(1_A(M_t-M_s)) \geqq 0,\ E(1_A(M_t-M_s)) \leqq 0) .$$

In this definition the expectation $E(1_A(M_t-M_s))$ is just a hyperfinite sum $\sum_{\omega\in\Omega} 1_A(\omega)(M_t(\omega)-M_s(\omega))P\{\omega\}$.

This paper is a study of hyperfinite martingales, and we have already encountered one; Anderson's process χ is a martingale with respect to the basis $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$ constructed in Example 1.

We shall need the following technical lemma relating the expectation of the quadratic variation of a martingale to the expectation of the square of the martingale.

Lemma 3. *Let $M\colon T\times\Omega \to {}^*\mathbf{R}$ be a hyperfinite martingale adapted to the internal basis $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$, and let σ,τ be two internal stopping times adapted to the same basis. Assume that $\sigma(\omega)\geqq\tau(\omega)$ for $\omega \in \Omega$. Then*:

$$\begin{aligned} E\big(M(\sigma(\omega),\omega)^2 - M(\tau(\omega),\omega)^2\big) &= E\big((M(\sigma(\omega),\omega)-M(\tau(\omega),\omega))^2\big) \\ &= E\bigg(\sum_{\tau(\omega)}^{\sigma(\omega)} \Delta M(s,\omega)^2\bigg) . \end{aligned}$$

The proof is an easy exercise in the martingale property and the definition of stopping times, and is left to the reader.

We end this section by quoting two results from the standard theory of martingales which we shall use in the sequel. The first is a famous inequality due to J. L. Doob which is proved in almost any text on stochastic processes; see e.g. Doob [6] or Métivier [18]:

PROPOSITION 4. (Doob's inequality). *Let* $X: J \times Z \to \mathbf{R}$ *be a positive (standard) submartingale with respect to a stochastic basis* $\langle Z, \{\mathscr{F}_t\}_{t\in J}, \mu\rangle$, *where* $J \subset \mathbf{R}_+$. *Let* I *be a countable subset of* J *and assume that* $b = \sup I \in J$. *Then for all* $p \in \langle 1, \infty\rangle$ *such that* $E(X_b^p) < \infty$, *we have*:

$$\left\| \sup_{t\in I} X_t \right\|_p \leqq \frac{p}{p-1} \|X_b\|_p .$$

In this paper we shall use the *-transfer of Proposition 4 to deal with hyperfinite martingales, while in [12] and [13] we shall need the standard version above.

The next proposition is *-transfer of well-known inequalities in the theory for discrete martingales.

PROPOSITION 5. *Let* $M: T \times \Omega \to {}^*\mathbf{R}$ *be a hyperfinite martingale. Then for all* $p \in \langle 1, \infty\rangle$ *and all* $t \in T$:

$$\|\sqrt{M_0^2 + [M](t)}\|_p \leqq 10p \left\| \sup_{s\leqq t} (M_t) \right\|_p$$

and

$$\left\| \sup_{s\leqq t} (M_t) \right\|_p \leqq p\sqrt{12}\, \|\sqrt{M_0^2 + [M](t)}\|_p .$$

For a proof, see Neveŭ [21] (Corollary VIII-3-18) or Meyer [19]. Lest the reader should feel discouraged by the length and difficulty of these proofs, we hasten to assure him that we shall only need the result for $p = 4$, and for some constants $c_4, d_4 \in \mathbf{R}$ in stead of $10\cdot 4$ and $4\sqrt{12}$. A simple proof may then be obtained from Meyer's comments in [19], and this was done in [11].

3. The structure of λ^2-martingales.

We define the classes of hyperfinite martingales, which we shall study in this paper. Compare the definition of L^2-martingales in Métivier [18, page 16]:

DEFINITION 6. Let $M: T \times \Omega \to {}^*\mathbf{R}$ be a hyperfinite martingale adapted to the internal basis $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$. M is called a *λ^2-martingale* if for all finite $t \in T$, $E(M_t^2)$ is finite. M is called a *local λ^2-martingale* if there exists an increasing sequence $\{\tau_n\}_{n\in \mathbf{N}}$ of internal stopping times adapted to $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$ such that

${}^\circ\tau_n \to \infty$ a.e. in Loeb-measure, and such that for each n the martingale M_{τ_n} defined by $M_{\tau_n}(t,\omega)=M(\tau_n(\omega)\wedge t,\omega)$ is a λ^2-martingale. The sequence $\{\tau_n\}$ is then called a *localizing sequence* for M.

In this section we shall use Lemma 3 and Proposition 4 to derive some properties of the paths of λ^2-martingales. An obvious consequence of Lemma 3 is that a hyperfinite martingale is a λ^2-martingale if and only if $E(M_0^2+[M](t))$ is finite for all finite $t \in T$.

PROPOSITION 7. *Let M: $T\times\Omega \to {}^*\mathrm{R}$ be a local λ^2-martingale. Then the set of all $\omega \in \Omega$ such that $M(t,\omega)$ and $[M](t,\omega)$ are finite for all finite $t \in T$ has Loeb-measure one.*

PROOF. Let $\{\tau_n\}$ be a localizing sequence for M. For each finite $t \in T$ and each $n \in \mathrm{N}$, $E([M_{\tau_n}](t))$ is finite. Consequently $[M_{\tau_n}](t)$ is finite a.e. Since $[M_{\tau_n}]$ is an increasing process $\max_{s\leqq t}[M_{\tau_n}](s)$ is finite $L(P)$-a.e. Now ${}^\circ\tau_n \to \infty$ $L(P)$-a.e. and hence $\max_{s\leqq t}[M](s)$ is finite $L(P)$-a.e. Since this is true for an arbitrary finite $t \in T$, the $[M]$-part of the proposition follows.

By Proposition 3

$$E\left(\max_{s\leqq t} M^2_{\tau_n}(s)\right) \leqq 4E(M^2_{\tau_n}(t))$$

and consequently $\max_{s\leqq t} M^2_{\tau_n}(s)$ is finite $L(P)$-a.e. for all $n \in \mathrm{N}$ and all finite $t \in T$. Repeating the argument of the first part of the proof, we prove the M-part of the proposition.

Proposition 7 tells us that the local λ^2-martingales live where we want them to live; in the finite part of ${}^*\mathrm{R}$. Our next result deals with the behaviour of the paths as we approach a point t in T. We need some definitions.

DEFINITION 8. If f: $T \to {}^*\mathrm{R}$ is an internal function we define the *upper S-left-limit* of f at t by:

$$\overline{S\text{-lim}}_{s\uparrow t} f(s) = \inf_{\substack{s<t \\ s\notin \mathrm{mon}(t)}} \sup\{{}^\circ f(r) : s\leqq r<t,\ r \notin \mathrm{mon}(t)\}\ .$$

The *lower S-left-limit* of f at t is defined by:

$$\underline{S\text{-lim}}_{s\uparrow t} f(s) = \sup_{\substack{s<t \\ s\notin \mathrm{mon}(t)}} \inf\{{}^\circ f(r) : s\leqq r<t,\ r \notin \mathrm{mon}(t)\}\ .$$

The *upper S-right limit* $\overline{S\text{-lim}}_{s\downarrow t} f(s)$ and the *lower S-right limit* $\underline{S\text{-lim}}_{s\downarrow t} f(s)$ are defined in a similar way.

The function f is said to have an *S-left limit* at t if

$$\overline{S\text{-lim}}_{s\uparrow t} f(s) = \underline{S\text{-lim}}_{s\uparrow t} f(s) \neq \pm\infty$$

and an *S-right limit* if

$$\overline{S\text{-lim}}_{s\downarrow t} f(s) = \underline{S\text{-lim}}_{s\downarrow t} f(s) \neq \pm\infty .$$

Notice that these limits only depend on the monad of t and not on which point in that monad we use.

A process $X\colon T\times\Omega \to {}^*\mathbf{R}$ is said to have *S-left limits* (*S-right limits*) a.e. if for $L(P)$-almost all $\omega \in \Omega$ the function $t \to X(t,\omega)$ has S-left limits (S-right limits) for all finite $t \in T$.

We have the following:

THEOREM 9. *Let M be a local λ^2-martingale; then M has S-left- and S-right limits* a.e.

PROOF. We prove the S-left limit case; the S-right limit case is similar.

There are three ways in which M can fail to have S-left limits on a path $s \to M(s,\omega)$; there must exist a finite $t \in T$ such that either

$$\overline{S\text{-lim}}_{s\uparrow t} M(s,\omega) = \infty \quad\text{or}\quad \underline{S\text{-lim}}_{s\uparrow t} M(s,\omega) = -\infty$$

or such that

$$\overline{S\text{-lim}}_{s\uparrow t} M(s,\omega) - \underline{S\text{-lim}}_{s\uparrow t} M(s,\omega) > \varepsilon$$

for an $\varepsilon \in \mathbf{R}$, $\varepsilon>0$.

In the first two cases $\max_{s\leqq t}|M(s,\omega)|$ must be infinite and according to Proposition 7 this can only happen on a set of measure zero.

In the third case we may for each $n \in \mathbf{N}$ find a sequence $t_1<t_2<\ldots<t_n<t$ such that $|M(t_{i+1},\omega)-M(t_i,\omega)|\geqq\varepsilon$, for $1\leqq i<n$, by definition of $\overline{S\text{-lim}}_{s\uparrow t}$ and $\underline{S\text{-lim}}_{s\uparrow t}$. If we can prove that this only happens on a set of measure zero, we have proved the theorem. Since it will be useful in the sequel, we formulate this statement as a separate lemma:

LEMMA 10. *Let $M\colon T\times\Omega \to {}^*\mathbf{R}$ be a local λ^2-martingale. Let A be the set of all ω such that there exists an $\varepsilon \in \mathbf{R}$, $\varepsilon>0$, and a finite $t \in T$ such that for all $n \in \mathbf{N}$ there are n elements $t_1<t_2<\ldots<t_n<t$ in T with $|M(t_{i+1},\omega)-M(t_i,\omega)|\geqq\varepsilon$ for all i, $1\leqq i<n$. Then $L(P)(A)=0$.*

PROOF. It is enough to prove the lemma for λ^2-martingales, since we can then use localizing sequences of stopping times to prove it for local λ^2-martingales.

Moreover, it is enough to prove that for all $\varepsilon > 0$ and all finite $t \in T$, the set B of all $\omega \in \Omega$ such that for all $n \in \mathbf{N}$ there are elements $t_1 < t_2 < \ldots < t_n < t$ in T with $|M(t_{i+1}, \omega) - M(t_i, \omega)| \geqq \varepsilon$ has Loeb-measure zero.

