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HYPERFINITE STOCHASTIC INTEGRATION II: COMPARISON WITH THE STANDARD THEORY

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Introduction.

This is the second of three papers on the nonstandard theory for stochastic integration. In the first paper we studied the nonstandard theory in its own right, and we shall now compare that theory with the standard one.

In the first section we define the SL^2-martingales, which constitute the class of hyperfinite martingales we shall work with in this paper. This class is a little smaller than the class of λ^2-martingales which we studied in the first paper, and the SL^2-martingales behave more regularly under standard parts. In the second section we use the fact that an SL^2-martingale M has S-right limits to introduce what we call its right standard part ${}^\circ M^+$; and we prove that this is an L^2-martingale. Given a process X which is integrable (standard sense) with respect to ${}^\circ M^+$, we show how to construct a hyperfinite process Y (called a 2-lifting of X) such that ${}^\circ(\int Y\,dM)^+ = \int X\,d{}^\circ M^+$; and we see that standard stochastic integration with respect to ${}^\circ M^+$ may be obtained from the nonstandard theory with respect to M. This is a generalization of Anderson's work in [1], and since the argument is rather similar to his, we have left some of the more technical lemmas to the reader. In the last section we introduce what we have called the "well-behaved" martingales; that is martingales which on a set of measure one have only one noninfinitesimal jump in each monad. These martingales are particularly easy to work with, and we obtain their basic properties and show that any SL^2-martingale M has a restriction M^s to a subline S which is well-behaved. Well-behaved martingales will be useful in the next paper.

To prove that the standard theory for stochastic integration can be obtained from the nonstandard theory, it still remains to show that all L^2-martingales can in a suitable sense be represented as the right standard parts of SL^2-martingales. This is the problem we shall study in the third paper [6].

We shall use the same symbols and terminology as in the first paper [5], and the reader is referred to the introduction of that paper for some remarks on the literature. We only mention that we still work with polysaturated models for nonstandard analysis (see [11]).

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1. SL^2-martingales.

In [5] we studied the class of λ^2-martingales, and we now define an important subclass. The reader should recall the definitions of S-integrability and SL^p-spaces (see [5, section 4]).

DEFINITION 1. A hyperfinite martingale $M: T \times \Omega \to {}^*\mathbf{R}$ adapted to an internal basis $\langle \Omega, \{G_t\}, P\rangle$ will be called an *SL^2-martingale* if $M_t \in SL^2(\Omega, G_t, P)$ for each finite $t \in T$. M is called a local *SL^2-martingale* if there exists an increasing sequence $\{\tau_n\}_{n\in\mathbf{N}}$ of stopping times adapted to $\langle \Omega, \{G_t\}, P\rangle$ such that ${}^\circ\tau_n \to \infty$ a.e. and M_{τ_n} is an SL^2-martingale for each $n \in \mathbf{N}$. The sequence $\{\tau_n\}$ is then called a *localizing sequence* for M.

Due to the S-integrability, the SL^2-martingales behave more regularly under standard-parts than the λ^2-martingales. However, S-integrability is difficult to check and consequently some properties of SL^2-martingales are harder to prove than the corresponding properties of λ^2-martingales. It was, for example, an easy consequence of [5, Lemma 3] that a hyperfinite martingale M is a λ^2-martingale if and only if $E(M_0^2+[M](t))$ is finite for all finite $t \in T$. The corresponding result for SL^2-martingales would be that M is an SL^2-martingale if and only if $M_0^2+[M](t)$ is S-integrable for each finite $t \in T$. But this is not obvious, and the reason is that we must now consider $\int_A (M_0^2 + [M](t))\,dP$ for $A \in \mathcal{G}_t$, $P(A)\approx 0$. Since A need not be in $\mathcal{G}_0$, we cannot use [5, Lemma 3] as above, and we seem to be stuck. Using another characterization of S-integrability, we may, however, prove the assertion:

THEOREM 2. *Let M be a hyperfinite martingale adapted to the internal basis $\langle \Omega, \{\mathcal{G}_t\}, P\rangle$. Then M is an SL^2-martingale if and only if $M_0^2+[M](t) \in SL^1(\Omega, \mathcal{G}_t, P)$ for each finite $t \in T$.*

PROOF. Since in either case M is a λ^2-martingale, it is enough to prove the theorem for such martingales.

We shall use the following characterization due to Anderson [1]: If $f: \Omega \to {}^*\mathbf{R}$ is an internal, *non-negative, $\mathcal{G}_t$-measurable function, then $f \in SL^1(\Omega, \mathcal{G}_t, P)$ if and only if ${}^\circ\int f\,dP = \int {}^\circ f\,dL(P)$.

By [5, Proposition 17]

$$M_0^2+[M](t) = M_t^2 - 2\int_0^t M\,dM$$

and taking *-expectations we get:

$$E(M_0^2+[M](t)) = E(M_t^2) \tag{1}$$

since $\int M\,dM$ is a martingale. Define a sequence $\{\tau_n\}_{n\in\mathbf{N}}$ of internal stopping times by

$$\tau_n(\omega) = \min\{s \in T:\ |M(s,\omega|\geqq n\} .$$

Then $\int M_{\tau_n}\,dM_{\tau_n}$ is a λ^2-martingale for each $n \in \mathbf{N}$, and it follows from [5, Lemma 12] that $\int_0^t M_{\tau_n}\,dM_{\tau_n}$ is S-integrable for each finite $t \in T$. By the result of Anderson quoted above

$$E\left(^\circ\int_0^t M_{\tau_n}\,dM_{\tau_n}\right) = {}^\circ E\left(\int_0^t M_{\tau_n}\,dM_{\tau_n}\right) = 0 .$$

Since

$$^\circ M_{\tau_n}(0)^2 + {}^\circ[M_{\tau_n}](t) = {}^\circ M_{\tau_n}(t)^2 - 2\,{}^\circ\int_0^t M_{\tau_n}\,dM_{\tau_n} \quad \text{a.e.},$$

we get

$$E(^\circ M_{\tau_n}(0)^2 + {}^\circ[M_{\tau_n}](t)) = E(^\circ M_{\tau_n}(t)^2) . \tag{2}$$

For almost all ω there exists an $n \in \mathbf{N}$ such that $\tau_m(\omega)\geqq t$ for $m\geqq n$, and thus $^\circ[M_{\tau_n}](t) \to {}^\circ[M](t)$ and $M_{\tau_n}(t) \to M(t)$ a.e. The sequence $^\circ[M_{\tau_n}](t)$ is increasing and bounded by $^\circ[M](t)$ which is integrable since $E(^\circ[M](t)) \leqq {}^\circ E([M](t)) < \infty$. We also have $^\circ(M_{\tau_n}(t)^2) \leqq {}^\circ(\max_{s\leqq t} M_s^2)$ and by Doob's inequality

$$E\left(^\circ\max_{s\leqq t} M_s^2\right) \leqq {}^\circ E\left(\max_{s\leqq t} M_s^2\right) \leqq 4\,{}^\circ E(M_t^2) < \infty$$

and thus $^\circ(\max_{s\leqq t} M_s^2)$ is integrable. Applying Lebesgue's Convergence Theorem to both sides of (2), we obtain

$$E(^\circ M(0)^2 + {}^\circ[M](t)) = E(^\circ M_t^2) . \tag{3}$$

Combining (1) and (3) we see that

$$^\circ E(M_0^2+[M](t)) = E(^\circ(M_0^2+[M](t))) \quad \text{if and only if} \quad {}^\circ E(M_t^2) = E(^\circ M_t^2) .$$

Anderson's characterization now tells us that $M_0^2+[M](t)$ is S-integrable if M_t^2 is, and the theorem is proved.

We just mention one other result of the same type which was proved in [4]: If M is an SL^2-martingale, then $\max_{s\leqq t} M_s^2$ is S-integrable for each finite $t \in T$.

Also notice the combination of Doob's inequality and Lebesgue's Convergence Theorem in the proof above, it will reappear several times in the sequel.

Our next result shows that the class of SL^2-martingales is reasonably closed under stochastic integration. Recall [5, Definition 18]:

PROPOSITION 3. *If M is an SL^2-martingale and $X \in SL^2(M)$, then $\int X\,dM$ is an SL^2-martingale. If M is a local SL^2-martingale and $X \in SL(M)$, then $\int X\,dM$ is a local SL^2-martingale.*

PROOF. The second assertion follows from the first by definition of $SL(M)$. Assume that M is an SL^2-martingale, and let us first consider the case where $X \in SL^2(M)$ is finite, i.e. there is an $n \in \mathbf{N}$ such that $|X| \leqq n$. Then

$$0 \leqq \left[\int X\,dM\right](t) = \sum_0^t X^2 \Delta M^2 \leqq n^2 \sum_0^t \Delta M^2 = n^2[M](t)$$

and it follows from Theorem 2 that $\int X\,dM$ is an SL^2-martingale.

