

## HYPERFINITE STOCHASTIC INTEGRATION III: HYPERFINITE REPRESENTATIONS OF STANDARD MARTINGALES

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**Introduction.**

In the second of the papers in this series [8], we showed that a stochastic integral with respect to the right standard part  ${}^{\circ}M^+$  of an  $SL^2$ -martingale  $M$  could be obtained from a nonstandard stochastic integral with respect to  $M$ . We did so in order to show that the standard and nonstandard theory for stochastic integration are equivalent, and we shall now complete our programme by showing that all (standard)  $L^2$ -martingales can be represented as right standard parts of  $SL^2$ -martingales.

The argument is in two steps. If  $\langle Z, \{\mathcal{F}_t\}, \mu \rangle$  is the stochastic basis of an  $L^2$ -martingale  $N$ , we first find a hyperfinite probability space  $\langle \Omega, \mathcal{G}, P \rangle$ , a family  $\{\mathcal{F}'_t\}_{t \in \mathbb{R}_+}$  of  $\sigma$ -algebras on  $\Omega$ , and a measure-preserving  $\sigma$ -homomorphism  $\theta: \langle \Omega, \mathcal{F}'_{\infty}, L(P) \rangle \rightarrow \langle Z, \mathcal{F}_{\infty}, \mu \rangle$  which maps each  $\mathcal{F}'_t$  onto  $\mathcal{F}_t$ . Each  $\mathcal{F}'_t$  is a sub- $\sigma$ -algebra of  $L(\mathcal{G}_t)$  for some  $\mathcal{G}_t \subset \mathcal{G}$ , and we find an  $L^2$ -martingale  $N^{\theta}: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  adapted to  $\langle \Omega, \{L(\mathcal{G}_t)\}, L(P) \rangle$  such that each  $N^{\theta}_t$  is  $\mathcal{F}'_t$ -measurable, and  $\theta[N^{\theta}_t \geq \alpha] = [N_t \geq \alpha]$  for all  $t \in \mathbb{R}_+$  and all  $\alpha \in \mathbb{R}$ . This  $L^2$ -martingale  $N^{\theta}$  is called a weak Loeb-space representation of  $N$ .

The second step is to construct an  $SL^2$ -martingale  $M$  adapted to  $\langle \Omega, \{\mathcal{G}_t\}, P \rangle$ , such that  $N^{\theta} = {}^{\circ}M^+$ . This  $M$  is called a weak hyperfinite representation of  $N$ . We shall prove that if  $X \in \mathcal{A}^2(N)$ , then we may find  $Y \in SL^2(M)$  such that

$${}^{\circ}\left(\int Y dM\right)^+ = \left(\int X dN\right)^{\theta}.$$

This will complete our programme, since we may now obtain  $\int X dN$  from  $\int Y dM$ .

We shall use the same conventions and terminology as in the two previous papers; and we only remind the reader that we are working with polysaturated models for nonstandard analysis (see Stroyan and Luxemburg [13]); the saturation property will be used repeatedly in this paper.

**1. Weak Loeb-space representations of measure-spaces.**

We want to represent arbitrary  $L^2$ -martingales as the right standard parts of  $SL^2$ -martingales, and to do this we first represent the corresponding measure spaces by hyperfinite Loeb-spaces.

DEFINITION 1. Let  $\langle Z, \mathcal{F}, \mu \rangle$  be a propability space. By a *weak Loeb-space representation* of  $\langle Z, \mathcal{F}, \mu \rangle$  we shall mean a hyperfinite probability space  $\langle \Omega, \mathcal{G}, P \rangle$ ; a sub- $\sigma$ -algebra  $\mathcal{F}'$  of  $L(\mathcal{G})$ ; and a measure-preserving  $\sigma$ -homomorphism  $\theta: \langle \Omega, \mathcal{F}', L(P) \rangle \rightarrow \langle Z, \mathcal{F}, \mu \rangle$  which is *onto*  $\mathcal{F}$ . The representation is called a *strong Loeb-space representation* if there is a partial mapping  $\Theta: \Omega \rightarrow Z$ , such that  $\Theta^{-1}(A) \in \mathcal{F}'$  for all  $A \in \mathcal{F}$  and  $\theta(\Theta^{-1}(A)) = A$  for all such  $A$ . Obviously, the domain of  $\Theta$  must have measure one.