So let us assume that M is a λ^2-martingale with ε, t and B as above. Define an internal sequence of stopping times by: $\tau_0(\omega) = 0$ and

$$\tau_{n+1}(\omega) = \min\{s \in T:\ s > \tau_n(\omega) \wedge |M(s,\omega) - M(\tau_n(\omega), \omega)| \geqq \tfrac{1}{2}\varepsilon\} \wedge t .$$

For $k \in \mathbf{N}$ let B_k be the set of all $\omega \in \Omega$ such that there exist $t_1 < t_2 < \ldots < t_k < t$ with $|M(t_{i+1}, \omega) - M(t_i, \omega)| \geqq \varepsilon$. If $\langle \Omega, \{\mathcal{G}_t\}, P \rangle$ is the internal basis of M, then $B_k \in \mathcal{G}_t$ and is hence measurable. For each $\omega \in B_k$ and each t_i in the sequence defining ω as an element of B_k there is an $m \in {}^*\mathbf{N}$ such that $t_i \leqq \tau_m(\omega) \leqq t_{i+1}$, and if $\tau_m(\omega) \leqq t_{i+1}$ then $\tau_{m+1}(\omega) \leqq t_{i+2}$. Consequently $\tau_{k-1}(\omega) \leqq t_k < t$.

By Lemma 3 we have:

$$E\left(\sum_{n \in {}^*\mathbf{N}} (M(\tau_{n+1}(\omega), \omega) - M(\tau_n(\omega), \omega))^2 \right)$$
$$= E\left(\sum_{n \in {}^*\mathbf{N}} \sum_{\tau_n(\omega)}^{\tau_{n+1}(\omega)} (\Delta M(s,\omega))^2 \right) = E([M](t)) .$$

(These sums are really hyperfine since $\tau_n(\omega) = t$ for some $n \in {}^*\mathbf{N}$, and so there is no problem of convergence.)

For $\omega \in B_k$, we have by definition of τ_{n+1} and the fact that $\tau_{k-1}(\omega) < t$:

$$\sum_{n \in {}^*\mathbf{N}} (M(\tau_{n+1}(\omega), \omega) - M(\tau_n(\omega), \omega))^2 \geqq (k-1)(\varepsilon/2)^2 .$$

Consequently:

$$P(B_k) \cdot (k-1)(\varepsilon/2)^2 \leqq E([M](t)) \quad \text{or} \quad P(B_k) \leqq \frac{4E([M](t))}{\varepsilon^2(k-1)} .$$

Since M is a λ^2-martingale $E([M](t))$ is finite and hence $L(P)(B_k) = {}^\circ P(B_k) \to 0$ as $k \to \infty$. Since $B = \bigcap B_k$, $L(P)(B) = 0$. This proves the lemma, and completes the proof of Theorem 9.

4. S-continuous λ^2-martingales.

Definition 11. An internal function f: $T \to {}^*\mathbf{R}$ is called *S-continuous* if for all finite $s, t \in T$ such that $s \approx t$, the values $f(s)$ and $f(t)$ are finite and infinitesimally close. A hyperfinite process X: $T \times \Omega \to {}^*\mathbf{R}$ is *S-continuous* if for all ω in a set of Loeb-measure one, the function $t \to X(t, \omega)$ is S-continuous.

Using the internal definition principle it is easy to show that f: $T \to {}^*\mathbf{R}$ is S-continuous if and only if $f(t)$ is finite for all finite $t \in T$, and for all such t and all positive $\varepsilon \in \mathbf{R}$ there is a $\delta \in \mathbf{R}$, $\delta>0$, such that for all $s \in T$ with $|s-t|<\delta$, we have $|f(s)-f(t)|<\varepsilon$.

In this section we want to prove that a local λ^2-martingale is S-continuous if and only if its quadratic variation is S-continuous. This is an example of the main strategy of this paper; to relate properties of a hyperfinite martingale to properties of its quadratic variation. Since it is an increasing process, the quadratic variation is often much easier to work with than the wildly oscillating martingale. A first feeble attempt in carrying out this programme was made in Lemma 3 and some of its consequences gathered in Lemma 10 and Theorem 9. Using Proposition 5 we shall now launch a more serious attack leading up to the result announced above (Theorem 14). We shall see more consequences later when we use Theorem 14 to prove Theorem 21.

Let us return to the problem in question. We shall use a sequence $\{\tau_n\}$ of internal stopping times defined by

$$\tau_n(\omega) = \min\{t \in T:\ |M(t,\omega)| \geqq n\}\ ,$$

and then apply Proposition 4 to the martingales M_{τ_n} with $p=4$. We shall thus consider expressions of the form $E([M_{\tau_n}](t)^2)$. However, this may be infinite, and in that case we should not be able to complete our calculations. Hence we loook for conditions that will ensure us that it is finite. One of them is obviously the following:

A process X: $T\times\Omega \to {}^*\mathbf{R}$ is said to have *infinitesimal increments* if for all finite $t \in T$ and all $\omega \in \Omega$, $\Delta M(t,\omega)\approx 0$.

We shall show that all S-continuous λ^2-martingales (or λ^2-martingales having S-continuous quadratic variation) can be modified into λ^2-martingales with infinitesimal increments, and then use Proposition 5 and the stopping times above to prove our results.

We shall need some technical preliminaries: Recall from Anderson [1], that a $\mathscr{G}$-measurable internal function f on a hyperfinite measure space $\langle\Omega,\mathscr{G},m\rangle$ is called *S-integrable* if:

(i) ${}^\circ\!\int |f|\,dm<\infty$.
(ii) If $A \in \mathscr{G}$ and $m(A)\approx 0$, then $\int_A |f|\,dm\approx 0$.
(iii) If $A \in \mathscr{G}$ and $f(x)\approx 0$ for all $x \in A$, then $\int_A |f|\,dm\approx 0$.

Clearly, (iii) is redundant if $\langle\Omega,\mathscr{G},m\rangle$ is a probability space. Anderson further defined $SL^p(\Omega,\mathscr{G},m)$ to be the set of all internal $\mathscr{G}$-measurable functions such that $|f|^p$ is S-integrable, $p>0$.

LEMMA 12. *Let $\langle\Omega,\mathscr{G},P\rangle$ be a hyperfinite probability space. Let f: $\Omega \to {}^*\mathbf{R}$ be a $\mathscr{G}$-measurable internal function such that ${}^\circ\!\int |f|^2\,dP<\infty$. Then f is S-integrable.*

PROOF. It is enough to prove that f satisfies (i) and (ii) above. By Hölder's inequality we have for $A \in \mathscr{G}$:

$$0 \leqq \int_A |f|\,dP = \int 1_A \cdot |f|\,dP \leqq \left(\int 1_A^2\,dP\right)^{\frac{1}{2}}\left(\int f^2\,dP\right)^{\frac{1}{2}} = P(A)^{\frac{1}{2}}\left(\int f^2\,dP\right)^{\frac{1}{2}}.$$

Taking $A=\Omega$ we have (i), and letting $P(A)\approx 0$ we get (ii). This proves the lemma.

PROPOSITION 13. *Let $M\colon T\times\Omega \to {}^*\mathbf{R}$ be a λ^2-martingale adapted to the internal basis $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$, and assume that the set*

$$\{\omega \in \Omega : \exists t \in T (t \text{ finite and } {}^\circ\Delta M(t,\omega) \neq 0)\}$$

has Loeb-measure zero. Then there exist an internal probability measure $\tilde{P}$ on $\mathscr{G}_{t_\infty}$ and a λ^2-martingale $\tilde{M}$ adapted to $\langle \Omega, \{\mathscr{G}_t\}, \tilde{P}\rangle$ such that:

(i) *$\tilde{M}$ has infinitesimal increments.*
(ii) *For all $B \in \mathscr{G}_{t_\infty}$, $P(B) \approx \tilde{P}(B)$.*
(iii) *There is a subset Ω' of Ω of Loeb-measure one, such that for all $\omega \in \Omega'$ and all $t \in T$, $\tilde{M}(t,\omega) = M(t,\omega)$.*

PROOF. Let $a \in \mathbf{R}^+$ and t a finite element of T. We shall construct a process $\bar{M}$ such that $|\Delta\bar{M}(s,\omega)| \leqq a$ for all $\omega \in \Omega$ and all $s<t$, and a probability measure $\bar{P}$ on $\mathscr{G}_{t_\infty}$ such that $\bar{M}$ is a martingale with respect to $\langle \Omega, \{\mathscr{G}_t\}, \bar{P}\rangle$. For all $s \in T$, we shall have $\int [M](s)\,dP \geqq \int [\bar{M}](s)\,d\bar{P}$. Finally, replacing $\tilde{M}$ and $\tilde{P}$ by $\bar{M}$ and $\bar{P}$ respectively, (ii) and (iii) hold.

Let us first show that if we can construct such $\bar{M}$ and $\bar{P}$ for all a and t, then we can prove the proposition: Consider the set of all $n \in {}^*\mathbf{N}$ such that there exist $\bar{M}^n$ and $\bar{P}^n$ with the following properties:

> $\bar{M}^n\colon T\times\Omega \to {}^*\mathbf{R}$ is a hyperfinite process, and $\bar{P}^n$ is an internal probability measure on $\mathscr{G}_{t_\infty}$ such that $\bar{M}^n$ is a hyperfinite martingale adapted to $\langle \Omega, \{\mathscr{G}_t\}, \bar{P}^n\rangle$. There exists a set $A_n \in \mathscr{G}_{t_\infty}$, $P(A_n) < 1/n$, such that for all $t<n$ and all $\omega \notin A_n$, $M(t,\omega) = \bar{M}^n(t,\omega)$. For all $B \in \mathscr{G}_{t_\infty}$, we have $|P(B) - \bar{P}^n(B)| < 1/n$, and for all $t<n$,
>
> $$\int [M](t)\,dP \geqq \int [\bar{M}^n](t)\,d\bar{P}^n.$$
>
> Finally, for all $\omega \in \Omega$ and all $t<n$, $|\Delta\bar{M}^n(t,\omega)| \leqq 1/n$.

This is an internal set, and putting $a = 1/n$ and t equal to the least element in T larger than n, we see that it contains all finite n. Thus it contains an infinite $\gamma \in {}^*\mathbf{N}$, and taking $\tilde{M} = \bar{M}^\gamma$, $\tilde{P} = \bar{P}^\gamma$ and $\Omega' = \Omega \setminus A_\gamma$, we get the proposition.

Let a and t be given; we shall construct $\bar{M}$ and $\bar{P}$. The proof is long and tedious, and we divide it into several steps:

(1). *Definition of* $\bar{M}$. Let A be the subset of Ω consisting of all ω such that there exists an $s<t$ with $|\Delta M(s,\omega)|>a$. The set A is internal and $\mathscr{G}_t$-measurable. For $\omega \in A$, let s_ω be the first element s in T such that $|\Delta M(s,\omega)|>a$.