Let us now consider the general case $X \in SL^2(M)$. Then there exists a sequence $\{X_n\}_{n\in\mathbf{N}}$ of finite elements in $SL^2(M)$ such that

$$\circ\int_{T_t\times\Omega} |X^2 - X_n^2|\,dv_{M_t} \to 0 \quad \text{as} \quad n \to \infty\,,$$

and $X_n^2 \leqq X^2$ (cf. the comments leading up to [5, Theorem 21], or see Anderson [1]). We have

$$0 \leqq {}^\circ E\left(\left[\int X\,dM\right](t) - \left[\int X_n\,dM\right](t)\right) = {}^\circ E\left(\sum_0^t X^2\Delta M^2 - \sum_0^t X_n^2 \Delta M^2\right)$$

$$= \circ\int_{T_t\times\Omega} (X^2 - X_n^2)\,dv_{M_t} \to 0\,,$$

as $n \to \infty$. Since each $[\int X_n\,dM](t)$ is S-integrable it follows that $[\int X\,dM](t)$ is S-integrable, and hence by Theorem 2 that $\int X\,dM$ is an SL^2-martingale.

2. The right standard part of λ^2-martingales.

According to [5, Theorem 9], local λ^2-martingales have S-right- and S-left-limits a.e., and thus the following definition makes sense.

DEFINITION 4. Let $M\colon T\times\Omega \to {}^*\mathbf{R}$ be a local λ^2-martingale. Define a process ${}^\circ M^+\colon \mathbf{R}_+ \times \Omega \to \mathbf{R}$ by letting ${}^\circ M^+(r,\omega)$ be equal to the S-right-limit of $t \to M(t,\omega)$ at the monad of r. The process ${}^\circ M^+$ is called the *right standard*

part of M. In a similar way we define the *left standard part of M* to be the S-left-limit ${}^\circ M^-$ of M.

We want to make a martingale of ${}^\circ M^+$, and we first construct the stochastic basis:

If $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$ is the internal basis of M, define $\mathscr{H}_\infty = \bar{\sigma}(\bigcup \{L(\mathscr{G}_t) : t \text{ finite}\})$; i.e. the completion with respect to $L(P)$ of the σ-algebra generated by all the Loeb-algebras $L(\mathscr{G}_t)$ for finite $t \in T$. Let $\mathscr{N}$ be the set of all null-sets of $\mathscr{H}_\infty$. We define a family $\{\mathscr{H}_t\}_{t\in \mathbf{R}_+}$ of σ-algebras on Ω by:

$$\mathscr{H}_t = \sigma(\mathscr{N} \cup (\bigcup \{L(\mathscr{G}_s) : s \in T,\ s \approx t\}))\,.$$

A family $\{\mathscr{H}'_t\}_{t\in \mathbf{R}_+}$ of smaller algebras is defined by

$$\mathscr{H}'_t = \sigma(\bigcup \{L(\mathscr{G}_s) : s \in T,\ s \approx t\})\,.$$

LEMMA 5. *For each* $t \in \mathbf{R}_+$

$$\mathscr{H}_t = \bigcup \{\sigma(L(\mathscr{G}_s) \cup \mathscr{N}) : s \in T,\ s \approx t\}$$

and

$$\mathscr{H}'_t = \bigcup \{L(\mathscr{G}_s) : s \in T,\ s \approx t\}\,.$$

PROOF. Obviously

$$\bigcup \{\sigma(L(\mathscr{G}_s) \cup \mathscr{N}) : s \in T, s \approx t\} \subset \mathscr{H}_t \subset \sigma(\bigcup \{\sigma(L(\mathscr{G}_s) \cup \mathscr{N}) : s \in T, s \approx t\})$$

and it is enough to prove that $\bigcup \{\sigma(L(\mathscr{G}_s) \cup \mathscr{N}) : s \in T,\ s \approx t\}$ is a σ-algebra. Let $\{A_n\}$ be a countable family of sets from $\bigcup \{\sigma(L(\mathscr{G}_s) \cup \mathscr{N}) : s \in T,\ s \approx t\}$. Assume $A_n \in \sigma(L(\mathscr{G}_{s_n}) \cup \mathscr{N})$. The family $S_n = {}^*[s_n, t + 1/n] \cap T$ is a countable family of internal sets with the finite intersection property, and by saturation $\bigcap_{n\in \mathbf{N}} S_n \neq \varnothing$. Let $\tilde{s} \in \bigcap S_n$, then $s \approx T$ and $\tilde{s} \geqq s_n$ for all n. Consequently $A_n \in \sigma(L(\mathscr{G}_{\tilde{s}}) \cup \mathscr{N})$ for each $n \in \mathbf{N}$ and hence $\bigcup A_n \in \sigma(L(\mathscr{G}_{\tilde{s}}) \cup \mathscr{N}) \subset \bigcup_{s \approx t} \sigma(L(\mathscr{G}_s) \cup \mathscr{N})$. Since $\bigcup \sigma(L(\mathscr{G}_s) \cup \mathscr{N})$ clearly has got the other properties of σ-algebras, the $\mathscr{H}_t$-part of the lemma is proved. The $\mathscr{H}'_t$-part is similar.

Using Lemma 5 and basic properties of the Loeb-measure we have

LEMMA 6. *Let the family* $\{\mathscr{F}_t\}_{t\in \mathbf{R}_+}$ *be either* $\{\mathscr{H}_t\}$ *or* $\{\mathscr{H}'_t\}$. *Then*

(a) *If* $A \in \mathscr{F}_t$, *there exist an* $s \in T$, $s \approx t$, *and a set* $B \in \mathscr{G}_s$ *such that* $L(P)(A \Delta B) = 0$.

(b) $\mathscr{F}_t = \mathscr{F}_t^+ = \bigcap_{s>t} \mathscr{F}_s$ *for all* $t \in \mathbf{R}_+$.

If $\{\mathscr{F}_t\}$ *is* $\{\mathscr{H}_t\}$ *we also have*

(c) *For all null-sets* N *in* $\mathscr{F}_\infty$, $N \in \mathscr{F}_t$ *for all* $t \in \mathbf{R}_+$.

The properties (b) and (c) above are usually called "Meyer's usual conditions" and are assumed in most standard theory for stochastic integration (see Métivier [8]).

The following nonstandard version of Egoroff's Theorem is often useful:

PROPOSION 7. *Let* $\langle \Omega, \mathscr{G}, P\rangle$ *be a hyperfinite probability space and let* $\{X_n\}_{n\leqq\xi}$ *be an internal sequence of* $\mathscr{G}$*-measurable functions* $X_n\colon \Omega \to {}^*\mathbf{R}$ *with* $\xi \in {}^*\mathbf{N}\setminus \mathbf{N}$. *Let* $Y_n = {}^\circ X_n$ *and assume that the sequence* $\{Y_n\}$ *converges* a.e. *to a random variable* Y *on* $\langle \Omega, L(\mathscr{G}), L(P)\rangle$. *Then there exist a set* Ω' *of Loeb-measure one and a* $\nu \in {}^*\mathbf{N}\setminus\mathbf{N}$ *such that for all* $n \in {}^*\mathbf{N}\setminus\mathbf{N}$, $n \leqq \nu$, *and all* $\omega \in \Omega'$ *we have* ${}^\circ X_n(\omega) = Y(\omega)$.

PROOF. Let X be an internal random variable on $\langle \Omega, \mathscr{G}, P\rangle$ such that ${}^\circ X = Y$ $L(P)$-a.e.; such an X exists by Proposition 2 of Loeb [7]. By Egoroff's Theorem, (see e.g. Royden [10, page 72]), there exists for each $m \in \mathbf{N}$ a set A_m with $L(P)(A_m) > 1 - 1/m$ such that $Y_n \to Y$ uniformly on A_m. We may find a $B_m \in \mathscr{G}$ with $P(B_m) > 1 - 2/m$ such that $B_m \subset A_m$ and ${}^\circ X = Y$ on B_m. For each $k \in \mathbf{N}$ let $n_{k,m}$ be such that for all $\omega \in B_m$ and all $n \in \mathbf{N}$, $n \geqq n_{k,m}$ implies

$$|Y_n(\omega) - Y(\omega)| < \frac{1}{k}.$$

Let

$$\mu_{k,m} = \max\{n \in {}^*\mathbf{N} : |X_q(\omega) - X(\omega)| < 2/k \text{ for all } \omega \in B_m \text{ and}$$

$$\text{all } q \in {}^*\mathbf{N} \text{ such that } n_{k,m} \leqq q \leqq n\}.$$

By definition of $n_{k,m}$ we must have $\mu_{k,m} \in {}^*\mathbf{N}\setminus\mathbf{N}$. Using saturation we may find a $\nu \in {}^*\mathbf{N}\setminus\mathbf{N}$ less than $\mu_{k,m}$ for all $k, m \in \mathbf{N}$. It follows that for all $\omega \in \bigcup B_m$ and all $n \in {}^*\mathbf{N}\setminus\mathbf{N}$, $n < \nu$, we have ${}^\circ X_n(\omega) = Y(\omega)$. Putting $\Omega' = \bigcup_{m\in\mathbf{N}} B_m$ we have proved the proposition.