Using the standard part map of the defining topology, Anderson proved in [2] that all Radon-spaces have strong Loeb-space representations. We shall prove that all probability spaces have weak Loeb-space representations. To do this we need the following theorem which we take from Sikorski's book [12, pp. 144–145]:

PROPOSITION 2. *Let  $Z$  and  $Z'$  be sets and let  $\mathcal{A}$  and  $\mathcal{A}'$  be  $\sigma$ -algebras on  $Z$  and  $Z'$  respectively. Let  $\mathcal{S}$  be a generator set for  $\mathcal{A}$  and let  $f: \mathcal{S} \rightarrow \mathcal{A}'$  be a mapping. Then  $f$  has an extension to  $\mathcal{A}$  which is a  $\sigma$ -homomorphism if and only if for each countable family  $\{A_i\}_{i \in \mathbf{N}}$  of elements from  $\mathcal{S}$ :*

$$\bigcap_{i \in \mathbf{N}} \varepsilon(i)A_i = \emptyset \Rightarrow \bigcap_{i \in \mathbf{N}} \varepsilon(i)f(A_i) = \emptyset,$$

where  $\varepsilon: \mathbf{N} \rightarrow \{1, -1\}$ .

We may now use Proposition 2 to prove

THEOREM 3. *Each probability space has a weak Loeb-space representation.*

PROOF. Let  $\langle Z, \mathcal{F}, \mu \rangle$  be a probability space and let  $\langle *Z, *\mathcal{F}, *\mu \rangle$  be its nonstandard version. Let  $\mathcal{S} = \{ *F : F \in \mathcal{F} \}$ , and let  $\mathcal{F}^*$  be the  $\sigma$ -algebra generated by  $\mathcal{S}$ .

Define a mapping  $f: \mathcal{S} \rightarrow \mathcal{F}$  by  $f(*F) = F$ . We shall use Proposition 2 to extend  $f$  to a  $\sigma$ -homomorphism  $h: \mathcal{F}^* \rightarrow \mathcal{F}$ . Since  $\mathcal{F}^*$  is closed under complements, it is enough to show that for each countable family  $\{ *F_i \}_{i \in \mathbf{N}}$  of sets from  $\mathcal{F}$  such that  $\bigcap_{i \in \mathbf{N}} *F_i = \emptyset$ , then  $\bigcap_{i \in \mathbf{N}} f(*F_i) = \bigcap_{i \in \mathbf{N}} F_i = \emptyset$ . But this is easy since saturation and  $\bigcap_{i \in \mathbf{N}} *F_i = \emptyset$  implies  $\bigcap_{i \leq n} *F_i = \emptyset$  for some  $n \in \mathbf{N}$ . Since  $*(\bigcap_{i \leq n} F_i) = \bigcap_{i \leq n} *F_i = \emptyset$ , we have  $\bigcap_{i \leq n} F_i = \emptyset$  and consequently  $\bigcap_{i \in \mathbf{N}} F_i = \emptyset$ . Hence  $f$  can be extended to a  $\sigma$ -homomorphism  $h: \mathcal{F}^* \rightarrow \mathcal{F}$ .

Let us show that  $h$  is measure-preserving, that is,  $L(*\mu)(A) = \mu(h(A))$  for

$A \in \tilde{\mathcal{F}}$ . If  $A \in \mathcal{S}$  this is obvious by definition of the Loeb-measure. Since  $\nu(A) = \mu(h(A))$  defines a measure on  $\tilde{\mathcal{F}}$ , it follows from Caratheodory's Extension Theorem (Royden [11, page 257]) that  $L(*\mu)(A) = \mu(h(A))$  for all  $A \in \tilde{\mathcal{F}}$ , since  $\mathcal{S}$  is an algebra of sets.