If $\omega \in \Omega \setminus A$, let $\bar{M}(s,\omega)=M(s,\omega)$ for all $s \in T$.

If $\omega \in A$, let $M(s,\omega) = M(s,\omega)$ for $s \leqq s_\omega$,

let $\Delta\bar{M}(s_\omega,\omega) = a\dfrac{\Delta M(s_\omega,\omega)}{|\Delta M(s_\omega,\omega)|}$ and

let $\Delta M(s,\omega) = 0$ for $s>s_\omega$.

By hypothesis, A has Loeb-measure zero and $\bar{M}$ thus satisfies (iii).

(2). *Definition of* $\bar{P}$. We must change P into $\bar{P}$ such that $\bar{M}$ becomes a martingale with respect to $\langle \Omega, \{\mathscr{G}_t\}, \bar{P}\rangle$. This is done by defining a measure P^{t_i} for all $t_i \in T$, $t_i \leqq t$, by induction. The measure P^t obtained by this procedure will be our measure $\bar{P}$.

Let $P^0=P$. Assume that P^{t_i} is defined, we shall construct $P^{t_{i+1}}$:

For each $\omega \in \Omega$, let $[\omega]_{t_i}=\bigcap\{B \in \mathscr{G}_{t_i} : \omega \in B\}$. Given such an equivalence class we define $\varepsilon_{[\omega]_{t_i}} \in {}^*\mathbf{R}$ by:

$$a\cdot P^{t_i}([\omega]_{t_i})\cdot \varepsilon_{[\omega]_{t_i}} = \sum \{(|\Delta M(t_i,\tilde{\omega})|-a)P^{t_i}\{\tilde{\omega}\} \;:\; \tilde{\omega} \in [\omega]_{t_i} \wedge s_{\tilde{\omega}}=t_i\}\,.$$

If $\tilde{\omega} \in [\omega]_{t_i}$ and $s_{\tilde{\omega}} \neq t_i$, let $P^{t_{i+1}}(\{\tilde{\omega}\}) = P^{t_i}\{\tilde{\omega}\}/1+\varepsilon_{[\omega]_{t_i}}$.

If $\tilde{\omega} \in [\omega]_{t_i}$ and $s_{\tilde{\omega}}=t_i$, let $P^{t_{i+1}}(\{\tilde{\omega}\})=\dfrac{|\Delta M(t_i,\tilde{\omega})|}{a(1+\varepsilon_{[\omega]_{t_i}})}\cdot P^{t_i}\{\tilde{\omega}\}$.

This defines $P^{t_{i+1}}$, and hence $\bar{P}=P^t$ by induction.

(3). $\bar{P}$ *is a probability measure*: Let us calculate $P^{t_{i+1}}([\omega]_{t_i})$. We write:

$$B = \{\tilde{\omega} \in \Omega : \tilde{\omega} \in [\omega]_{t_i} \wedge s_{\tilde{\omega}} \neq t_i\}$$

and

$$C = \{\tilde{\omega} \in \Omega : \tilde{\omega} \in [\omega]_{t_i} \wedge s_{\tilde{\omega}}=t_i\}\,.$$

We have:

$$\begin{aligned} P^{t_{i+1}}([\omega]_{t_i}) &= \sum_{\tilde{\omega}\in B\cup C} P^{t_{i+1}}(\{\tilde{\omega}\}) = \sum_{\tilde{\omega}\in B} P^{t_{i+1}}(\{\tilde{\omega}\}) + \sum_{\tilde{\omega}\in C} P^{t_{i+1}}(\{\tilde{\omega}\}) \\ &= \sum_{\tilde{\omega}\in B} \frac{P^{t_i}\{\tilde{\omega}\}}{1+\varepsilon_{[\omega]_{t_i}}} + \sum_{\tilde{\omega}\in C} \frac{|\Delta M(t_i,\tilde{\omega})|}{a(1+\varepsilon_{[\omega]_{t_i}})} P^{t_i}\{\tilde{\omega}\} \end{aligned}$$

$$= \frac{1}{1+\varepsilon_{[\omega]_{t_i}}}\left(\sum_{\tilde{\omega}\in B} P^{t_i}\{\tilde{\omega}\} + \frac{1}{a}\sum_{\tilde{\omega}\in C} |\Delta M(t_i,\tilde{\omega})| P^{t_i}\{\tilde{\omega}\}\right)$$

$$= \frac{1}{1+\varepsilon_{[\omega]_{t_i}}}\left(\sum_{\tilde{\omega}\in B\cup C} P^{t_i}\{\tilde{\omega}\} + \frac{1}{a}\sum_{\tilde{\omega}\in C} (|\Delta M(t_i,\tilde{\omega})| - a) P^{t_i}\{\tilde{\omega}\}\right)$$

$$= \frac{1}{1+\varepsilon_{[\omega]_{t_i}}}\left(P^{t_i}([\omega]_{t_i}) + P^{t_i}([\omega]_{t_i})\varepsilon_{[\omega]_{t_i}}\right) = P^{t_i}([\omega]_{t_i}) .$$

Thus $P^{t_{i+1}}([\omega]_{t_i}) = P^{t_i}([\omega]_{t_i})$, and summing over all equivalence classes we see that if P^{t_i} is a probability measure, so is $P^{t_{i+1}}$. By induction $\bar{P} = P^t$ is a probability measure.

(4). *$\bar{M}$ is a martingale with respect to $\langle \Omega, \{\mathscr{G}_t\}, \bar{P}\rangle$.* It is enough to prove

$$\sum_{\tilde{\omega}\in[\omega]_{t_i}} \Delta\bar{M}(t_i,\tilde{\omega}) P^t\{\tilde{\omega}] = 0 \quad \text{for all } \omega\in\Omega \text{ and all } t_i \in T .$$

By definition of $\bar{M}$ and $\bar{P}$ this is obvious if $t_i \geqq t$ or $t_i > s_\omega$, and hence we only have to prove the formula for $t_i < t$, $t_i \leqq s_\omega$. In that case

$$P^{t_i}\{\tilde{\omega}\} = P\{\tilde{\omega}\}/\alpha, \text{ where } \alpha = \prod_{j<i} (1+\varepsilon_{[\omega]_{t_j}}) ,$$

for all $\tilde{\omega} \in [\omega]_{t_i}$.

Let $D = \{[\tilde{\omega}]_{t_{i+1}} : \tilde{\omega} \in [\omega]_{t_i}\}$. Then:

$$\sum_{\tilde{\omega}\in[\omega]_{t_i}} \Delta\bar{M}(t_i,\tilde{\omega}) P^t\{\tilde{\omega}\} = \sum_{[\tilde{\omega}]_{t_{i+1}}\in D} \Delta\bar{M}(t_i,\tilde{\omega}) P^t([\tilde{\omega}]_{t_{i+1}})$$

$$= \sum_{[\tilde{\omega}]_{t_{i+1}}\in D} \Delta\bar{M}(t_i,\tilde{\omega}) P^{t_{i+1}}([\tilde{\omega}]_{t_{i+1}}) ,$$

since it follows from the calculation in (3) that $P^t([\tilde{\omega}]_{t_{i+1}}) = P^{t_{i+1}}([\tilde{\omega}]_{t_{i+1}})$ for $t \geqq t_{i+1}$.

Let now

$$D_1 = \{[\tilde{\omega}]_{t_{i+1}} \in D : s_{\tilde{\omega}} \neq t_i\} \quad \text{and} \quad D_2 = \{[\tilde{\omega}]_{t_{i+1}} \in D : s_{\tilde{\omega}} = t_i\} .$$

We get

$$\sum_{[\tilde{\omega}]_{t_{i+1}}\in D} \Delta\bar{M}(t_i,\tilde{\omega}) P^{t_{i+1}}([\tilde{\omega}]_{t_{i+1}})$$

$$= \sum_{[\tilde{\omega}]_{t_{i+1}}\in D_1} \Delta\bar{M}(t_i,\tilde{\omega}) P^{t_{i+1}}([\tilde{\omega}]_{t_{i+1}}) + \sum_{[\tilde{\omega}]_{t_{i+1}}\in D_2} \Delta\bar{M}(t_i,\tilde{\omega}) P^{t_{i+1}}([\tilde{\omega}]_{t_{i+1}})$$

$$= \sum_{[\tilde{\omega}]_{t_{i+1}}\in D_1} \Delta\bar{M}(t_i,\tilde{\omega}) \frac{P^{t_i}([\tilde{\omega}]_{t_{i+1}})}{1+\varepsilon_{[\omega]_{t_i}}} + \sum_{[\tilde{\omega}]_{t_{i+1}}\in D_2} \Delta\bar{M}(t_i,\tilde{\omega}) \frac{|\Delta M(t_i,\tilde{\omega})|}{a(1+\varepsilon_{[\omega]_{t_i}})} P^{t_i}([\tilde{\omega}]_{t_{i+1}})$$

$$= \sum_{[\tilde{\omega}]_{t_{i+1}}\in D_1} \Delta M(t_i,\tilde{\omega}) \frac{P^{t_i}([\tilde{\omega}]_{t_{i+1}})}{1+\varepsilon_{[\omega]_{t_i}}} +$$

$$+\sum_{[\tilde\omega]_{t_{i+1}}\in D_2} \Delta M(t_i,\tilde\omega)\frac{P^{t_i}[\tilde\omega]_{t_{i+1}}}{1+\varepsilon_{[\omega]_{t_i}}}$$

$$=\frac{1}{\alpha(1+\varepsilon_{[\omega]_{t_i}})}\sum_{[\tilde\omega]_{t_{i+1}}\in D} \Delta M(t_i,\tilde\omega)P([\tilde\omega]_{t_{i+1}})$$

$$=\frac{1}{\alpha(1+\varepsilon_{[\omega]_{t_i}})}\sum_{\tilde\omega\in[\omega]_{t_i}} \Delta M(t_i,\tilde\omega)P\{\tilde\omega\} = 0$$

since M is a martingale. This proves that $\bar M$ is a martingale with respect to $\langle\Omega,\{\mathscr{G}_t\},\bar P\rangle$.