LEMMA 8. *Let* $M\colon T \times \Omega \to {}^*\mathbf{R}$ *be a local* λ^2*-martingale adapted to the internal basis* $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$. *Then the right standard part* ${}^\circ M^+$ *is a right continuous process adapted to the stochastic basis* $\langle \Omega, \{\mathscr{H}_t\}, L(P)\rangle$. *Moreover, for each* $t \in \mathbf{R}$ *and each* $s \in T$, $s \approx t$, *there exists* $\tilde{t} \in T$, $\tilde{t} \approx t$, $\tilde{t} > s$ *such that* ${}^\circ(M(\tilde{t}, \omega)) = {}^\circ M^+(t, \omega)$ $L(P)$-a.e.

PROOF. The right continuity of ${}^\circ M^+$ follows immediately from the definition. The rest of the lemma follows by applying Proposition 7 to the sequence $\{M(\overline{t+1/n},\omega)\}_{n\in{}^*\mathbf{N}}$, where $\overline{t+1/n}$ denotes the least element of T larger than $t+1/n$, and using $\mathcal{N}\subset\mathcal{H}_t$.

Recall from Métivier [8] that if $\langle Z,\{\mathcal{F}_t\},\mu\rangle$ is a stochastic basis, a process $M\colon \mathbf{R}_+\times Z\to\mathbf{R}$ is called an *L^2-martingale* if M is a martingale with respect to the basis and if for all $t\in\mathbf{R}_+$, the random variable $\omega\to M(t,\omega)$ is an element of $L^2(Z,\mathcal{F}_t,\mu)$.

We may now conclude:

PROPOSITION 9. *Let $M\colon T\times\Omega\to{}^*\mathbf{R}$ be a λ^2-martingale with respect to the internal basis $\langle\Omega,\{\mathcal{G}_t\},P\rangle$. Then ${}^\circ M^+$ is a right continuous L^2-martingale with respect to the stochastic basis $\langle\Omega,\{\mathcal{H}_t\},L(P)\rangle$.*

PROOF. We already know that ${}^\circ M^+$ is a right continuous process adapted to $\langle\Omega,\{\mathcal{H}_t\},L(P)\rangle$, by Lemma 8. Also, given a $t\in\mathbf{R}_+$ we may find a $\tilde t\in T$, $\tilde t\approx t$, such that ${}^\circ M^+(t)={}^\circ M(\tilde t)$ a.e. But then $E({}^\circ M^+(t)^2)=E({}^\circ M(\tilde t)^2)\leqq{}^\circ E(M(\tilde t)^2)<\infty$ since M is a λ^2-martingale, and hence ${}^\circ M^+(t)\in L^2(\Omega,\mathcal{H}_t,L(P))$.

Let $A\in\mathcal{H}_s$, and let $s,t\in\mathbf{R}_+$, $s<t$. We shall show that $\int_A({}^\circ M^+(t)-{}^\circ M^+(s))\,dL(P)=0$. By Lemma 6 (a) and Lemma 8 we may find $\tilde s,\tilde t\in T$ such that ${}^\circ M(\tilde s)={}^\circ M^+(s)$, ${}^\circ M(\tilde t)={}^\circ M^+(t)$ and such that there exists a $B\in\mathcal{G}_{\tilde s}$ with $L(P)(B\Delta A)=0$. We have:

$$\int_A({}^\circ M^+(t)-{}^\circ M^+(s))\,dL(P)=\int_B({}^\circ M^+(t)-{}^\circ M^+(s))\,dL(P)$$

$$=\int_B({}^\circ M(\tilde t)-{}^\circ M(\tilde s))\,dL(P)={}^\circ\int_B(M(\tilde t)-M(\tilde s))\,dP=0$$

since M is a martingale and $B\in\mathcal{G}_{\tilde s}$. To get the standard part outside the integral we have used [5, Lemma 12] which implies that $M(\tilde t)$ and $M(\tilde s)$ are S-integrable. This proves the proposition.

A process $M\colon\mathbf{R}_+\times Z\to\mathbf{R}$ is called a *local L^2-martingale* if there exists a sequence $\{\sigma_n\}$ of stopping times such that ${}^\circ\sigma_n\to\infty$ a.e. and each M_{σ_n} is an L^2-martingale. In view of Proposition 9 it is natural to guess that the right standard part of a local λ^2-martingale is a local L^2-martingale. But this is false as the following example shows:

EXAMPLE 10. Choose $\gamma\in{}^*\mathbf{N}\setminus\mathbf{N}$ and let Ω consist of 2γ elements: ω_{1+},ω_{1-}; $\omega_{2+},\omega_{2-};\ldots,\omega_{\gamma+},\omega_{\gamma-}$. Let $\varepsilon=\sum_{n=1}^{\gamma}1/n^2$, then $\varepsilon\approx\pi^2/6$. The internal measure

P on Ω is defined by $P\{\omega_{n+}\}=P\{\omega_{n-}\}=1/2\varepsilon n^2$. We shall use the time-line $T=\{k/\eta : k\in {}^*\mathbf{N}, k<\eta^2\}$ for some $\eta\in {}^*\mathbf{N}\setminus\mathbf{N}$. The family $\{\mathscr{G}_t\}$ of internal algebras is defined by: $\mathscr{G}_t=\{\varnothing,\Omega\}$ for $t<1-1/\eta$; $\mathscr{G}_{1-1/\eta}$ is the internal algebra generated by the sets $\{\omega_{n+},\omega_{n-}\}$; and $\mathscr{G}_t$ is the internal power set of Ω for $t\geqq 1$. The martingale M is defined as follows:

$$M(t,\omega)=0 \text{ for all } \omega \text{ if } t\leqq 1-\frac{1}{\eta};\ M(t,\omega_{n+})=-M(t,\omega_{n-})=n \text{ if } t\geqq 1\,.$$

M is obviously a hyperfinite martingale adapted to the internal basis $\langle\Omega,\{\mathscr{G}_t\},P\rangle$, and using the sequence $\{\tau_n\}$, where

$$\tau_n(\omega_{k+})=\tau_n(\omega_{k-})=1-1/\eta \qquad \text{if } k>n$$

$$\tau_n(\omega_{k+})=\tau_n(\omega_{k-})=1+(n-k) \qquad \text{if } k\leqq n\,,$$

we see that M is a local SL^2-martingale. Let us show that ${}^\circ M^+$ is not a local L^2-martingale. Let $\{\sigma_n\}$ be an increasing sequence of stopping times such that $\sigma_n\to\infty$ a.e. For $t<1$, the σ-algebra $\mathscr{H}_t$ consists only of sets of Loeb-measure 0 and 1. Since $\sigma_n\to\infty$ a.e. there must be an $n\in\mathbf{N}$ such that $L(P)\{\omega : \sigma_n(\omega)<1\}<1$, and consequently this set must have measure zero. But then $\sigma_n(\omega_{k+})\geqq 1$ for all $k\in\mathbf{N}$. Consequently $E({}^\circ M^+_{\sigma_n}(1)^2)=\infty$ and $\{\sigma_n\}$ is not a localizing sequence for ${}^\circ M^+$. This proves that ${}^\circ M^+$ is not a local L^2-martingale.

We can get more out of this example: Let $S=T\setminus\{1-1/\eta\}$ be a subline of T. Then the restriction of M to S is not a local λ^2-martingale. Thus a restriction of a local SL^2-martingale is not necessarily a local λ^2-martingale.

3. Stochastic integration with respect to ${}^\circ M^+$.

In this section we compare the nonstandard theory for stochastic integration with respect to M with the standard theory for integration with respect to ${}^\circ M^+$.

Let us briefly review the standard definition of a stochastic integral. (Métivier [8]):

Let $N\colon \mathbf{R}_+\times Z\to\mathbf{R}$ be a right-continuous L^2-martingale with respect to the stochastic basis $\langle Z,\{\mathscr{F}_t\},\mu\rangle$. The σ-algebra $\mathscr{P}$ of *predictable sets* with respect to $\langle Z,\{\mathscr{F}_t\},\mu\rangle$ is the σ-algebra of subsets of $\mathbf{R}_+\times Z$ generated by the sets $\langle s,t]\times F_s$ and $\{0\}\times F_0$ for $s,t\in\mathbf{R}_+$, $s<t$, $F_s\in\mathscr{F}_s$ and $F_0\in\mathscr{F}_0$. A process is called *predictable* if it is measurable with respect to $\mathscr{P}$.

We may define a unique measure m_N on $\mathscr{P}$ by

$$m_N(\langle s,t]\times F_s)=\int_{F_s}(N_t^2-N_s^2)\,d\mu$$

and $m_N(\{0\}\times F_0)=0$.