We have thus constructed an internal probability space  $\langle *Z, *\mathcal{F}, *\mu \rangle$ ; a sub- $\sigma$ -algebra  $\tilde{\mathcal{F}}$  of  $L(*\mathcal{F})$ ; and a measure-preserving  $\sigma$ -homomorphism  $h: \tilde{\mathcal{F}} \rightarrow \mathcal{F}$ . It only remains to turn  $\langle *Z, *\mathcal{F}, *\mu \rangle$  into a hyperfinite probability space:

For each finite set  $F_1, \dots, F_n$  of elements of  $\mathcal{F}$ , there is a finite partition  $\mathcal{P}$  of  $*Z$  such that if  $P \in \mathcal{P}$  and  $P \cap *F_i \neq \emptyset$ , then  $P \subset *F_i$ . By saturation there exists a hyperfinite partition  $\mathcal{P}$  of  $*Z$  such that for each  $F \in \mathcal{F}$  if  $P \cap *F \neq \emptyset$  for a  $P \in \mathcal{P}$ , then  $P \subset *F$ . Let  $\sim$  be the equivalence relation generated by  $\mathcal{P}$ , let  $\Omega = *Z/\sim$ , and let  $\pi: *Z \rightarrow \Omega$  be the quotient map. We may choose  $\mathcal{P}$  such that the equivalence classes of  $\mathcal{P}$  are elements of  $*\mathcal{F}$ . Let  $\mathcal{G} = \pi(*\mathcal{F})$  and let  $P = \pi(*\mu)$ , then  $\langle \Omega, \mathcal{G}, P \rangle$  is a hyperfinite probability space. Let  $\mathcal{F}'$  be  $\pi(\tilde{\mathcal{F}})$ , it follows easily from the definitions of  $\tilde{\mathcal{F}}$  and  $\mathcal{P}$  that  $\mathcal{F}' \subset L(\mathcal{G})$ . Define

$$\theta: \mathcal{F}' \rightarrow \mathcal{F} \quad \text{by } \theta = h \circ \pi^{-1}.$$

Then  $\{\langle \Omega, \mathcal{G}, P \rangle, \mathcal{F}', \theta\}$  is a weak Loeb-space representation of  $\langle Z, \mathcal{F}, \mu \rangle$ , and we have proved the theorem.

If  $\{\langle \Omega, \mathcal{G}, P \rangle, \mathcal{F}', \theta\}$  is a weak Loeb-space representation of  $\langle Z, \mathcal{F}, \mu \rangle$ , we have for each set  $F'$  in  $\mathcal{F}'$  a set  $\theta(F') \in \mathcal{F}$  such that  $L(P)(F') = \mu(\theta(F'))$ . On the other hand, for each  $F \in \mathcal{F}$ , we have a non-empty subset  $\theta^{-1}(\{F\})$  of  $\mathcal{F}'$  consisting of sets that are equal to  $L(P)$ -a.e. It is easy to prove that we have a similar correspondence for random variables. If  $f: \Omega \rightarrow \mathbb{R}$  is  $\mathcal{F}'$ -measurable, then there exists a uniquely determined  $f_\theta: Z \rightarrow \mathbb{R}$  such that for each  $\alpha \in \mathbb{R}$ ,  $\theta[f \geq \alpha] = [f_\theta \geq \alpha]$ . Conversely, if  $f: Z \rightarrow \mathbb{R}$  is  $\mathcal{F}$ -measurable there is a nonempty set of  $\mathcal{F}'$ -measurable functions  $f^\theta$  such that  $\theta[f^\theta \geq \alpha] = [f \geq \alpha]$ . Two such  $f^\theta$ 's are different only on a set which  $\theta$  maps on  $\emptyset$ , and which consequently has measure zero. Since  $\theta$  is measure-preserving we have  $\int f^\theta dL(P) = \int f d\mu$  and  $\int f dL(P) = \int f_\theta d\mu$ . If  $\{f_n\}$  is a sequence of random variables on  $\langle \Omega, \mathcal{F}', L(P) \rangle$  which converges a.e. to  $f$ , then  $\{(f_n)_\theta\}$  converges a.e. to  $f_\theta$ . If  $\{f_n\}$  is a sequence of random variables on  $\langle Z, \mathcal{F}, \mu \rangle$  that converges a.e. to  $f$ , then  $\{f_n^\theta\}$  converges a.e. to  $f^\theta$ . We shall use these simple facts about  $f_\theta$  and  $f^\theta$  in the sequel.