(5). *An estimate*: To prove that $P(B)\approx\bar P(B)$, we shall need the following estimate (where $t=t_n$):

$$0\leqq\sum_{\omega\in\Omega\setminus A}\left(\sum_{i=0}^{n-1}\varepsilon_{[\omega]_{t_i}}\right)P(\{\omega\}) = \sum_{i=0}^{n-1}\sum_{\omega\in\Omega\setminus A}\left[\prod_{j<i}(1+\varepsilon_{[\omega]_{t_j}})\right]\varepsilon_{[\omega]_{t_i}}P^{t_i}\{\omega\}$$

$$\leqq\sum_{i=0}^{n-1}\sum_{[\omega]_{t_i}\subset\Omega}\left[\prod_{j<i}(1+\varepsilon_{[\omega]_{t_j}})\right]\varepsilon_{[\omega]_{t_i}}\cdot P^{t_i}([\omega]_{t_i})$$

$$=\sum_{i=0}^{n-1}\sum_{[\omega]_{t_i}\subset\Omega}\left[\prod_{j<i}(1+\varepsilon_{[\omega]_{t_j}})\right]\left(\frac{1}{a}\sum\{(|\Delta M(t_i,\tilde\omega)|-a)\cdot P^{t_i}\{\tilde\omega\}\ :\ \tilde\omega\in[\omega]_{t_i}\wedge s_{\tilde\omega}=t_i\}\right)$$

$$=\frac{1}{a}\sum_{i=0}^{n-1}\sum_{[\omega]_{t_i}\subset\Omega}\{(|\Delta M(t_i,\tilde\omega)|-a)\cdot P\{\tilde\omega\}\ :\ \tilde\omega\in[\omega]_{t_i}\wedge s_{\tilde\omega}=t_i\}$$

$$=\frac{1}{a}\int_A(|\Delta M(s_\omega,\omega)|-a)\,dP\leqq\frac{1}{a}\int_A|\Delta M(s_\omega,\omega)|\,dP\leqq\frac{2}{a}\int_A\max_{s\leqq t}|M(s,\omega)|\,dP\,.$$

By Doob's inequality we see that $E(\max_{s\leqq t}|M(s,\omega)|^2)$ is finite, and it follows by Lemma 12 that $\max_{s\leqq t}|M(s,\omega)|$ is S-integrable. Since $P(A)\approx 0$, we get $2a^{-1}\int_A\max_{s\leqq t}|M(s,\omega)|\,dP\approx 0$, and consequently

$$\sum_{\omega\in\Omega\setminus A}\left(\sum_{i=0}^{n-1}\varepsilon_{[\omega]_{t_i}}\right)P(\omega)\approx 0\,.$$

(6). $P(B)\approx\bar P(B)$ *for all* $B\in\mathscr{G}_{t_\infty}$: Let C be the set of all $\omega\in\Omega\setminus A$ such that $\sum_{i=0}^{n-1}\varepsilon_{[\omega]_{t_i}}\leqq\frac{1}{2}$. It follows from (5) that $P(C)\approx 1$.

Let $D\subset C$ be an element of $\mathscr{G}_{t_\infty}$. For each $\omega\in D$, we have $P^t\{\omega\}=P\{\omega\}/\prod_{s<t}(1+\varepsilon_{[\omega]_s})$, and consequently $P^t(D)\leqq P(D)$. But on the other hand

$$P^t(D) = \sum_{\omega\in D} P\{\omega\}\Big/\prod_{s<t}(1+\varepsilon_{[\omega]_s}) \geqq \sum_{\omega\in D} P\{\omega\}/(1+\varepsilon)^n\,,$$

$$\text{where } \varepsilon = (1/n)(\varepsilon_{[\omega]_0}+\ldots+\varepsilon_{[\omega]_{t_{n-1}}})\,.$$

This may be seen as follows:

$$\left(\prod(1+\varepsilon_{[\omega]_s})\right)^{1/n} = e^{(1/n)\sum\log(1+\varepsilon_{[\omega]_s})} \leqq e^{\log(1+\varepsilon)} = 1+\varepsilon$$

since the logarithm is concave. Moreover,

$$(1+\varepsilon)^n = 1+n\varepsilon+\frac{n(n-1)}{1\cdot 2}\varepsilon^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\varepsilon^3+\ldots \leqq \sum_{k=0}^{\infty}(n\varepsilon)^k = \frac{1}{1-n\varepsilon}$$

(since $D\subset C$, $n\varepsilon=\sum_{i=0}^{n-1}\varepsilon_{[\omega]_{t_i}}\leqq\frac{1}{2}$ snd the sum converges). We get

$$P^t(D) \geqq \sum_{\omega\in D} P\{\omega\}(1-n\varepsilon) = P(D)-\sum_{\omega\in D}P\{\omega\}\sum_{i=0}^{n-1}\varepsilon_{[\omega]_{t_i}} \approx P(D)$$

by (5). Hence $P^t(D)\approx P(D)$. This proves the assertion when $B\subset C$. Let now B be an arbitrary element of $\mathscr{G}_{t_\infty}$; then $B=(B\cap C)\cup(B\setminus C)$. By what we have just proved $\bar{P}(C)\approx 1$, and since $\bar{P}$ is probability measure $\bar{P}(B\setminus C)\approx 0\approx P(B\setminus C)$. Since $B\cap C\subset C$, we have already seen that $\bar{P}(B\cap C)\approx P(B\cap C)$. We get $\bar{P}(B)$ $=\bar{P}(B\cap C)+\bar{P}(B\setminus C)\approx P(B\cap C)+P(B\setminus C)=P(B)$.

(7). $\int[M](r)\,dP\geqq\int[\bar{M}](r)\,d\bar{P}$: It is enough to prove

$$\int_{[\omega]_{t_i}} \Delta M(t_i,\tilde{\omega})^2\,dP \geqq \int_{[\omega]_{t_i}} \Delta\bar{M}(t_i,\tilde{\omega})^2\,d\bar{P}$$

for each $t_i<t$ and all equivalence classes $[\omega]_{t_i}$. We define

$$E_1 = \{[\tilde{\omega}]_{t_{i+1}}\subset[\omega]_{t_i} : s_{\tilde{\omega}}\neq t_i\} \qquad E_2 = \{[\tilde{\omega}]_{t_{i+1}}\subset[\omega]_{t_i} : s_{\tilde{\omega}}=t_i\}\,.$$

Then

$$\int_{[\omega]_{t_i}} \Delta M(t_i,\tilde{\omega})^2\,dP = \sum_{[\tilde{\omega}]_{t_{i+1}}\in E_1} \Delta M(t_i,\tilde{\omega})^2 P[\tilde{\omega}]_{t_{i+1}} + \sum_{[\tilde{\omega}]_{t_{i+1}}\in E_2} \Delta M(t_i,\tilde{\omega})^2 P[\tilde{\omega}]_{t_{i+1}}\,.$$

If $s_\omega<t_i$, then $\Delta\bar{M}(t_i,\tilde{\omega})=0$ for all $\tilde{\omega}\in[\omega]_{t_i}$ and the inequality is obvious. If not, we have

$$\sum_{[\tilde{\omega}]_{t_{i+1}}\in E_1} \Delta M(t_i,\tilde{\omega})^2 P[\tilde{\omega}]_{t_{i+1}} = \prod_{j\leqq i}(1+\varepsilon_{[\omega]_{t_j}}) \sum_{[\tilde{\omega}]_{t_{i+1}}\in E_i} \Delta M(t_i,\tilde{\omega})^2 P^{t_{i+1}}[\tilde{\omega}]_{t_{i+1}}$$

$$= \prod_{j\leqq i}(1+\varepsilon_{[\omega]_{t_j}}) \sum_{[\tilde{\omega}]_{t_{i+1}}\in E_1} \Delta\bar{M}(t_i,\tilde{\omega})^2\bar{P}[\tilde{\omega}]_{t_{i+1}} \geqq \sum_{[\tilde{\omega}]_{t_{i+1}}\in E_1} \Delta\bar{M}(t_i,\tilde{\omega})^2\bar{P}[\tilde{\omega}]_{t_{i+1}}$$

since $\Delta\bar{M}(t_i, \tilde{\omega}) = \Delta M(t_i, \tilde{\omega})$ by definition of $\bar{M}$ and $\bar{P}[\tilde{\omega}]_{t_{i+1}} = P^{t_{i+1}}[\tilde{\omega}]_{t_{i+1}}$ by the computation in (3). We also have

$$\sum_{[\tilde{\omega}]_{t_{i+1}} \in E_2} \Delta M(t_i, \tilde{\omega})^2 P[\tilde{\omega}]_{t_{i+1}} = \prod_{j<i} (1+\varepsilon_{[\omega]_{t_j}}) \sum_{[\tilde{\omega}]_{t_{i+1}} \in E_2} \Delta M(t_i, \tilde{\omega})^2 P^{t_i}[\tilde{\omega}]_{t_{i+1}}$$

$$= \prod_{j<i} (1+\varepsilon_{[\omega]_{t_j}}) \sum_{[\tilde{\omega}]_{t_{i+1}} \in E_2} \Delta M(t_i, \tilde{\omega})^2 \frac{a(1+\varepsilon_{[\omega]_{t_i}})}{|\Delta M(t_i, \tilde{\omega})|} P^{t_{i+1}}[\tilde{\omega}]_{t_{i+1}}$$

$$\geqq \sum_{[\tilde{\omega}]_{t_{i+1}} \in E_2} a^2 P^{t_{i+1}}[\tilde{\omega}]_{t_{i+1}} = \sum_{[\tilde{\omega}]_{t_{i+1}} \in E_2} \Delta\bar{M}(t_i, \tilde{\omega})^2 \bar{P}[\tilde{\omega}]_{t_{i+1}} .$$

Combining our results we obtain the inequality. We have now proved all assertions about $\bar{M}$ and $\bar{P}$ made in the beginning of the proof, and we are through.

Notice the use of Proposition 13 in the proof of the main result of this paper:

THEOREM 14. *Let $M: T \times \Omega \to {}^*\mathbf{R}$ be a local λ^2-martingale. Then M is S-continuous if and only if $[M]$ is.*

PROOF. (i) Assume that $[M]$ is S-continuous. Using a localizing sequence of stopping times we may assume that M is a λ^2-martingale. Then M satisfies the hypothesis of Proposition 13, and $\tilde{M}$ of that proposition is S-continuous if and only if M is; and $[\tilde{M}]$ is S-continuous since $[M]$ is. Let τ_n be the internal stopping time

$$\tau_n(\omega) = \min\{t \in T : [\tilde{M}](t,\omega) \geqq n\} .$$

Since $\tilde{M}$ has infinitesimal increments, $[\tilde{M}_{\tau_n}] < n+1$.