If X is a predictable process of the form

$$X(t,\omega) = \sum_{i=1}^{n} a_i 1_{\langle s_i, t_i]} \times F_{s_i}(t,\omega)$$

for $a_i \in \mathbf{R}$, we define the *stochastic integral* of X with respect to N to be the process $\int X\,dN$ defined by

$$(t,\omega) \to \sum_{i=1}^{n} a_i 1_{F_{s_i}}(\omega)(N_{t_i \wedge t} - N_{s_i \wedge t}) = \int_0^t X\,dN\,.$$

If X is a predictable process let $X^{(t)}$ be the process defined by $X^{(t)}(s,\omega) = X(s,\omega)$ for $s \leqq t$, $X^{(t)}(s,\omega) = 0$ for $s > t$. Let $\Lambda^2(N)$ be the set of all predictable processes X such that

$$X^{(t)} \in L^2(\mathbf{R}_+ \times Z, \mathscr{P}, m_N) \quad \text{for all } t \in \mathbf{R}_+\,.$$

We extend the stochastic integral to the class $\Lambda^2(N)$ by noticing that the processes of the form

$$X = \sum_{i=0}^{n} a_i 1_{\langle s_i, t_i]} \times F_{s_i}$$

are dense in $L^2(\mathbf{R}_+ \times Z, \mathscr{P}, m_N)$, and that the mapping $X \to \int_0^\infty X\,dN$ is an isometry from this dense subset into $L^2(Z, \mathscr{F}_\infty, \mu)$. If we denote the unique extension of this isometry to the whole of $L^2(Z, \mathscr{F}_\infty, \mu)$ also by $X \to \int_0^\infty X\,dN$, we define the stochastic integral $\int X\,dN$ for $X \in \Lambda^2(N)$ to be the process $\int_0^t X\,dN = \int_0^\infty X^{(t)}\,dN$.

Let us return to our nonstandard setting: Let $M\colon T \times \Omega \to {}^*\mathbf{R}$ be a λ^2-martingale adapted to the internal basis $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$. By Proposition 9, ${}^\circ M^+$ is a right-continuous L^2-martingale with respect to $\langle \Omega, \{\mathscr{H}_t\}, L(P)\rangle$. We write $\mathscr{P}$ for the predictable sets with respect to $\{\mathscr{H}_t\}$ and $\mathscr{P}'$ for the predictable sets with respect to $\{\mathscr{H}'_t\}$. The difference between the two classes is not large:

LEMMA 11. *For each $A \in \mathscr{P}$ there exists a $B \in \mathscr{P}'$ such that $m_N(A \Delta B) = 0$ for all L^2-martingales N adapted to $\langle \Omega, \{\mathscr{H}_t\}, L(P)\rangle$.*

PROOF. Follows from the fact that for $C \in \mathscr{H}_t$ there is a $D \in \mathscr{H}'_t$ with $L(P)(C \Delta D) = 0$.

Let T_f denote the set of finite elements of the time-line T. We define a mapping π from the power set of $\mathbf{R}_+ \times \Omega$ into the power set of $T_f \times \Omega$ by:

$$\pi(A) = \{(t,\omega) \in T \times \Omega : ({}^\circ t, \omega) \in A\}\,.$$

Since π is really an inverse image operation, it is easy to see that π commutes with all countable Boolean operations.

A set $A \subset T \times \Omega$ is called *adapted* (to the basis $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$) if each section A_t is in the corresponding $\mathscr{G}_t$, $t \in T$. Let $\mathscr{A}$ be the internal algebra of adapted sets, and let $L(\mathscr{A})$ be its Loeb-algebra with respect to the internal measure ν_M (remember the definition of ν_M preceding [5, Definition 18]).

Our task is to compare stochastic integration with respect to M with stochastic integration with respect to ${}^\circ M^+$; since ${}^\circ M^+(t,\omega) = S\text{-}\lim_{s \downarrow t} M(s,\omega)$, ${}^\circ M^+$ cannot register what happens to M in the monad of 0. To make sure nothing significant happens there, we define: A local λ^2-martingale M is said to be *S-right-continuous at* 0 if ${}^\circ M(0,\omega) = {}^\circ M^+(0,\omega)$ a.e.

We may prove:

LEMMA 12, *Let M be an SL^2-martingale which is S-right-continuous at* 0. *Then the image under π of the predictable sets $\mathscr{P}'$ with respect to $\mathscr{H}'_t$ is contained in the Loeb-algebra $L(\mathscr{A})$ of the adapted sets (with respect to $L(\nu_M)$).*

PROOF. By the definition of the predictable sets, it is enough to prove that any set of the form $\pi(\langle s,t] \times F_s)$ or $\pi(\{0\} \times F_0)$ is in $L(\mathscr{A})$, $F_s \in \mathscr{H}'_s$. We leave the details to the reader, and only remark that the conditions that M is an SL^2-martingale and that M is S-right-continuous at 0 both are needed in the proof.

LEMMA 13. *Let M be an SL^2-martingale which is S-right-continuous at* 0. *Then the mapping π from $\mathscr{P}'$ to $L(\mathscr{A})$ is measure-preserving, i.e. for all $A \in \mathscr{P}'$, $m_{\circ M^+}(A) = L(\nu_M)(\pi(A))$.*

PROOF. The measure $m_{\circ M^+}$ is uniquely determined by its values on the sets of the form $\langle s,t] \times F_s$, $\{0\} \times F_0$, and since π commutes with countable Boolean operations, it is enough to prove $m_{\circ M^+}(A) = L(\nu_M)(\pi(A))$ for A of this form. Again we leave the details to the reader, but remark that we make vital use of the two conditions on M.

Let M be as in Lemma 13. We define a measure $\tilde{m}_{\circ M^+}$ on the image $\pi(\mathscr{P})$ of the predictable sets under π by $\tilde{m}_{\circ M^+}(\pi A) = m_{\circ M^+}(A)$. From Lemmas 11 and 12 we see that for each set $B \in \pi(\mathscr{P})$ there is a set $C \in \pi(P) \cap L(\mathscr{A})$ such that $\tilde{m}_{\circ M^+}(B \Delta C) = 0$. It follows from this that if $f\colon T_f \times \Omega \to \mathbf{R}$ is $\pi(\mathscr{P})$-measurable, then the conditional expectation $E(f \mid \pi(\mathscr{P}) \cap L(\mathscr{A}))$ of f with respect to the sub-σ-algebra $\pi(\mathscr{P}) \cap L(\mathscr{A})$ and the measure $\tilde{m}_{\circ M^+}$ equals f a.e. By Lemma 13 we see that $\tilde{m}_{\circ M^+}$ and $L(\nu_M)$ agree on $\pi(\mathscr{P}) \cap L(\mathscr{A})$.

If $X \in \Lambda^2({}^\circ M^+)$, we define $X'\colon T_f \times \Omega \to \mathbf{R}$ by $X'(t,\omega) = X({}^\circ t, \omega)$. The process X' will obviously be $\pi(\mathscr{P})$-measurable, and if X'' is the conditional expectation of X' with respect to $\pi(\mathscr{P}) \cap L(\mathscr{A})$, the random variables X' and X'' are equal $\tilde{m}_{\circ M^+}$-a.e.

DEFINITION 14. An adapted process Y: $T\times\Omega \to {}^*\mathbf{R}$ is called a 2-*lifting* of X if $Y \in SL^2(M)$ and Y and X'' are infinitesimally close $L(\nu_M)$-a.e. on $T_f \times \Omega$.

Our purpose is to show that $\int X\, d^\circ M^+ = {}^\circ(\int Y dM)^+$ when Y is a 2-lifting of X, and hence to derive the standard theory for stochastic integration with respect to ${}^\circ M^+$ from the nonstandard theory for integration with respect to M.

The notion of lifting is central in the theory of Loeb-spaces; liftings of random variables were first studied by Loeb in [7] and further developed by Anderson in [1] and [2]. Liftings of processes were introduced by Anderson in [1] to treat stochastic integration with respect to Brownian motions. The importance of the concept is perhaps most easily seen from Keisler [3] where different classes of processes are characterized by what kind of liftings they allow, and this again is used to prove properties of the solutions of stochastic differential equations.

But let us return to our problem. We first prove that there are enough 2-liftings.

LEMMA 15. *Let M be an SL^2-martingale S-right-continuous at 0, and let $X \in \Lambda^2({}^\circ M^+)$. Then X has a 2-lifting with respect to M.*

PROOF. Let $\{t_n\}$ be an increasing sequence of finite elements of T such that ${}^\circ t_n \to \infty$. Let $X^{(n)}$ be the restriction of X'' to $A_n = ({}^*[0, t_n] \cap T)\times\Omega$. Then $X^{(n)} \in L^2(A_n, L(\mathscr{A}_n), L(\nu_{M_n}))$, where $\mathscr{A}_n$ and ν_{M_n} are the restrictions of $\mathscr{A}$ and ν_M respectively to A_n. By Theorem 11 (ii) of Anderson [1] there exists for each n a $Y^{(n)} \in SL^2(A_n, \mathscr{A}_n, \nu_{M_n})$ such that ${}^\circ Y^{(n)} = X^{(n)}$ $L(\nu_{M_n})$-a.e. We may choose the $Y^{(n)}$'s such that $Y^{(m)}\restriction A_n = Y^{(n)}$ for $m > n$. By saturation we may extend the sequence $\{Y^{(n)}\}_{n\in\mathbf{N}}$ to an internal sequence $\{Y^{(n)}\}_{n\leqq\eta}$ with the same property. Choosing $Y = Y^{(\eta)}$ we prove the lemma.