## 2. Weak Loeb-space representations of martingales.

In this section we use the representation of measure spaces found in Theorem 3, to represent arbitrary  $L^2$ -martingales by  $L^2$ -martingales on Loeb-

spaces. These are still real-valued martingales parameterized by  $\mathbb{R}_+$ , but in the next section we shall see how to replace them by hyperfinite martingales.

We first extend the notion of weak Loeb-space representations to stochastic bases:

DEFINITION 4. By a *weak Loeb-space representation* for a stochastic basis  $\langle Z, \{\mathcal{F}_t\}, \mu \rangle$ , we mean a sequence  $\langle \Omega, \{\mathcal{G}_t\}_{t \in \mathbb{R}_+}, P \rangle$  of internal probability spaces with  $\mathcal{G}_s \subset \mathcal{G}_t$  for  $s < t$ ; an increasing family  $\{\mathcal{F}'_t\}_{t \in \mathbb{R}_+ \cup \{\infty\}}$  of  $\sigma$ -algebras on  $\Omega$  with  $\mathcal{F}'_t \subset L(\mathcal{G}_t)$  for  $t \in \mathbb{R}_+$ ; and a measure-preserving  $\sigma$ -homomorphism  $\theta: \mathcal{F}'_\infty \rightarrow \mathcal{F}_\infty$  such that  $\theta$  is surjective and for each  $t \in \mathbb{R}_+ \cup \{\infty\}$ , the tuple  $\langle \Omega, \mathcal{G}_t, P \rangle, \mathcal{F}'_t, \theta|_{\mathcal{F}'_t}$  is a weak Loeb-space representation for  $\langle Z, \mathcal{F}_t, \mu \rangle$ .

Let  $M: \mathbb{R}_+ \times Z \rightarrow \mathbb{R}$  be an  $L^2$ -martingale adapted to the stochastic basis  $\langle Z, \{\mathcal{F}_t\}, \mu \rangle$ , and let  $\langle \Omega, \{\mathcal{G}_t\}, P \rangle, \{\mathcal{F}'_t\}, \theta$  be a weak Loeb-space representation of this basis. For each  $t \in \mathbb{R}_+$  we may find a random variable  $M_t^\theta: \Omega \rightarrow \mathbb{R}$  such that  $\theta[M_t^\theta \geq \alpha] = [M_t \geq \alpha]$  for all  $\alpha \in \mathbb{R}$ , and  $[M_t^\theta \geq \alpha] \in \mathcal{F}'_t$ . Since  $\theta$  is measure-preserving, the process  $M^\theta: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  defined by  $M^\theta(t, \omega) = M_t^\theta(\omega)$  is an  $L^2$ -martingale adapted to the stochastic basis  $\langle \Omega, \{\mathcal{F}'_t\}, L(P) \rangle$ .