If $r \in {}^*\mathbf{R}$, let $\bar{r}$ be the smallest element of T larger than r. For $m, n \in {}^*\mathbf{N}$ and $t \in T$, let

$$A^{(t)}_{m,n} = \left\{\omega \in \Omega : \exists i \leqq [t \cdot n] \left(\sup_{\overline{i/n} \leqq s \leqq \overline{i+1/n}} (\tilde{M}_{\tau_l}(s) - \tilde{M}_{\tau_l}(\overline{i/n})) \right)^4 > 1/m \right\}$$

where $[t \cdot n]$ is the integer part of $t \cdot n$. If A is the set of all ω such that $s \to \tilde{M}_{\tau_l}(s,\omega)$ is not S-continuous, then $A = \bigcup_{k \in \mathbf{N}} \bigcup_{m \in \mathbf{N}} \bigcap_{n \in \mathbf{N}} A^{(k)}_{m,n}$. Since ${}^\circ\tau_l \to \infty$ a.e. as $l \to \infty$, it is enough for us to prove $L(P)(\bigcap_{n \in \mathbf{N}} A^{(k)}_{m,n}) = 0$ for all $k, m, l \in \mathbf{N}$; and to prove this it is enough to prove $P(A^{(k)}_{m,\gamma}) \approx 0$ for $k, m, l \in \mathbf{N}$, $\gamma \in {}^*\mathbf{N} \setminus \mathbf{N}$. We have:

$$0 \leqq P(A^{(k)}_{m,\gamma}) \leqq \sum_{i=0}^{[k\gamma]} P\left\{\omega : \sup_{\overline{i/\gamma} \leqq s \leqq \overline{i+1/\gamma}} (\tilde{M}_{\tau_l}(s) - \tilde{M}_{\tau_l}(\overline{i/\gamma}))^4 > 1/m\right\}$$

$$\leqq m \sum_{i=0}^{[k\gamma]} E\left(\sup_{\overline{i/\gamma} \leqq s \leqq \overline{i+1/\gamma}} (\tilde{M}_{\tau_l}(s) - \tilde{M}_{\tau_l}(\overline{i/\gamma}))^4 \right)$$

$$\leqq md_4 \sum_{i=0}^{[k\gamma]} E(([\tilde{M}_{\tau_l}](\overline{i+1/\gamma}) - [\tilde{M}_{\tau_l}](\overline{i/\gamma}))^2)$$

(By Proposition 5, for suitable $d_4 \in \mathbf{R}$)

$$= E\left(md_4 \sum_{i=0}^{[k\gamma]} ([\tilde{M}_{\tau_l}](\overline{i+1/\gamma}) - [\tilde{M}_{\tau_l}](\overline{i/\gamma}))^2\right).$$

Since $[\tilde{M}_{\tau_l}] < l+1$, we have:

$$md_4 \sum_{i=0}^{[k\gamma]} ([\tilde{M}_{\tau_l}](\overline{i+1/\gamma}) - [\tilde{M}_{\tau_l}](\overline{i/\gamma}))^2$$

$$\leqq md_4(l+1) \sum_{i=0}^{[k\gamma]} ([\tilde{M}_{\tau_l}](\overline{i+1/\gamma}) - [\tilde{M}_{\tau_l}](\overline{i/\gamma}))$$

$$\leqq md(l+1)^2 .$$

There exists a set of Loeb-measure one where $[\tilde{M}_{\tau_l}](\overline{i+1/\gamma}) - [\tilde{M}_{\tau_l}](\overline{i/\gamma}) < \delta$ for an infinitesimal δ, when $\overline{i+1/\gamma} \leqq [k\gamma]$. For ω in this set

$$md_4 \sum_{i=0}^{[k\gamma]} ([\tilde{M}_{\tau_l}](\overline{i+1/\gamma}, \omega) - [\tilde{M}_{\tau_l}](\overline{i/\gamma}, \omega))^2$$

$$\leqq md_4\delta \sum_{i=0}^{[k\gamma]} ([\tilde{M}_{\tau_l}](\overline{i+1/\gamma}, \omega) - [\tilde{M}_{\tau_l}](\overline{i/\gamma}, \omega))$$

$$\leqq m\delta d_4(l+1) \approx 0 .$$

Thus the expression inside the expectation

$$E\left(md_4 \sum_{i=0}^{[k\gamma]} ([\tilde{M}_{\tau_l}](\overline{i+1/\gamma}) - [\tilde{M}_{\tau_l}](\overline{i/\gamma}))^2\right)$$

is finite everywhere and infinitesimal a.e., and hence the expectation is infinitesimal. It follows that $P(A^{(k)}_{m,\gamma}) \approx 0$, and as we have already noticed, this proves one part of the theorem.

(ii) Assume that M is S-continuous. As above we may assume that M is a λ^2-martingale, and since M satisfies the hypothesis of Proposition 13, $\tilde{M}$ is S-continuous. It is enough to prove that $[\tilde{M}]$ is S-continuous.

We define the sets $B^{(k)}_{m,n}$ for $m, n \in {}^*\mathbf{N}$, $k \in \mathbf{N}$ by

$$B^{(k)}_{m,n} = \{\omega \in \Omega : \exists i \leqq [k \cdot n]((([\tilde{M}](\overline{i+1/n}) - [\tilde{M}](\overline{i/n}))^2) > 1/m)\} .$$

As above it is enough to prove $P(B^{(k)}_{m,\gamma}) \approx 0$ for all $m, k \in \mathbf{N}$, $\gamma \in {}^*\mathbf{N} \setminus \mathbf{N}$. For given $\gamma \in {}^*\mathbf{N} \setminus \mathbf{N}$ we define a new martingale $\tilde{M}^{(\gamma)}$ by: For each $\omega \in \Omega$ such that there is an $i < [k\gamma]$ with

$$\sup_{\overline{i/\gamma} \leqq s \leqq \overline{i+1/\gamma}} (|\tilde{M}(s, \omega) - \tilde{M}(\overline{i/\gamma}, \omega)|) \geqq 1 ,$$

Let s_ω be the least s satisfying this formula. We put $\tilde{M}^{(\gamma)}(s,\omega)=\tilde{M}(s,\omega)$ for $s \leqq s_\omega$, but let $\Delta\tilde{M}^{(\gamma)}(s,\omega)=0$ for $s \geqq s_\omega$. Then $\tilde{M}^{(\gamma)}$ is obviously a martingale, and since $\tilde{M}$ and $\tilde{M}^{(\gamma)}$ have the same paths outside a set of infinitesimal measure, it is enough to prove $P(C^{(k)}_{m,\gamma})\approx 0$, where $C^{(k)}_{m,\gamma}$ is the set obtained from $\tilde{M}^{(\gamma)}$ in the same way that $B^{(k)}_{m,\gamma}$ was obtained from $\tilde{M}$.

Let $S=\{\overline{i/\gamma} : i \in {}^*\mathrm{N}\}$ be a subline of T. Let $N\colon S\times\Omega \to {}^*\mathrm{R}$ be the restriction of $\tilde{M}^{(\gamma)}$ to S. We define a sequence $\{\sigma_n\}$ of internal stopping times by

$$\sigma_n(\omega) \;=\; \min\{s \in S \;:\; [N](s,\omega)\geqq n\}\;.$$

By the definitions of $\tilde{M}^{(\gamma)}$ and σ_n, we have $[N_{\sigma_n}]\leqq n+2$. Since ${}^\circ\sigma_n \to \infty$ as $n \to \infty$, we only have to prove that $P(C^{(k)}_{m,\gamma,n})\approx 0$ for each n, where $C^{(k)}_{m,\gamma,n}$ is the set obtained from $\tilde{M}^{(\gamma)}_{\sigma_n}$ in the same way that $B^{(k)}_{m,\gamma}$ was obtained from $\tilde{M}$. We have

$$
\begin{aligned}
0 \leqq P(C^{(k)}_{m,\gamma,n}) &\leqq \sum_{i=0}^{[k\gamma]} P\{\omega \;:\; ([\tilde{M}^{(\gamma)}_{\sigma_n}](\overline{i+1/\gamma})-[\tilde{M}^{(\gamma)}_{\sigma_n}](\overline{i/\gamma}))^2>1/m\}\\
&\leqq m\sum_{i=0}^{[k\gamma]} E(([\tilde{M}^{(\gamma)}_{\sigma_n}](\overline{i+1/\gamma})-[\tilde{M}^{(\gamma)}_{\sigma_n}](\overline{i/\gamma}))^2)\\
&\leqq mc_4\sum_{i=0}^{[k\gamma]} E\Big(\sup_{\overline{i/\gamma}\leqq s\leqq \overline{i+1/\gamma}} (\tilde{M}^{(\gamma)}_{\sigma_n}(s)-\tilde{M}^{(\gamma)}_{\sigma_n}(\overline{i/\gamma}))^4\Big) \qquad \text{(by Proposition 5)}\\
&\leqq mc_4(4/3)^4\sum_{i=0}^{[k\gamma]} E((\tilde{M}^{(\gamma)}_{\sigma_n}(\overline{i+1/\gamma})-\tilde{M}^{(\gamma)}_{\sigma_n}(\overline{i/\gamma}))^4) \qquad \text{(by Doob's inequality)}\\
&= mc_4(4/3)^4\sum_{i=0}^{[k\gamma]} E((N_{\sigma_n}(\overline{i+1/\gamma})-N_{\sigma_n}(\overline{i/\gamma}))^4)\\
&= \mathrm{E}\Big(mc_4(4/3)^4\sum_{i=0}^{[k\gamma]} (N_{\sigma_n}(\overline{i+1/\gamma})-N_{\sigma_n}(\overline{i/\gamma}))^4\Big)\,.
\end{aligned}
$$

Now $(N_{\sigma_n}(\overline{i+1/\gamma})-N_{\sigma_n}(\overline{i/\gamma}))^2\leqq[N_{\sigma_n}]\leqq n+2$, and thus the expression inside the last expectation is always less than:

$$mc_4(4/3)^4(n+2)\sum_{i=0}^{[k\gamma]} (N_{\sigma_n}(\overline{i+1/\gamma})-N_{\sigma_n}(\overline{i/\gamma}))^2 \;\leqq\; mc_4(4/3)^4(n+2)^2$$

which is finite. On the set of measure one where $\tilde{M}^{(\gamma)}_{\sigma_n}$ is S-continuous each $(N_{\sigma_n}(\overline{i+1/\gamma})-N_{\sigma_n}(\overline{i/\gamma}))^2$ is infinitesimal, and repeating the argument we see that the expression inside the last expectation is infinitesimal a.e.

It follows that the expectation is infinitesimal, and so is $P(C^{(k)}_{m,\gamma,n})$. Hence $[\tilde{M}]$ is S-continuous and so is $[M]$. This proves the theorem.