The next lemma tells us that which lifting we choose does not matter:

LEMMA 16. *Let M be an SL^2-martingale which is S-right continuous* a.e. *at 0, and let $X \in \Lambda^2({}^\circ M^+)$. If Y and Y' are 2-liftings of X, there exists a set $\Omega' \subset \Omega$ of Loeb-measure one such that*

$$^\circ\left(\int_0^t Y(s,\omega)\, dM(s,\omega)\right) = {}^\circ\left(\int_0^t Y'(s,\omega)\, dM(s,\omega)\right) \quad \textit{for all } t \in T_f,\ \omega \in \Omega' .$$

PROOF. Applying Doob's inequality to the positive *-sub-martingale

$$(t,\omega) \to \left|\int_0^t Y(s,\omega)\, dM(s,\omega) - \int_0^t Y'(s,\omega)\, dM(s,\omega)\right|$$

we get:

$$E\left(\sup_{s\leqq t}\left|\int_0^s Y(r,\omega)\,dM(r,\omega)-\int_0^s Y'(r,\omega)\,dM(r,\omega)\right|^2\right)$$
$$\leqq 4E\left(\left(\int_0^t (Y-Y')\,dM\right)^2\right).$$

But

$$^\circ E\left(\left(\int_0^t (Y-Y')\,dM\right)^2\right) = {}^\circ E\left(\sum_0^t (Y-Y')^2\,\Delta M^2\right) = {}^\circ\int_{T_t\times\Omega}(Y-Y')^2\,dv_M$$
$$= \int_{T_t\times\Omega} {}^\circ(Y-Y')^2\,dL(v_M) = 0$$

for all finite $t\in T$. It follows that $^\circ E(\sup_{s\leqq t}(\int_0^s Y\,dM-\int_0^s Y'\,dM)^2)=0$ for all finite $t\in T$ and the lemma is immediate.

We may now obtain the main result of this section, the comparison between standard and nonstandard stochastic integration:

THEOREM 17. *Let M be an SL^2-martingale which is S-right-continuous at 0, and let $X\in\Lambda^2(M)$. Let Y be a 2-lifting of X with respect to M. Then*

$$\int X\,d^\circ M^+ = {}^\circ\left(\int Y\,dM\right)^+.$$

PROOF. We first remark that since the process $\int X\,d^\circ M^+$ is only defined up to equivalence, the equality in the theorem must be interpreted as equivalence; by Lemma 16 the equivalence class of $^\circ(\int Y\,dM)^+$ is independent of the choice of Y. It is therefore enough to prove the theorem for some lifting Y.

Let us first assume that X is of the form $X=1_{\langle s,t]\times F_s}$, where $s,t\in\mathbf{R}_+$, $s<t$, and $F_s\in\mathscr{H}_s$. By Lemma 6 (a) there are an $\tilde{s}\in T$, $\tilde{s}\approx s$ and a $G_s\in\mathscr{G}_{\tilde{s}}$ such that $L(P)(F_s\Delta G_s)=0$. By Lemma 8 we may choose $\tilde{s}$ such that $^\circ M(\tilde{s},\omega)={}^\circ M^+(s,\omega)$ $L(P)$-a.e., and by the same lemma we find a $\tilde{t}\in T$, $\tilde{t}\approx t$, such that $^\circ M(\tilde{t},\omega)$ $={}^\circ M^+(t,\omega)$. Define

$$Y = 1_{(^*\langle\tilde{s},\tilde{t}]\cap T)\times G_s}.$$

We shall prove Y is a 2-lifting of X; since Y obviously is an element of $SL^2(M)$ it is enough to prove that Y is infinitesimally close to X'' $L(v_M)$-a.e. Let

$$A = \bigcup_{n\in\mathbf{N}}\bigcap_{m\in\mathbf{N}}\left[\left({}^*\left\langle s+\frac{1}{n},t+\frac{1}{m}\right]\cap T\right)\times G_s\right],$$

then the process 1_A is a version of X'', and by the choice of $\tilde{s}$ and $\tilde{t}$ it follows that $Y \approx 1_A$ a.e. and hence that Y is a 2-lifting of X.

For each $r \in \mathbf{R}_+$ we have

$$\int_0^r X \, d{}^\circ M^+ = 1_{F_s}({}^\circ M^+_{r\wedge t} - {}^\circ M^+_{r\wedge s}),$$

while for each $\tilde{r} \in T$

$$\int_0^{\tilde{r}} Y \, dM = 1_{G_s}(M_{\tilde{r}\wedge\tilde{t}} - M_{\tilde{r}\wedge\tilde{s}})$$

and by the choice of $\tilde{s}$ and $\tilde{t}$:

$${}^\circ\left(\int Y \, dM\right)^+(r) = 1_{G_s}({}^\circ M^+_{r\wedge t} - {}^\circ M^+_{r\wedge s}) L(P)\text{-a.e.}$$

This proves the theorem for X of the form $1_{\langle s,t]\times F_s}$, and by linearity the theorem holds for all X of the form $X = \sum_{i=0}^n a_i 1_{\langle s_i, t_i]\times F_{s_i}}$.

To prove the theorem it is then enough to show that the mapping $X^{(t)} \to {}^\circ(\int Y \, dM)^+(t)$ is an L^2-isometry for all $t \in \mathbf{R}_+$. We have

$$\begin{aligned}
E\left({}^\circ\left(\int Y \, dM\right)^+(t)^2\right) &= \lim_{n\to\infty} E\left(\left({}^\circ\int_0^{t+1/n} Y \, dM\right)^2\right) \\
&= \lim_{n\to\infty} {}^\circ E\left(\left(\int_0^{t+1/n} Y \, dM\right)^2\right) \\
&= \lim_{n\to\infty} {}^\circ E\left(\sum_0^{t+1/n} Y^2 \, \Delta M^2\right) = \lim_{n\to\infty} {}^\circ\int_{T_{t+1/n}\times\Omega} Y^2 \, dv_M \\
&= \lim_{n\to\infty} \int_{T_{t+1/n}\times\Omega} {}^\circ Y^2 \, dL(v_M) = \int_{\pi([0,t]\times\Omega)} {}^\circ Y^2 \, dL(v_M) \\
&= \int_{\pi([0,t]\times\Omega)} X'^2 \, d\tilde{m}_{\circ M^+} = \int_{[0,t]\times\Omega} X^2 \, dm_{\circ M^+} \\
&= \int_{\mathbf{R}_+\times\Omega} X^{(t)^2} \, dm_{\circ M^+},
\end{aligned}$$

where we have used the usual combination of Doob's inequality and Lebesgue's Convergence Theorem to introduce the limit; the fact that $\int Y \, dM$ is an SL^2-martingale (Proposition 3) to get the standard part outside the expectation; and that $Y \in SL^2(M)$ to move it inside again. The rest of the equalities follows from the assumption that Y is a 2-lifting of X, and the basic relations between the measures $L(v_M)$, $\tilde{m}_{\circ M^+}$ and $m_{\circ M^+}$.

As we have already noticed, the equality

$$E\left(\circ\left(\left(\int Y\,dM\right)^{+}(t)\right)^{2}\right) = \int_{\mathbf{R}_+ \times \Omega} X^{(t)^2}\,dm_{\circ M^+}$$

establishes the theorem.

So far we have only dealt with integrals of the form $\int X\,dN$ where N is an L^2-martingale and $X \in \Lambda^2(N)$. If N is a local L^2-martingale and X is predictable, we define X to be in $\Lambda(N)$ if there exists a localizing sequence $\{\sigma_n\}$ for N such that $X \in \Lambda^2(N_{\sigma_n})$ for each $n \in \mathbf{N}$. The stochastic integral $\int X\,dN$ is then defined as the limit $\lim_{n\to\infty} \int X\,dN_{\sigma_n}$. The theory developed above can be extended to the case where M is an SL^2-martingale and $X \in \Lambda(\circ M^+)$. We sketch the construction: For each n we may find an internal stopping time τ_n adapted to the internal basis $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$ such that $\circ\tau_n = \sigma_n$ $L(P)$-a.e. We can then prove that $X \in \Lambda^2(\circ(M_{\tau_n})^+)$ for each $n \in \mathbf{N}$, and we can find an adapted process Y: $T \times \Omega \to {}^*\mathbf{R}$ such that Y is a 2-lifting of X with respect to each M_{τ_n}. Such a Y is called a *local 2-lifting* of X. The result follows from Theorem 17:

COROLLARY 18. *Let M be an SL^2-martingale which is S-right-continuous at 0. Let $X \in \Lambda(\circ M^+)$. Then X has a local 2-lifting $Y \in SL(M)$ and*

$$\circ\left(\int Y\,dM\right)^{+} = \int X\,d\circ M^{+}\,.$$

The statements above were proved in [4]. The problem for M a local SL^2-martingale is more troublesome since by Example 10 $\circ M^+$ need not be a local λ^2-martingale, but by using a localizing sequence of stopping times, we should usually be able to reduce the problem to the SL^2-martingale case.