DEFINITION 5. Let  $M: \mathbb{R}_+ \times Z \rightarrow \mathbb{R}$  be an  $L^2$ -martingale adapted to the stochastic basis  $\langle Z, \{\mathcal{F}_t\}, \mu \rangle$ , and let  $\langle \Omega, \{\mathcal{G}_t\}, P \rangle, \{\mathcal{F}'_t\}, \theta$  be a weak Loeb-space representation of  $\langle Z, \{\mathcal{F}_t\}, \mu \rangle$ . Then  $\langle \Omega, \{\mathcal{G}_t\}, P \rangle, \{\mathcal{F}'_t\}, \theta, M^\theta$  is a *weak Loeb-space representation of  $M$*  if  $M^\theta$  is an  $L^2$ -martingale with respect to  $\langle \Omega, \{L(\mathcal{G}_t)\}, L(P) \rangle$ .

Notice that the martingale  $M^\theta$  in Definition 5 is required to be adapted to the basis  $\langle \Omega, \{L(\mathcal{G}_t)\}, L(P) \rangle$  and not only to  $\langle \Omega, \{\mathcal{F}'_t\}, L(P) \rangle$ . We shall need this to be able to replace  $M^\theta$  by a hyperfinite martingale adapted to an extension of  $\langle \Omega, \{\mathcal{G}_t\}, P \rangle$ . But by this requirement it is no longer obvious that  $M$  has a weak Loeb-space representation; however, we shall prove in Theorem 7 that it does have one.

Let us make the connection to the setting of [8]: Let  $\{\mathcal{G}_t\}_{t \in T}$  be an extension of  $\{\mathcal{G}_t\}_{t \in \mathbb{R}_+}$  to an increasing sequence of internal algebras indexed by a hyperfinite time-line  $T$ . From [8, Lemma 5 and Lemma 6], we see that

$$\mathcal{H}_t = \bigcap_{n \in \mathbb{N}} \sigma(L(\mathcal{G}_{s_n}) \cup \mathcal{N})$$

for each sequence  $\{s_n\}$  of elements from  $T$  such that  $\{s_n\}$  decreases strictly to  $t$ . This tells us that the family  $\{\mathcal{H}_t\}$  does not change if we pass to a sub-line, and that it is uniquely determined by the original family  $\{\mathcal{G}_t\}_{t \in \mathbb{R}_+}$  and does not depend on the extension  $\{\mathcal{G}_t\}_{t \in T}$ .

LEMMA 6. Let  $M: \mathbb{R}_+ \times Z \rightarrow \mathbb{R}$  be a right-continuous  $L^2$ -martingale with a weak Loeb-space representation  $\langle \Omega, \{\mathcal{G}_t\}, P \rangle, \{\mathcal{F}'_t\}, \theta, M^0 \rangle$ . Then  $M^0$  is an  $L^2$ -martingale adapted to  $\langle \Omega, \{\mathcal{H}_t\}, L(P) \rangle$ .

PROOF. Let  $A \in \mathcal{H}_s$ , then by [8, Lemma 6] there exists a  $B$  such that  $B \in L(\mathcal{G}_{s+1/n})$  for all  $n \in \mathbb{N}$  and  $L(P)(A \Delta B) = 0$ . For  $t > s$  we get:

$$\begin{aligned} \int_A (M_t^0 - M_s^0) dL(P) &= \int_B (M_t^0 - M_s^0) dL(P) = \int_B \lim_{n \rightarrow \infty} (M_t^0 - M_{s+1/n}^0) dL(P) \\ &= \lim_{n \rightarrow \infty} \int_B (M_t^0 - M_{s+1/n}^0) dL(P) = 0, \end{aligned}$$

where we have used  $M_{s+1/n}^0 \rightarrow M_s^0$  since  $M$  is right-continuous, and the usual combination of Doob's inequality and Lebesgue's Convergence Theorem to get the limit outside the integral.

We now prove

THEOREM 7. If  $M$  is an  $L^2$ -martingale, then  $M$  has a weak Loeb-space representation.