We end this section with an example:

EXAMPLE 15. Let χ be Anderson's process from Example 1, and let us see how much information we may obtain about χ using the theory we have developed so far: Obviously χ is a martingale and $[\chi](t)=t$. By Lemma 3 χ is a λ^2-martingale, and from Proposition 7 we see that χ is finite a.e. Since $[\chi]$ obviously is S-continuous it follows from Theorem 14 that so is χ.

5. The stochastic integral.

DEFINITION 16. Let X, Y: $T \times \Omega \to {}^*\mathrm{R}$ be two hyperfinite processes. By the *stochastic integral* of X with respect to Y, we shall mean the process $\int X\,dY$ defined by

$$(t,\omega) \to \sum_0^t X(s,\omega)\Delta Y(s,\omega)\,.$$

We shall write $\int_0^t X(s,\omega)\,dY(s,\omega)$ or $\int_0^t X\,dY$ for the value of the process $\int X\,dY$ at time t. Obviously, $[\int X\,dY](t)) = \sum_0^t X^2(s)\Delta Y(s)^2$.

We shall mainly be concerned with stochastic integrals of the form $\int X\,dM$ where M is a hyperfinite martingale with respect to an internal basis $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$, and X is a process adapted to the same basis. It follows from the martingale property that $\int X\,dM$ then is a martingale adapted to $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$.

The following simple identity will be useful in [12] and [13]. For $X=\chi$ (the process of Example 1) it was proved by Anderson in [1]; the proof in the general case is similar to his:

PROPOSITION 17. *Let X be a hyperfinite process. Then*

$$[X](t) = X(t)^2 - X(0)^2 - 2\int_0^t X\,dX\,.$$

PROOF. We have

$$\begin{aligned} X_{t_k}^2 - X_0^2 - 2\int_0^{t_k} X\,dX &= \Big(X_0 + \sum_{i=0}^{k-1} \Delta X_{t_i}\Big)^2 - X_0^2 - 2\sum_{i=0}^{k-1} X_{t_i}\Delta X_{t_i} \\ &= X_0^2 + 2X_0 \sum_{i=0}^{k-1} \Delta X_{t_i} + \Big(\sum_{i=0}^{k-1} \Delta X_{t_i}\Big)^2 - X_0^2 - \\ &\quad - 2X_0 \sum_{i=0}^{k-1} \Delta X_{t_i} - 2\sum_{i=0}^{k-1}\Big(\sum_{j=0}^{i-1} \Delta X_{t_j}\Big)\Delta X_{t_i} \\ &= \Big(\sum_{i=0}^{k-1} \Delta X_{t_i}\Big)^2 - 2\sum_{i=0}^{k-1}\sum_{j=0}^{i-1} \Delta X_{t_i}\Delta X_{t_j}\,. \end{aligned}$$

Now we have

$$2\sum_{i=0}^{k-1}\sum_{j=0}^{i-1} \Delta X_{t_i}\Delta X_{t_j} = \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \Delta X_{t_i}\Delta X_{t_j} - \sum_{i=0}^{k-1} (\Delta X_{t_i})^2 ,$$

and

$$\left(\sum_{i=0}^{k-1} \Delta X_{t_i}\right)^2 = \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \Delta X_{t_i}\Delta X_{t_j} .$$

Thus

$$X_{t_k}^2 - X_0^2 - 2\int_0^{t_k} X\,dX = \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \Delta X_{t_i}\Delta X_{t_j} - \left(\sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \Delta X_{t_i}\Delta X_{t_j} - \sum_{i=0}^{k-1} (\Delta X_{t_i})^2\right) = [X](t_k) ,$$

which proves the theorem.

If M is a λ^2-martingale, we define $\lambda^2(M)$ to be the set of all those internal processes X adapted to the internal basis of M such that $°E((\int_0^t X\,dM)^2) < \infty$ for all finite $t \in T$. Since $\int X\,dM$ is a hyperfinite martingale with quadratic variation $\sum X^2 \Delta M^2$, this is by Lemma 3 equivalent to $°E(\sum_0^t X^2 \Delta M^2) < \infty$ for all finite $t \in T$.

If M is a local λ^2-martingale, we define $\lambda(M)$ to be the set of all X adapted to the basis of M such that there exists a localizing sequence $\{\tau_n\}_{n \in \mathbf{N}}$ for M with $X \in \lambda^2(M_{\tau_n})$ for each $n \in \mathbf{N}$.

The classes $\lambda^2(M)$ and $\lambda(M)$ are perhaps too large to be really useful, and we now construct two smaller classes $SL^2(M)$ and $SL(M)$ which consist — in some sense — of the square S-integrable elements of $\lambda^2(M)$ and $\lambda(M)$:

For each $t \in T$, let $T_t = T \cap {}^*[0, t\rangle$ and let $\mathscr{T}_t$ be the internal power-set of T_t. If M is a λ^2-martingale with respect to $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$, we define an internal measure ν_{M_t} on the *-measurable space $\langle T_t \times \Omega, \mathscr{T}_t \times \mathscr{G}_{t_\infty}\rangle$ by:

$$\nu_{M_t}\{\langle s,\omega\rangle\} = P\{\omega\}(\Delta M(s,\omega))^2 \quad \text{for } \omega \in \Omega,\ s \in T_t .$$

If t is finite:

$$\nu_{M_t}(T_t \times \Omega) = \sum_{\omega\in\Omega}\sum_{0}^{t} \Delta M(s,\omega)^2 P\{\omega\} = E([M](t))$$

which is finite.

Definition 18. Let $M\colon T\times\Omega \to {}^*\mathbf{R}$ be a λ^2-martingale adapted to $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$ and let X be a process adapted to the same basis. Then $X \in SL^2(M)$ if and only if $X{\upharpoonright}T_t \times \Omega \in SL^2(T_t \times \Omega, \mathscr{T}_t \times \mathscr{G}_{t_\infty}, \nu_{M_t})$ for all finite $t \in T$.

If M is a local λ^2-martingale, $X \in SL(M)$ if and only if there is a localizing sequence $\{\tau_n\}$ for M such that $X \in SL^2(M_{\tau_n})$ for all $n \in \mathbb{N}$.

Since the process X in Definition 18 is adapted, each $X \upharpoonright T_t \times \Omega$ is measurable with respect to a*-algebra $\mathscr{P}_t$ much smaller than $\mathscr{T}_t \times \mathscr{G}_{t_\infty}$. The question naturally arises whether $X \upharpoonright T_t \times \Omega \in SL^2(T_t \times \Omega, \mathscr{T}_t \times \mathscr{G}_{t_\infty}, \nu_{M_t})$ if and only if $X \upharpoonright T_t \times \Omega \in SL^2(T_t \times \Omega, \mathscr{P}_t, \nu_{M_t} \upharpoonright \mathscr{P}_t)$. The answer is not quite obvious since there are more sets of infinitesimal measure in $\mathscr{T}_t \times \mathscr{G}_{t_\infty}$ than in $\mathscr{P}_t$, but it is not difficult to prove:

LEMMA 19. *Let $\langle \Omega, \mathscr{G}, \mu \rangle$ be a hyperfinite measure-space, and let $\mathscr{B}$ be a *-subalgebra of $\mathscr{G}$. Let $f\colon \Omega \to {}^*\mathbb{R}$ be $\mathscr{B}$-measurable. Then $f \in SL^1(\Omega, \mathscr{G}, \mu)$ if and only if $f \in SL^1(\Omega, \mathscr{B}, \mu)$.*

The next proposition shows us that the class $SL(M)$ is large enough for most purposes.

PROPOSITION 20. *Let M be a local λ^2-martingale and X a process adapted to the internal basis of M. Assume that on a set of Loebmeasure one $\max_{s \leqq t} |X(s)|$ is finite for all finite $t \in T$. Then $X \in SL(M)$. In particular if X is a local λ^2-martingale, then $X \in SL(M)$.*

PROOF. Let $\{\tau_n\}$ be a localizing sequence for M, and define another sequence $\{\sigma_n\}$ of internal stopping times by

$$\sigma_n(\omega) = \min\{s \in T : |X(s,\omega)| \geqq n\} .$$

By the hypothesis, ${}^\circ\tau_n(\omega) \to \infty$ a.e. Let $\tau'_n = \sigma_n \wedge \tau_n$; then $\{\tau'_n\}$ is a localizing sequence for M and $|X(s,\omega)| < n$ for $s < \tau'_n(\omega)$. Consequently $X \in SL^2(M_{\tau'_n})$, and $X \in SL(M)$.

If X is a local λ^2-martingale, then X satisfies the hypothesis of Proposition 20 by Proposition 7.

Before we turn to the next theorem let us recall some results about S-integration. Let $\langle \Omega, \mathscr{G}, \mu \rangle$ be a hyperfinite measure space with $\mu(\Omega)$ finite. Anderson [1] showed that an internal $\mathscr{G}$-measurable function $f\colon \Omega \to {}^*\mathbb{R}$ is S-integrable if and only if there exists a sequence $\{f_n\}$ of $\mathscr{G}$-measurable finite functions such that $\lim {}^\circ (\int |f - f_n| \, d\mu) = 0$.

If f is *-non-negative it follows from the proof that we may take $f_n = f \wedge n$. Anderson also proved that if f is $\mathscr{G}$-measurable and *-non-negative, then ${}^\circ\!\int f \, d\mu \geqq \int {}^\circ f \, dL(\mu)$ with equality if and only if f is S-integrable.

We now use Theorem 14 to prove:

THEOREM 21. *Let M be an S-continuous local λ^2-martingale, and let $X \in SL(M)$. Then $\int X \, dM$ is S-continuous.*

PROOF. It is clearly enough to prove the theorem when M is a λ^2-martingale and $X \in SL^2(M)$.

For each $n \in \mathbf{N}$ define an adapted process X_n: $T \times \Omega \to {}^*\mathbf{R}$ by: $X_n(t,\omega) = X(t,\omega)$ if $|X(t,\omega)| \leqq \sqrt{n}$; $X_n(t,\omega) = \sqrt{n}$ if $X(t,\omega) > \sqrt{n}$; and $X_n(t,\omega) = -\sqrt{n}$ if $X(t,\omega) < -\sqrt{n}$.

Since M is S-continuous, it follows from Theorem 13 that $[M]$ is S-continuous. Now

$$\left[\int X_n\,dM\right](t) - \left[\int X_n\,dM\right](s) = \sum_s^t X_n^2 \Delta M^2 \leqq n \sum_s^t \Delta M^2$$
$$= n([M](t) - [M](s)) \quad \text{for } s,t \in T,\ s<t\,,$$

and hence we see that $[\int X_n\,dM]$ is S-continuous for each $n \in \mathbf{N}$.