We end this section by a remark on the right standard part. It may seem that the use of this process has been a little unnatural, and that it would have been better to work with the standard part process $\circ M$: $\mathbf{R}_+ \times \Omega \to \mathbf{R}$ defined by $\circ M(t,\omega) = \circ(M(\bar{t},\omega))$. The advantage of our procedure is that the right standard part is always right continuous, and—since the standard theory for stochastic integration is developed only for right continuous martingales—this makes comparison with standard treatments easier. Also, our method leaves the standard martingale invariant under restriction of M to a subline, and we shall see in the next section that this may be of importance.

4. Well-behaved martingales and the quadratic variation.

A central notion in this paper and in [5] is that of the quadratic variation of a hyperfinite martingale. We have a similar notion for real-valued L^2-

martingales $N: \mathbf{R}_+ \times Z \to \mathbf{R}$: If $\{\pi_n\}$ is an increasing sequence of partitions $\pi_n = \{0 = t_0 < t_1 < \ldots < t_k < \ldots\}$ of the positive real numbers such that the diameter $\delta(\pi_n) = \sup_{t_i \in \pi_n}(t_{i+1} - t_i)$ tends to zero as $n \to \infty$, and $\sup \pi_n = \infty$, then the *quadratic variation* $[N]$ of N is defined by

$$[N](t) = \lim_n (L^1) \sum_{t_i \in \pi_n} (N_{t_{i+1} \wedge t} - N_{t_i \wedge t})^2 .$$

For a proof that $[N]$ exists and is well-defined, see Métivier [8, page 234].

If $M: T \times \Omega \to {}^*\mathbf{R}$ is an SL^2-martingale, we form the right standard part ${}^\circ M^+$ and its quadratic variation $[{}^\circ M^+]$. We may also form the quadratic variation $[M]$ of M; since this is an increasing process it must have S-right-limits, and we can define its right standard part ${}^\circ[M]^+$. In the spirit of Section 3 it is natural to ask when the processes $[{}^\circ M^+]$ and ${}^\circ[M]^+$ are equal. That this question has some real importance may be seen by comparing the nonstandard version of the Transformation formula ([5, Theorem 22]) with the standard version on page 265 in Métivier [8] applied to ${}^\circ M^+$; in the first case we integrate with respect to $[M]$ in the second with respect to $[{}^\circ M^+]$. If we want to deduce the standard form from the nonstandard, we must know the relationship between $[M]$ and $[{}^\circ M^+]$.

It is easy to make examples of SL^2-martingales where $[{}^\circ M^+] \neq {}^\circ[M]^+$. A closer inspection of such examples makes the following definition natural.

DEFINITION 19. Let $M: T \times \Omega \to {}^*\mathbf{R}$ be a hyperfinite martingale. For $(t, \omega) \in \mathbf{R}_+ \times \Omega$ we say that M is *well-behaved at* (t, ω) if there exists an $s \in T$, $s \approx t$, such that for all $r \in T$, $r \approx t$, we have ${}^\circ M(r, \omega) = S - \lim_{u \uparrow t} M(u, \omega)$ for $r \leqq s$ and ${}^\circ M(r, \omega) = S - \lim_{u \downarrow t} M(u, \omega)$ for $r > s$.

The martingale M is called *well-behaved* if there exists a subset Ω' of Ω of Loeb-measure one, such that for all $t \in \mathbf{R}_+$ and all $\omega \in \Omega'$, M is well-behaved at (t, ω). In particular all S-continuous martingales are well-behaved.

In this section we shall sketch three results about well-behaved martingales; the first is that if M is a well-behaved martingale which is S-right continuous at 0, then $[{}^\circ M^+] = {}^\circ[M]^+$. The second is that if M is a λ^2-martingale, then there exists a subline S of T such that the restriction M^s of M to S is well-behaved; and the third is that if M is well-behaved, and X is a 2-lifting of a process $Y \in \Lambda^2({}^\circ M^+)$, then $\int X\, dM$ is well-behaved.

To solve the first problem we first prove the following result which should be of independent interest:

LEMMA 20. *Let M be a well-behaved SL^2-martingale, S-right-continuous at 0. Then the left standard part ${}^\circ M^-$ is an element of $\Lambda({}^\circ M^+)$, and M is a local 2-lifting of ${}^\circ M^-$ with respect to ${}^\circ M^+$. In particular*:

$$^\circ\left(\int M\,dM\right)^+ = \int {}^\circ M^-\, d{}^\circ M^+ .$$

PROOF. We only outline the idea of the proof and leave the details to the reader. Let $f\colon \mathbf{N} \to \mathbf{N}^2$ be a bijection and let $(f(n))_1$ be the first component of $f(n)$. Define internal stopping times

$$\tau_n(\omega) = \min\{t_{i+1} \in T\colon\ |\Delta M(t_i,\omega)| > 1/(f(n))_1 \text{ and } t_{i+1} \neq \tau_k(\omega) \text{ for } k<n\} .$$

As in the proof of [5, Theorem 22], we see by [5, Lemma 10] that outside a set of measure zero, the sequence $\{\tau_n\}$ enumerates the non-infinitesimal jumps of M. It follows that the points where $({}^\circ M^-)'$ and M differ are the union of a null-set and the set $\bigcup_n \bigcap_m {}^\circ[\tau_n, \tau_n + 1/\mathrm{m}\rangle$. But since M is well-behaved $\lim_{m\to\infty} {}^\circ M(\tau_n + 1/m) = {}^\circ M(\tau_n) = {}^\circ M^+(\tau_n)$, which implies that $\bigcap_m {}^*[\tau_n, \tau_n + 1/m\rangle$ has Loeb-measure zero. Thus $({}^\circ M^-)' = {}^\circ M$ $L(\nu_M)$-a.e.

The last part of the lemma follows from the first by Corollary 18.

THEOREM 21. *Let M be a well-behaved SL^2-martingale which is S-right continuous at 0. Then* ${}^\circ[M]^+ = [{}^\circ M^+]$.

PROOF. Métivier [8] proves (Korollar 1 and 2 on page 267) that if N is an L^2-martingale, right-continuous and with left limits, then:

$$[N](t) = N(t)^2 - N(0)^2 - 2\int_0^t N^-\, dN ,$$

where N^- is the left limit of N. Applying this to ${}^\circ M^+$ we have

$$\begin{aligned}[{}^\circ M^+](t) &= {}^\circ M^+(t)^2 - {}^\circ M^+(0)^2 - 2\int_0^t {}^\circ M^-\, d{}^\circ M^+ \\ &= {}^\circ M^+(t)^2 - {}^\circ M(0)^2 - 2\,{}^\circ\left(\int_0^t M\,dM\right)^+ = {}^\circ\left(M^2 - M_0^2 - \int M\,dM\right)^+(t) \\ &= {}^\circ[M]^+(t)\end{aligned}$$

where we have used Lemma 20 and [5, Proposition 17].

The proof above is the only place in this paper where we use a result from the standard theory for stochastic integration beyond the mere definitions. However, we shall prove the formula $[N](t) = N(t)^2 - N(0)^2 - 2\int_0^t N^-\, dN$ by nonstandard methods in [6] (independently of Theorem 21).

To prove our second result we need some preliminaries. In [8], Métivier proves (Satz 17.5) that if N is a right continuous process with left limits adapted to a stochastic basis $\langle Z, \{\mathcal{F}_t\}, \mu\rangle$, then there exists a sequence $\{\sigma_n\}_{n\in\mathbf{N}}$

of stopping times adapted to the same basis such that N has no jumps outside the graphs of the σ_n's. If $M: T\times\Omega \to {}^*\mathbf{R}$ is a λ^2-martingale, let $\delta M: \mathbf{R}_+ \times \Omega \to \mathbf{R}$ be defined by

$$\delta M(t,\omega) = \sup\{{}^\circ M(s,\omega) : s \in T \wedge s \approx t\} - \inf\{{}^\circ M(s,\omega) : s \in T \wedge s \approx t\} .$$

In a way entirely similar to that of Métivier, we may prove that there exist a set Ω' of Loeb-measure one and a sequence $\{\sigma_n\}_{n\in\mathbf{N}}$ of stopping times adapted to $\langle \Omega, \{\mathscr{H}_t\}, L(P)\rangle$ such that if $\omega \in \Omega'$ then $\delta M(t,\omega) \neq 0$ if and only if there exists an $n \in \mathbf{N}$ with $\sigma_n(\omega) = t$.