PROOF. Let  $\langle Z, \{\mathcal{F}_t\}, \mu \rangle$  be the stochastic basis of  $M$ . If  $\mathcal{F}_\infty$  is finite, the result is obvious and we hence assume that  $\mathcal{F}_\infty$  is infinite.

For each  $t \in \mathbb{R}_+ \cup \{\infty\}$  let  $\mathcal{F}_t$  be the  $\sigma$ -algebra generated by the sets  $\{ *F : F \in \mathcal{F}_t \}$ . In the proof of Theorem 3 we saw that for each  $t$  there exists a measure-preserving  $\sigma$ -homomorphism  $h_t: \mathcal{F}_t \rightarrow \mathcal{F}_t$ , such that  $h_t(*F) = F$  for all  $F \in \mathcal{F}_t$ . If  $t > s$ , it is clear that  $h_t|_{\mathcal{F}_s} = h_s$  since they agree on the generator set  $\{ *F : F \in \mathcal{F}_s \}$ . Hence all  $h_t$  are obtainable from  $h_\infty$  and we shall write  $h$  for  $h_\infty$ .

For each  $t \in \mathbb{R}_+$  we may find an  $\mathcal{F}_t$ -measurable random variable  $M'_t: *Z \rightarrow \mathbb{R}$  such that  $h[M'_t \geq \alpha] = h_t[M'_t \geq \alpha] = [M_t \geq \alpha]$  for all  $\alpha \in \mathbb{R}$ . Since  $\mathcal{F}_t \subset L(*\mathcal{F}_t)$  we may find an internal  $K_t: *Z \rightarrow \mathbb{R}$  such that  $K_t \in SL^2(*Z, *\mathcal{F}_t, *\mu)$  and  ${}^\circ K_t = M'_t$   $L(*\mu)$ -a.e. for all  $t \in \mathbb{R}_+$ .

For all  $s, t \in \mathbb{R}_+$ ,  $s < t$ , and all  $A \in \mathcal{F}_s \cap *\mathcal{F}_s$ , we have:

$$\begin{aligned} 0 &\leq \left| \int_A (K_t - K_s) d*\mu \right| = \left| \int_A ({}^\circ K_t - {}^\circ K_s) dL(*\mu) \right| \\ &= \left| \int_A (M'_t - M'_s) dL(*\mu) \right| = \left| \int_{h(A)} (M_t - M_s) d\mu \right| = 0. \end{aligned}$$

Let now  $B_1, B_2, \dots, B_n$  be a finite family of sets from  $\mathcal{F}_\infty$ , and let

$\langle s_1, t_1 \rangle, \dots, \langle s_m, t_m \rangle$  be a finite set of pairs  $\langle s_i, t_i \rangle \in \mathbb{R}_+^2$  such that  $s_i < t_i$ . Then there exists a finite partition  $\mathcal{P}$  of  $*Z$  such that if  $P \in \mathcal{P}$  and  $P \cap *B_i \neq \emptyset$  for some  $i \in \{1, \dots, n\}$ , then  $P \subset *B_i$ ; and if  $A$  is in the algebra generated by  $\mathcal{P}$ , and  $A \in *F_{s_i}$  for some  $i \in \{1, \dots, m\}$ , then

$$\left| \int_A (K_{t_i} - K_{s_i}) d*\mu \right| \leq \frac{1}{*\text{card}(\mathcal{P})}$$

where  $*\text{card}(\mathcal{P})$  denotes the internal cardinality of  $\mathcal{P}$ . To see that this statement is true, take  $\mathcal{P}$  to be the partition generated by  $B_1, \dots, B_n$ , and use the inequality we proved above.