By the last of the results of Anderson mentioned above:

$$\begin{aligned}
0 &\leqq E\left(\sup_{s \leqq t} {}^\circ\left(\left[\int_0^s X\,dM\right] - \left[\int_0^s X_n\,dM\right]\right)\right)\\
&\leqq {}^\circ E\left(\max_{s \leqq t}\left(\left[\int_0^s X\,dM\right] - \left[\int_0^s X_n\,dM\right]\right)\right)\\
&= {}^\circ E\left(\left[\int_0^t X\,dM\right] - \left[\int_0^t X_n\,dM\right]\right) = {}^\circ E\left(\sum_0^t X^2 \Delta M^2 - \sum_0^t X_n^2 \Delta M^2\right)\\
&= {}^\circ E\left(\sum_0^t (X^2 - X_n^2)\Delta M^2\right) = {}^\circ\int_{T_t \times \Omega} (X^2 - X_n^2)\,d\nu_{M_t}\,.
\end{aligned}$$

By definition of $X \in SL^2(M)$ and the first of the results of Anderson above, ${}^\circ\int (X^2 - X_n^2)\,d\nu_{M_t} \to 0$ as $n \to \infty$. A well-known result of measure-theory (see e.g. Bauer [3], Satz 15.5), tells us we may find a subsequence $\{\sup_{s \leqq t} {}^\circ([\int_0^s X\,dM] - [\int_0^s X_{n_k}\,dM])\}_{k \in \mathbf{N}}$ which converges to zero a.e.

Using the ε-δ-formulation of S-continuity (recall the comment following Definition 11) one may prove exactly as in the real-valued case that the S-uniform limit of a sequence of S-continuous functions is S-continuous. It thus follows that $[\int X\,dM]$ is S-continuous. Applying Theorem 14 again we see that $\int X\,dM$ is also S-continuous, and the theorem is proved.

If we replace $X \in SL(M)$ by $X \in \lambda(M)$ in Theorem 21, the resulting statement will be false as is easily seen.

Anderson [1] proved the theorem above for $M = \chi$ (the process of Example 1) when X was in a certain subclass of $SL^2(\chi)$ (the class of "2-liftings"). Keisler [10] used Bernstein's inequality to prove the theorem when $M = \chi$ and X is a finite adapted process.

6. The transformationformula and Itô's formula.

In this section we prove a nonstandard version of what may be called the fundamental theorem of stochastic analysis.

If $r \in T$, we write r^- and r^+ for respectively the predecessor and the successor of r in T.

THEOREM 22. (Nonstandard version of the Transformationformula). *Let* φ: $^*\mathrm{R} \to {}^*\mathrm{R}$ *be an internal function and assume that* $\varphi, \varphi', \varphi''$ *all exist and are S-continuous and finite for finite arguments. Let* M: $T \times \Omega \to {}^*\mathrm{R}$ *be a local* λ^2-*martingale. Then for all finite* $t \in T$:

$$^\circ\varphi(M_t) - {}^\circ\varphi(M_0) = {}^\circ\!\!\int_0^t \varphi'(M)\,dM + \tfrac{1}{2}\,{}^\circ\!\!\int_0^t \varphi''(M)\,d[M]$$

$$+\sum_0^t {}^\circ(\varphi(M_{s^+}) - \varphi(M_s) - \varphi'(M_s)(M_{s^+} - M_s) + \tfrac{1}{2}\varphi''(M_s)(M_{s^+} - M_s)^2) \quad \text{a.e.},$$

where the last sum is absolutely convergent.

PROOF. We begin by enumerating the points $s \in T$ such that $^\circ\Delta M(s^-, \omega) \neq 0$. By Lemma 10 this is possible for almost all $\omega \in \Omega$, since for each finite $t \in T$ and each positive $\varepsilon \in \mathrm{R}$ there are only a finite number of $s \in T$, $s < t$, such that $\Delta M(s^-, \omega) > \varepsilon$. To make the definition precise, it is convenient to define a bijection f: $\mathrm{N}_+ \to \mathrm{N}_+ \times \mathrm{N}_+$ and let $(f(n))_1$ denote the first component of $f(n)$.

Define a sequence $\{\tau_n\}_{n \in \mathrm{N}}$ of internal stopping times by

$$\tau_n(\omega) = \min\left\{s \in T:\ |\Delta M(s^-, \omega)| > \frac{1}{(f(n))_1} \wedge s \neq \tau_k(\omega) \text{ for } k < n\right\}.$$

Let Ω' be the set of Loeb-measure one where $M(t)$ is finite for each finite $t \in T$ and where the sequence $\{\tau_n(\omega)\}$ enumerates all finite $s \in T$ such that $^\circ\Delta M(s^-, \omega) \neq 0$. Let $\{\tau_n\}_{n \in {}^*\mathrm{N}}$ be an internal extension of $\{\tau_n\}_{n \in \mathrm{N}}$ such that $\tau_k \neq \tau_m$ when $k \neq m$. For each finite $t \in T$ and each $\nu \in {}^*\mathrm{N} \setminus \mathrm{N}$, let

$$A_{t,\nu} = \{s \in T_t:\ s^+ \notin \{\tau_n(\omega): n \leqq \nu\}\}.$$

For all $\omega \in \Omega'$ we get from Taylor's formula:

$$\varphi(M_t) - \varphi(M_0) = \sum_0^t (\varphi(M_{s^+}) - \varphi(M_s))$$

$$= \sum_{n \leqq \nu} (\varphi(M_{\tau_n \wedge t}) - \varphi(M_{\tau_n^- \wedge t})) + \sum_{s \in A_{t,\nu}} (\varphi(M_{s^+}) - \varphi(M_s))$$

$$= \sum_{n \leq \nu} (\varphi(M_{\tau_n \wedge t}) - \varphi(M_{\tau_n^- \wedge t})) + \sum_{s \in A_{t,\nu}} \{\varphi'(M_s)(M_{s^+} - M_s)$$
$$+ \tfrac{1}{2}\varphi''(M_s)(M_{s^+} - M_s)^2 + \tfrac{1}{2}(\varphi''(\xi_s) - \varphi''(M_s))(M_{s^+} - M_s)^2\} ,$$

where ξ_s lies between M_{s^+} and M_s.

By the S-continuity of φ'' the sum over the very last term is infinitesimal for $\omega \in \Omega'$ and thus

$$\varphi(M_t) - \varphi(M_0) \approx \sum_{n \leq \nu} (\varphi(M_{\tau_n \wedge t}) - \varphi(M_{\tau_n^- \wedge t}))$$
$$+ \sum_{s \in A_{t,\nu}} \{\varphi'(M_s)(M_{s^+} - M_s) + \tfrac{1}{2}\varphi''(M_s)(M_{s^+} - M_s)^2\} \quad \text{for } \omega \in \Omega' .$$

Adding and subtracting

$$\sum_{n \leq \nu} \{\varphi'(M_{\tau_n^- \wedge t})(M_{\tau_n \wedge t} - M_{\tau_n^- \wedge t}) + \tfrac{1}{2}\varphi''(M_{\tau_n^- \wedge t})(M_{\tau_n \wedge t} - M_{\tau_n^- \wedge t})^2\}$$

we get:

$$\varphi(M_t) - \varphi(M_0) \approx \int_0^t \varphi'(M)\, dM + \tfrac{1}{2} \int_0^t \varphi''(M)\, d[M]$$
$$+ \sum_{n \leq \nu} \{\varphi(M_{\tau_n \wedge t}) - \varphi(M_{\tau_n^- \wedge t}) - \varphi'(M_{\tau_n^- \wedge t})(M_{\tau_n \wedge t} - M_{\tau_n^- \wedge t})$$
$$- \tfrac{1}{2}\varphi''(M_{\tau_n^- \wedge t})(M_{\tau_n \wedge t} - M_{\tau_n^- \wedge t})^2\} \quad \text{on } \Omega' .$$

Since this is true for all $\nu \in {}^*\mathbf{N} \setminus \mathbf{N}$, the series

$$\sum_0^\infty {}^\circ\{\varphi(M_{\tau_n \wedge t}) - \varphi(M_{\tau_n^- \wedge t}) - \varphi'(M_{\tau_n^- \wedge t})(M_{\tau_n \wedge t} - M_{\tau_n^- \wedge t}) - \tfrac{1}{2}\varphi''(M_{\tau_n^- \wedge t})(M_{\tau_n \wedge t} - M_{\tau_n^- \wedge t})^2\}$$

converges to

$${}^\circ\varphi(M_t) - {}^\circ\varphi(M_0) - {}^\circ\int_0^t \varphi'(M)\, dM - \tfrac{1}{2}{}^\circ\int_0^t \varphi''(M)\, d[M] \quad \text{on } \Omega' .$$

The convergence is absolute since the order of the terms in the sequence $\{\tau_n\}$ does not matter. But the sum above is equal on Ω' to the sum in the theorem, and this completes the proof.

In the S-continuous case Theorem 22 becomes

COROLLARY 23. (Nonstandard version of Itô's formula). *Let φ: ${}^*\mathbf{R} \to {}^*\mathbf{R}$ be an internal function and assume that $\varphi, \varphi', \varphi''$ all exist, and are S-continuous and finite for finite arguments. Let M: $T \times \Omega \to {}^*\mathbf{R}$ be an S-continuous local λ^2-martingale. Then for all finite $t \in T$:*

$$\varphi(M_t)-\varphi(M_0) \approx \int_0^t \varphi'(M)\,dM+\tfrac{1}{2}\int_0^t \varphi''(M)\,d[M] \text{ a.e.}$$

Applying Corollary 23 to a suitable integral of Anderson's process χ, one may obtain a nonstandard version of Itô's lemma, as Anderson did in [1].

For standard versions of the transformationformula and Itô's formula, see Métivier [18, page 264 and page 49 respectively]. We shall deduce these results from the above versions in [13].

7. Doob–Meyer decomposition.

A process $X\colon T\times\Omega \to {}^*\mathrm{R}$ is said to be of *finite variation* if $|X_0|+\sum_0^t|\Delta X(s)|$ is finite a.e. for all finite $t \in T$. Such a process is easily decomposed into two monotone processes X^+ and X^-, by defining $X^+(t,\omega)=\sum_0^t(\Delta X(s,\omega)\vee 0)$ and $X^-(t,\omega)=\sum_0^t(\Delta X(s,\omega)\wedge 0)$. Integration with respect to monotone processes is easy to deal with, and since we in this paper have developed the theory for integration with respect to martingales, we should now be able to treat integration with respect to processes of the form $X+M$ where X is of finite variation and M is a local λ^2-martingale.