Using the lifting-techniques of Loeb [7] and Anderson [2] it is not difficult to prove the following:

LEMMA 22. *Let σ be a stopping time from Ω to $\mathbf{R}_+$ adapted to $\langle \Omega, \{\mathscr{H}_t\}, L(P)\rangle$. Then there exists an internal stopping time $\tau: \Omega \to T$ adapted to $\langle \Omega, \{\mathscr{G}_t\}, P\rangle$ such that ${}^\circ\tau = \sigma$ $L(P)$-a.e.*

We have already tacitly made use of this lemma in the argument leading up to Corollary 18.

Combining Lemma 22 with what we have already got, we obtain a sequence $\{\tau_n\}_{n\in\mathbf{N}}$ of internal stopping times enumerating the points where $\delta M \neq 0$ inside a set $\Omega_1 \subset \Omega$ of Loeb-measure one.

THEOREM 23. *Let $M: T\times\Omega \to {}^*\mathbf{R}$ be a λ^2-martingale. Then there exists a subline S of T such that the restriction M^S of M to $S\times\Omega$ is a well-behaved λ^2-martingale.*

PROOF. Let $\{\tau_l\}_{l\in\mathbf{N}}$ and Ω_1 be the sequence and the set constructed above. Clearly, if $\omega \in \Omega_1$ and M is not well-behaved at (t,ω) then $t = {}^\circ\tau_l(\omega)$ for some $l \in \mathbf{N}$. The proof is in two steps. First we prove that there are a $\gamma \in {}^*\mathbf{N} \setminus \mathbf{N}$ and a set $\Omega_2 \subset \Omega_1$ of Loeb-measure one such that if $\omega \in \Omega_2$ and $t = {}^\circ\tau_l(\omega)$, then

$${}^\circ M(s,\omega) = S\text{-}\lim_{r\uparrow t} M(r,\omega) \quad \text{for } s \approx t,\ s < \tau_l(\omega) - 1/\gamma \quad \text{and}$$

$${}^\circ M(s,\omega) = S\text{-}\lim_{r\downarrow t} M(r,\omega) \quad \text{for } s \approx t,\ s > \tau_l(\omega) + 1/\gamma .$$

Secondly, we prove that this is enough to construct the subline S. To do this it will be enough to construct S such that the set

$$\{\omega \in \Omega : \exists s \in S\ \exists n \in \mathbf{N}(\tau_n(\omega) - 1/\gamma < s < \tau_n(\omega) + 1/\gamma)\}$$

has Loeb-measure zero, since M then is well-behaved at all points (t,ω) where ω is not in the union of this set and the complement of Ω_2. We shall prove by a combinatorial argument that we can always find such an S.

(i) Let $\{B_m\}_{m\in\mathbf{N}}$ be a sequence of internal subsets of Ω_1 with $P(B_m)>1-1/m$. For each 5-tuple $(l,m,n,k,p)\in\mathbf{N}^5$ define the set

$$D^{(l)}_{m,n,k,p} = \{\omega\in B_m : \forall r,s\in T[((\tau_l(\omega)-1/k\leqq r,s\leqq\tau_l-1/p)$$
$$\vee\ (\tau_l(\omega)+1/p\leqq r,s\leqq\tau_l(\omega)+1/k))\to |M(r,\omega)-M(s,\omega)|<1/n]\}\ .$$

We claim that for all l and m

$$B_m = \bigcap_n \bigcup_k \bigcap_{p>k} D^{(l)}_{m,n,k,p}\ .$$

It is enough to prove that if $\omega\in B_m$ then for all l,m,n we have $\omega\in\bigcup_k\bigcap_{p>k}D^{(l)}_{m,n,k,p}$. But this is immediate since for $\omega\in\Omega_1$, $t\to M(t,\omega)$ has S-left and S-right-limits at $t={}^\circ\tau_l(\omega)$.

For each 4-tuple $(l,m,n,k)\in\mathbf{N}^4$ we pick an internal set $C^{(l)}_{m,n,k}\subset\bigcap_{p>k}D^{(l)}_{m,n,k,p}$ such that

$$L(P)\left(\bigcap_{p>k} D^{(l)}_{m,n,k,p}\setminus C^{(l)}_{m,n,k}\right) < 2^{-(m+n+k)}\ .$$

Since $B_m=\bigcap_n\bigcup_k\bigcap_{p>k}D^{(l)}_{m,n,k,p}$ we see that

$$L(P)\left(B_m\setminus\bigcap_n\bigcup_k C^{(l)}_{m,n,k}\right) < 2^{-m}$$

and consequently for each $l\in\mathbf{N}$

$$L(P)\left(\bigcup_m\bigcap_n\bigcup_k C^{(l)}_{m,n,k}\right) = 1\ .$$

For each $(l,m,n,k)\in\mathbf{N}^4$, let $\gamma^{(l)}_{m,n,k}$ be the largest element p in $^*\mathbf{N}$ such that for all $\omega\in C^{(l)}_{m,n,k}$ and all $r,s\in T$ if $t-1/k\leqq r,s\leqq t-1/p$ or $t+1/p\leqq r,s\leqq t+1/k$ then $|M(r,\omega)-M(s,\omega)|<1/n$. Since by definition $C^{(l)}_{m,n,k}\subset\bigcap_{p>k}D^{(l)}_{m,n,k}$ this is true for all finite p and hence for some infinite p since $C^{(l)}_{m,n,k}$ is internal. Thus each $\gamma^{(l)}_{m,n,k}$ is infinite and by saturation we may find an infinite γ less than all of them. Putting $\Omega_2=\bigcap_l\bigcup_m\bigcap_n\bigcup_k C^{(l)}_{m,n,k}$ we have finished the first part of the proof.

(ii) We may now construct the new time-line S. Since for $\omega\in\Omega_2$ the only points $(t,\omega)\in\mathbf{R}_+\times\Omega$ where M may fail to be well-behaved are the points $({}^\circ\tau_l(\omega),\omega)$, it is enough to construct S such that the set

$$\{\omega\in\Omega :\ \exists s\in S\ \exists n\in\mathbf{N}(\tau_n(\omega)-1/\gamma<s<\tau_n(\omega)+1/\gamma)\}$$

has Loeb-measure zero.

By choosing a smaller γ if necessary we may assume that $1/\gamma>\min\{t_{i+1}-t_i : t_{i+1},t_i\in T\}$. Define a subline $S'\subset T$ by putting $s'_0=0$ and $s'_{i+1}=$ the least

element of T larger than s'_i+2/γ. Obviously $2/\gamma<s'_{i+1}-s'_i\leqq 3/\gamma$, and for each $\omega\in\Omega$, $n\in\mathbf{N}$ there can be only one $s'\in S'$ in ${}^*\langle\tau_n(\omega)-1/\gamma,\tau_n(\omega)+1/\gamma\rangle$.

Let $\eta=\gamma^{4/5}$. Then η is infinite, $\eta^{5/4}/\gamma=1$, while $\eta/\gamma=1/\gamma^{1/5}$ is infinitesimal. Extend the sequence $\{\tau_n\}_{n\in\mathbf{N}}$ to an internal sequence $\{\tau_n\}_{n<\sqrt{\eta}}$. Let

$$p'_i = P\{\omega : \exists n < \sqrt{\eta}(s'_i \in {}^*\langle\tau_n(\omega)-1/\gamma,\tau_n(\omega)+1/\gamma\rangle)\} .$$

Since for each ω the sets ${}^*\langle\tau_n(\omega)-1/\gamma,\tau_n(\omega)+1/\gamma\rangle$, $n<\sqrt{\eta}$, can only cover less than $\sqrt{\eta}$ of the elements of S', we have $\sum p'_i<\sqrt{\eta}$.

We now define S: Let $s_0=0$ and if s_i is chosen consider the elements of S' between $s_i+\eta/\gamma$ and $s_i+2\eta/\gamma$. There must be $\eta/3$ or more such elements. Since

$$\sum_{s_i+\eta/\gamma}^{s_i+2\eta/\gamma} p'_j < \sqrt{\eta}$$

there must exist a j in this interval such that $p'_j<3/\sqrt{\eta}$. We choose the corresponding s'_j as s_{i+1}.