By polysaturation there then exists a hyperfinite partition  $\mathcal{P}$  of  $*Z$  such that for each  $B \in F_\infty$  if  $P \in \mathcal{P}$  and  $P \cap *B \neq \emptyset$ , then  $P \subset *B$ ; and if  $A$  is in the intersection of the internal algebra generated by  $\mathcal{P}$ , and  $*F_s$ , then for all  $t > s$

$$\left| \int_A (K_t - K_s) d*\mu \right| \leq \frac{1}{*\text{card} \mathcal{P}}.$$

Since  $F_\infty$  is infinite,  $*\text{card}(\mathcal{P}) \in *N \setminus N$ . Let  $\mathcal{A}$  be the internal algebra generated by  $\mathcal{P}$ , and for each  $t \in \mathbb{R}_+$  define

$$\mathcal{G}_t = \mathcal{A} \cap *F_t.$$

By definition of  $\mathcal{P}$ , we have  $*B \in \mathcal{A}$  for all  $B \in F_\infty$ , and consequently  $*F_t \subset L(\mathcal{G}_t)$ .

If  $\sim$  is the equivalence relation induced by  $\mathcal{P}$ , let  $\pi: *Z \rightarrow *Z/\sim$  be the quotient map. We write  $\Omega$  for  $*Z/\sim$ ;  $\mathcal{G}_t$  for  $\pi(\mathcal{G}_t)$ ; and  $P$  for  $\pi(*\mu)$ . We also put  $F'_t = \pi(*F_t)$  for  $t \in \mathbb{R}_+ \cup \{\infty\}$ , and let  $\theta = h \circ \pi^{-1}$ .

Then  $\{\langle \Omega, \{\mathcal{G}_t\}, P \rangle, \{F'_t\}, \theta\}$  is a weak Loeb-space representation of  $\langle Z, \{F_t\}, \mu \rangle$ . We must prove that  $M^\theta$  is a martingale with respect to the basis  $\langle \Omega, L(\mathcal{G}_t), L(P) \rangle$ :

If  $A \in L(\mathcal{G}_s)$ , then there exists  $B \in \mathcal{G}_s$  such that  $L(P)(A \Delta B) = 0$ . For  $t > s$  we have

$$\begin{aligned} 0 &\leq \left| \int_A (M_t^\theta - M_s^\theta) dL(P) \right| = \left| \int_B (M_t^\theta - M_s^\theta) dL(P) \right| \\ &= \left| \int_{\pi^{-1}(B)} (M'_t - M'_s) dL(*\mu) \right| = \left| \int_{\pi^{-1}(B)} ({}^\circ K_t - {}^\circ K_s) dL(*\mu) \right| \\ &= \left| {}^\circ \int_{\pi^{-1}(B)} (K_t - K_s) d*\mu \right| \leq {}^\circ \left( \frac{1}{*\text{card}(\mathcal{P})} \right) = 0, \end{aligned}$$

since  $\pi^{-1}(B) \in \mathcal{A} \cap *F_s$ . But this proves that  $M^\theta$  is an  $L^2$ -martingale with respect to the stochastic basis  $\langle \Omega, L(\mathcal{G}_t), L(P) \rangle$ , and hence the theorem.

If  $X$  is a predictable process with respect to  $\langle Z, \{\mathcal{F}_t\}, \mu \rangle$ , then we can find a predictable version of  $X^\theta$ , and if  $X$  is predictable with respect to  $\langle \Omega, \{\mathcal{F}'_t\}, L(P) \rangle$ , then  $X_\theta$  is predictable. If  $X \in \Lambda^2(M)$ , then  $X^\theta \in \Lambda^2(M^\theta)$  and for  $r \in \mathbf{R}_+$ :

$$\int_{[0,r] \times Z} X^2 dm_M = \int_{[0,r] \times \Omega} X^{\theta^2} dm_{M^\theta}.$$

We need these simple facts to prove the following commutation rules for  $\theta$  and stochastic integrals:

**PROPOSITION 8.** *Let  $M$  be a right-continuous  $L^2$ -martingale. If  $X \in \Lambda^2(M)$ , then  $X^\theta \in \Lambda^2(M)$  and  $(\int X dM)^\theta = \int X^\theta dM^\theta$ .*