Definition 24. Let $Y\colon T\times\Omega \to {}^*\mathrm{R}$ be a hyperfinite process adapted to the internal basis $\langle\Omega,\{\mathscr{G}_t\},P\rangle$. Y is said to have a *Doob–Meyer decomposition* if there exist a process X of finite variation and a local λ^2-martingale M both adapted to $\langle\Omega,\{\mathscr{G}_t\},P\rangle$ such that $Y=X+M$.

Proposition 25. *Let $Y\colon T\times\Omega \to {}^*\mathrm{R}$ be a hyperfinite process adapted to the internal basis $\langle\Omega, \{\mathscr{G}_t\},P\rangle$ such that $Y(0)$ is finite* a.e. *Assume that there exists a sequence $\{\tau_n\}$ of internal stopping times such that ${}^\circ\tau_n \to \infty$* a.e., *and $E(\sum_0^{t\wedge\tau_n}(\Delta Y(s))^2)$ is finite for all $n \in \mathrm{N}$ and finite $t \in T$. Assume further that $\sum_0^t|E(\Delta Y(s)|\mathscr{G}_s)|$ is finite for finite $t \in T$, $L(P)$*-a.e. *Then Y has a Doob–Meyer decomposition.*

Before we prove the proposition, we make two remarks on the conditions which Y must satisfy:

Remark 26. The conditional expectation $E(\Delta Y(s)\,|\,\mathscr{G}_s)$ is in the hyperfinite case just an average: As in Section 4 let

$$[\omega]_s = \bigcap\,\{B \in \mathscr{G}_s : \omega \in B\}\,.$$

Then we have:

$$E(\Delta Y(s)\,|\,\mathscr{G}_s)(\omega) = \left(\sum_{\tilde\omega\in[\omega]_s} \Delta Y(s,\tilde\omega)P\{\tilde\omega\}\right)\Big/P([\omega]_s)\,.$$

This definition makes some properties of the conditional expectation very easy to prove, e.g. will the inequality

$$E(E(\Delta Y(s)\,|\,\mathscr{G}_s)^2) \leqq E(\Delta Y(s)^2)\,,$$

which we shall need in the proof of the proposition, reduce to the algebraic inequality

$$[(a_1p_1+\ldots+a_np_n)/(p_1+\ldots+p_n)]^2(p_1+\ldots+p_n) \leqq a_1^2p_1+\ldots+a_n^2p_n\,.$$

REMARK 27. The condition $\sum_0^t |E(\Delta Y(s)\,|\,\mathscr{G}_s)|<\infty$ a.e. in the proposition is not much stronger than necessary, and we show that it is satisfied when Y has a Doob–Meyer decomposition $Y=X+M$ where X has reasonable integration properties: Assume $E(X_t^+)<\infty$ and $E(-X_t^-)<\infty$ for all finite $t\in T$. Then

$$E(\Delta Y(s)\,|\,\mathscr{G}_s) = E(\Delta X(s)\,|\,\mathscr{G}_s)+E(\Delta M(s)\,|\,\mathscr{G}_s) = E(\Delta X(s)\,|\,\mathscr{G}_s)$$

and

$$\begin{aligned} E\left(\textstyle\sum |E(\Delta Y(s)\,|\,\mathscr{G}_s)|\right) &= E\left(\textstyle\sum |E(\Delta X(s)\,|\,\mathscr{G}_s)|\right) \leqq E\left(\textstyle\sum E(|\Delta X(s)|\,|\,(\mathscr{G}_s))\right)\\ &= E\left(\textstyle\sum |\Delta X(s)|\right) = E(X_t^+)+E(-X_t^-) < \infty\,. \end{aligned}$$

Thus $\sum_0^t |E(\Delta Y(s)\,|\,\mathscr{G}_s)|<\infty$ a.e. This condition corresponds in the standard case to the condition that Y is a quasimartingale, i.e. that the Doleans-measure of Y is of finite variation (see Métivier [18]).

The condition that $E(\sum_0^{t\wedge\tau_n}\Delta Y(s)^2)$ is finite is just to ensure that the martingale we construct is a local λ^2-martingale. Without this condition we can only prove that M is a martingale.

PROOF OF PROPOSITION 25. Define M by $M(0)=0$ and $\Delta M(t)=\Delta Y(t)-E(\Delta Y(t)\,|\,\mathscr{G}_t)$; M is obviously a martingale. Define

$$X(t) = Y(0)+\sum_0^t E(\Delta Y(s)\,|\,\mathscr{G}_s)\;;$$

by hypothesis X is of finite variation. Obviously $Y=X+M$.

It remains to prove that M is a local λ^2-martingale, and to do this it is enough to prove that $E([M_{\tau_n}](t))=E(\sum_0^{t\wedge\tau_n}\Delta M(s)^2)$ is finite for all $n\in \mathbf{N}$ and all finite $t\in T$.

$$\begin{aligned} E\left(\sum_0^{t\wedge\tau_n}\Delta M(s)^2\right) &= E\left(\sum_0^{t\wedge\tau_n}\Delta Y(s)^2\right)-2E\left(\sum_0^{t\wedge\tau_n}\Delta Y(s)E(\Delta Y(s)\,|\,\mathscr{G}_s)\right)\\ &\quad+E\left(\sum_0^{t\wedge\tau_n}E(\Delta Y(s)\,|\,\mathscr{G}_s)^2\right). \end{aligned}$$

The first term is finite by assumption and the third is finite since we saw in Remark 26 that $E(E(\Delta Y(s) \mid \mathscr{G}_s)^2) \leqq E(\Delta Y(s)^2)$. Using Hölder's inequality on the second term we get

$$E\left(\sum_{0}^{t \wedge \tau_n} |\Delta Y(s) E(\Delta Y(s) \mid \mathscr{G}_s)|\right) \leqq \left(E\left(\sum_{0}^{t \wedge \tau_n} \Delta Y(s)^2\right)\right)^{\frac{1}{2}} \left(E\left(\sum_{0}^{t \wedge \tau_n} \Delta E(\Delta Y(s) \mid \mathscr{G}_s)^2\right)\right)^{\frac{1}{2}}$$

$$\leqq E\left(\sum_{0}^{t \wedge \tau_n} \Delta Y(s)^2\right),$$

which proves that the second term is finite. This proves the proposition.

NOTE ADDED IN PROOF. Most of the results in this paper have been independently rediscovered and extended by D. N. Hoover and E. Perkins: *Nonstandard construction of the stochastic integral and applications to stochastic differential equations* (preprint), 1980.

REFERENCES

1. R. M. Anderson, *A non-standard representation of Brownian motion and Itô integration*, Israel J. Math. 25 (1976), 15–46.
2. R. M. Anderson, *Star-finite Probability Theory*, Ph.D.-thesis, Yale University, 1977.
3. H. Bauer, *Wahrscheinlichkeitstheorie und Grundzüge der Masstheorie* (Sammlung Goschen 1216), 2. erweiterte Auflage, Walter de Gruyter, Berlin, 1974.
4. A. R. Bernstein and P. A. Loeb, *A nonstandard integration theory for unbounded functions*. In A. Hurd and P. Loeb (eds.): *Victoria Symposium on Nonstandard Analysis* (Lecture Notes in Mathematics 369) pp. 40–50, Springer-Verlag, Berlin - Heidelberg - New York, 1974.
5. A. R. Bernstein and F. Wattenberg, *Nonstandard measure theory*. In W. A. J. Luxemburg (ed.): *Applications of Model Theory to Algebra, Analysis and Probability*, pp. 171–185, Holt, Rinehart and Winston, New York - Chicago - London, 1969.
6. J. L. Doob, *Stochastic Processes*, John Wiley and Sons, New York - London, 1953.
7. D. N. Hoover, *Probability Logic*, Ann. Math. Logic 14 (1978), 287–313.
8. A. E. Hurd, *Nonstandard Analysis and Lattice Statistical Mechanics: A Variational Principle*, University of Victoria, 1977.
9. H. J. Keisler, *Hyperfinite model theory*. In R. O. Gandy and J. M. E. Hyland (eds.): *Logic Colloqium* 76, pp. 5–110, North-Holland Publ. Co. New York - Amsterdam, 1977.
10. H. J. Keisler, *An infinitesimal approach to stochastic analysis*, preliminary version, 1978.
11. T. L. Lindström, *Nonstandard theory for stochastic integration*, unpublished.
12. T. L. Lindström, *Hyperfinite stochastic integration* II: *Comparison with the standard theory*, Math. Scand. 46 (1980), 293–314.
13. T. L. Lindström, *Hyperfinite stochastic integration* III: *Hyperfinite representations of standard martingales*, Math. Scand. 46 (1980), 315–331.
14. P. A. Loeb, *A non-standard representation of measurable spaces, L_∞ and L_∞^**. In W. A. J. Luxemburg and A. Robinson (eds.): *Contributions to Non-standard Analysis*, pp. 65–80, North-Holland Publ. Co. New York - Amsterdam, 1972.
15. P. A. Loeb, *A Non-standard Representation of Borel Measures and σ-finite Measures*. In A. Hurd and P. Loeb (eds.): *Victoria Symposium on Non-standard Analysis* (Lecture Notes in Mathematics 369), pp. 144–152, Springer-Verlag, Berlin - Heidelberg - New York, 1974.

16. P. A. Loeb, *Conversion from non-standard to standard measure spaces and applications in probability theory*, Trans. Amer. Math. Soc. 211 (1975), 113–122.
17. P. A. Loeb, *An Introduction to Non-standard Analysis and Hyperfinite Probability Theory*, University of Illinois, 1978.
18. M. Métivier, *Reelle und Vektorwertige Quasimartingale und die Theorie der Stochastischen Integration* (Lecture Notes in Mathematics 607), Springer-Verlag, Berlin - Heidelberg - New York, 1977.
19. P.-A. Meyer, *Martingales and Stochastic Integrals* I (Lecture Notes in Mathematics 284), Springer-Verlag, Berlin - Heidelberg - New York, 1972.
20. P.-A. Meyer, *Un Cours sur les intégrales stochastiques.* In P.-A. Meyer (ed.): *Seminaire de Probabilité* X (Lecture Notes in Mathematics 511), pp. 245–400, Springer-Verlag, Berlin - Heidelberg - New York, 1976.
21. J. Neveu, *Discrete-Parameter Martingales*, North-Holland Publ. Co. New York - Amsterdam, 1975.
22. H. L. Royden, *Real Analysis.* Second Edition, Macmillan, London, 1968.
23. K. D. Stroyan und W. A. J. Luxemburg, *Introduction to the Theory of Infinitesimals*, Academic Press, New York - London, 1976.
24. C. W. Henson, *Unbounded Loeb measures*, Proc. Amer. Math. Soc. 74 (1979), 143–150.

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