Let $t\in T$; the number of elements in S less than t is less than $t/(\eta/\gamma)=t\gamma/\eta$. If

$$p_i = P\{\omega : \exists n<\sqrt{\eta}(s_i \in {}^*\langle\tau_n(\omega)-1/\gamma,\tau_n(\omega)+1/\gamma\rangle)\}$$

then

$$\sum_{s_i<t} p_i \leqq \frac{3}{\sqrt{\eta}}\cdot\frac{t\gamma}{\eta} = \frac{3t\cdot\gamma}{\eta^{3/2}} = \frac{3t}{\eta^{1/4}}\left(\frac{\gamma}{\eta^{5/4}}\right) = \frac{3t}{\eta^{1/4}} .$$

This tells us that if we cut off S at $t=\eta^{1/8}$, then

$$P\{\omega : \exists n<\sqrt{\eta}\ \exists s\in S(\tau_n(\omega)-1/\gamma<s<\tau_n(\omega)+1/\gamma)\} \leqq 3/\eta^{1/8} \approx 0$$

and consequently

$$L(P)\{\omega\in\Omega : \exists s\in S\ \exists n\in\mathbf{N}(\tau_n(\omega)-1/\gamma<s<\tau_n(\omega)+1/\gamma)\} = 0 .$$

We have already observed that this proves the theorem.

If we replace "λ^2-martingale" by "local λ^2-martingale" in the hypothesis and conclusion of Theorem 23, the resulting statement is false. In fact, by making a slight change in the martingale of Example 10, we may construct a local SL^2-martingale which does not have any restriction that is a well-behaved local λ^2-martingale. The point is that to make the martingale well-behaved we must remove a point on the time-line that is essential for making it a local λ^2-martingale.

The well-behaved martingales are "well-behaved" in the sense that they satisfy Lemma 20 and Theorem 21. These results are necessary to derive the standard Transformation formula from the non-standard version. Theorem 23 shows that the class of well-behaved martingales is "large enough". In other

respects the class is sadly irregular; for instance is the sum of two well-behaved martingales usually not well-behaved. Furthermore, there are examples of well-behaved martingales M and processes $X \in SL^2(M)$ such that $\int X\,dM$ is not well-behaved even when X is bounded.

We shall now prove that this can not happen if X is a lifting. The proof is a nice application of Theorem 23:

THEOREM 24. *Let $M\colon T\times\Omega \to {}^*\mathbf{R}$ be a well-behaved λ^2-martingale, and let $X \in SL^2(M)$ be a 2-lifting of some predictable process $Y \in \Lambda^2({}^\circ M^+)$. Then the martingale $\int X\,dM$ is well-behaved. Moreover, there exists a set Ω' of Loeb-measure one such that if $\omega \in \Omega'$ and ${}^\circ((\int X\,dM)(t^+,\omega)-(\int X\,dM)(t,\omega)) \neq 0$, then ${}^\circ(M(t^+,\omega)-M(t,\omega)) \neq 0$.*

PROOF. Let X and X' be two 2-liftings of Y. By Lemma 16, there is a set Ω_1 of Loeb-measure one such that $\int_0^t X(s,\omega)\,dM(s,\omega) \approx \int_0^t X'(s,\omega)\,dM(s,\omega)$ for all $\omega \in \Omega_1$ and all finite $t \in T$. Hence $\int X\,dM$ is well-behaved if and only if $\int X'\,dM$ is.

Assume first that Y is bounded, $|Y| \leqq \eta$. We may then choose $|X| \leqq n$. The process $\int X\,dM$ is λ^2-martingale, and by Theorem 23 it has a well-behaved restriction $(\int X\,dM)^S$ to a subline S.

Let ν_M^S be the internal measure on ${}^*\mathscr{P}(S\times\Omega)$ defined by $\nu_M^S(\langle s_i,\omega\rangle) = P\{\omega\}([M](s_{i+1})-[M](s_i))$. Let $\tilde{X}'$ be a 2-lifting of Y with respect to the internal basis $\langle \Omega, \{\mathscr{G}_s\}_{s\in S}, P\rangle$ and ν_M^S. Define $X'\colon T\times\Omega \to {}^*\mathbf{R}$ by $X'(t,\omega) = \tilde{X}'(s_i,\omega)$ for $s_i \leqq t < s_{i+1}$. Then X' is a 2-lifting of Y with respect to ν_M, and we may assume $|X'| \leqq n$.

Let Ω' be the set of Loeb-measure one where M is well-behaved, $(\int X\,dM)^S$ is well-behaved, and $\int_0^t X'\,dM \approx \int_0^t X\,dM$ for all finite $t \in T$.

Since X' is constant and less than n in absolute value on each interval ${}^*[s_i, s_{i+1}\rangle$, we see that if $r_1, r_2 \in {}^*[s_i, s_{i+1}\rangle$, $\omega \in \Omega'$, and $M(r_1,\omega) \approx M(r_2,\omega)$, then $\int_0^{r_1} X'\,dM \approx \int_0^{r_2} X'\,dM$. We thus see that $\int X'\,dM$ can jump in only one interval ${}^*[s_i, s_{i+1}\rangle$ in each monad; and the jump must be in the point where M itself jumps. The same must hold for $\int X\,dM$, and we have proved the theorem for bounded Y.

Let now Y be an arbitrary element of $\Lambda^2({}^\circ M^+)$, and let $X \in SL^2(M)$ be a 2-lifting of Y. For $n \in \mathbf{N}$, define X_n by

$$X_n(t,\omega) = \begin{cases} n & \text{if } X(t,\omega) > n \\ X(t,\omega) & \text{if } |X(t,\omega)| \leqq n \\ -n & \text{if } X(t,\omega) < -n \end{cases},$$

and define Y_n similarly. Then X_n is a 2-lifting of Y_n, and we have proved the theorem for X_n. Now

$$0 \leqq E\left(\circ\max_{s \leqq t}\left(\int_0^s X\,dM - \int_0^s X_n\,dM\right)^2\right) \leqq \circ E\left(\max_{s \leqq t}\left(\int_0^s (X - X_n)\,dM\right)^2\right)$$

$$\leqq 4\circ E\left(\left(\int_0^t (X - X_n)\,dM\right)^2\right) = 4\circ E\left(\sum_0^t (X - X_n)^2 \Delta M^2\right)$$

$$= 4\circ E\left(\sum_0^t (X^2 - 2XX_n + X_n^2)\Delta M^2\right) \leqq 4\circ E\left(\sum_0^t (X^2 - 2X_nX_n + X_n^2)\Delta M^2\right)$$

$$= 4\circ E\left(\sum_0^t (X^2 - X_n^2)\Delta M^2\right) = 4\circ \int_{T_t \times \Omega} (X^2 - X_n^2)\,dv_M \to 0 \quad \text{as } n \to \infty,$$

since $X \in SL^2(M)$.

By a result of measure theory, we may find a subsequence $\{\circ\max_{s \leqq t}\{\int_0^s X\,dM - \int_0^s X_{n_k}\,dM)^2\}_{k \in \mathbf{N}}$ which converges pointwise to zero a.e.

But it is easy to see that if $\int_0^{r_1} X_{n_k}\,dM \approx \int_0^{r_2} X_{n_k}\,dM$ for all $k \in \mathbf{N}$, then $\int_0^{r_1} X\,dM \approx \int_0^{r_2} X\,dM$ on this set of convergence.

This proves the theorem.

The last theorem in combination with Theorems 17, 21 and 23 indicates that the class of well-behaved martingales is a suitable class to work with, when using the nonstandard theory in dealing with standard situations. It may therefore also be a natural class when applying stochastic integration in the analysis of natural phenomena.

NOTE ADDED IN PROOF. Most of the results in this paper have been independently rediscovered and extended by D. N. Hoover and E. Perkins: *Nonstandard construction of the stochastic integral and applications to stochastic differential equations* (preprint), 1980.

REFERENCES

1. R. M. Anderson, *A nonstandard representation for Brownian motion and Itô integration*, Israel J. Math. 25 (1976), 15–46.
2. R. M. Anderson, *Star-finite Probability Theory*, Ph.D.-thesis Yale University, 1977.
3. H. J. Keisler, *An infinitesimal approach to stochastic analysis*, preliminary version, 1978.
4. T. L. Lindström, *Nonstandard theory for stochastic integration*, (unpublished).
5. T. L. Lindström, *Hyperfinite stochastic integration* I: *The nonstandard theory*, Math. Scand. 46 (1980), 265–292.
6. T. L. Lindström, *Hyperfinite stochastic integration* III: *Hyperfinite representations of standard martingales*, Math. Scand. 46 (1980), 315–331.
7. P. A. Loeb, *Conversion from Nonstandard to Standard Measure Spaces and Applications in Probability Theory*, Trans. Amer. Math. Soc. 211 (1975), 113–122.
8. M. Métivier, *Reelle und Vektorwertige Quasimartingale und die Theorie der Stochastischen Integration*, Lecture Notes in Mathematics 607, Springer-Verlag, Berlin - Heidelberg - New York, 1977.

9. E. Nelson, *Dynamical theories of Brownian motion*, Princeton University Press, Princeton, 1967.
10. H. L. Royden, *Real Analysis*. Second Edition, Macmillan, London, 1968.
11. K. D. Stroyan and W. A. J. Luxemburg, *Introduction to the Theory of Infinitesimals*, Academic Press, New York - London, 1976.

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