*If  $Y$  is predictable with respect to  $\langle \Omega, \{\mathcal{F}'_t\}, L(P) \rangle$  and  $Y \in \Lambda^2(M^\theta)$ , then  $Y_\theta \in \Lambda^2(M)$  and  $\int Y dM^\theta = (\int Y_\theta dM)^\theta$ .*

**PROOF.** Since  $M$  is right-continuous,  $\lim_{n \rightarrow \infty} M^\theta(t+r_n) = M^\theta(t)$  for each sequence  $\{r_n\}$  decreasing to zero, and thus stochastic integration with respect to  $M^\theta$  is well-defined.

Assume first that  $X$  is of the form  $X = 1_{\langle s,t \rangle \times F}$  where  $F \in \mathcal{F}_s$ . Then  $X^\theta = 1_{\langle s,t \rangle \times F'}$ , where  $F' \in \theta^{-1}(F) \cap \mathcal{F}'_s$ , and we have

$$\int_0^r X^\theta dM^\theta = 1_{F'}(M_{t \wedge r}^\theta - M_{s \wedge r}^\theta) = \left( \int_0^r X dM \right)^\theta.$$

By linearity the assertion holds for all  $X$  of the form

$$X = \sum a_i 1_{\langle s_i, t_i \rangle \times F_i}, \quad F_i \in \mathcal{F}_{s_i}.$$

If  $X$  is an arbitrary element of  $\Lambda^2(M)$ , there exists a sequence  $\{X_n\}$  of elements of the above form such that

$$\lim_{n \rightarrow \infty} (L^2) \left( \int_0^r X_n dM \right) = \int_0^r X dM \quad \text{for all } r \in \mathbf{R}_+.$$

Thus

$$\lim_{n \rightarrow \infty} (L^2) \left( \int_0^r X_n^\theta dM^\theta \right) = \lim_{n \rightarrow \infty} (L^2) \left( \int_0^r X_n dM \right)^\theta = \left( \int_0^r X dM \right)^\theta.$$

Moreover, the elements  $\{X_n\}$  are chosen such that

$$\lim_{n \rightarrow \infty} \int_{[0,r] \times Z} (X - X_n)^2 dm_M = 0,$$

but since

$$\int_{[0,r] \times Z} (X - X_n)^2 dm_M = \int_{[0,r] \times \Omega} (X^\theta - X_n^\theta)^2 dm_{M^\theta}$$

we also have

$$\lim_{n \rightarrow \infty} \int_{[0,r] \times \Omega} (X^\theta - X_n^\theta)^2 dm_{M^\theta} = 0.$$

But then  $\int_0^r X^\theta dM^\theta = \lim (L^2) \int_0^r X_n^\theta dM^\theta$  and hence  $\int_0^r X^\theta dM^\theta = (\int_0^r X dM)^\theta$ .

The second part follows immediately from the first since  $(Y_\theta)^\theta = Y$ .

We have similar results for local  $L$ -martingales:

Let  $\mathcal{F}_t'' = (\mathcal{F}_t')^+ = \bigcap_{s>t} \mathcal{F}_s'$ . Then the following result was proved in [6]:

**PROPOSITION 9.** *Let  $M: \mathbb{R}_+ \times Z \rightarrow \mathbb{R}$  be a right continuous local  $L^2$ -martingale adapted to the basis  $\langle Z, \{\mathcal{F}_t\}, \mu \rangle$ , and let  $X \in \Lambda(M)$ . There exist a weak Loeb-space representation  $\langle \Omega, \{\mathcal{G}_t\}, P, \{\mathcal{F}_t'\}, \theta \rangle$  of  $\langle Z, \{\mathcal{F}_t\}, \mu \rangle$  and a version of  $M^\theta$  which is a local  $L^2$ -martingale with respect to  $\langle \Omega, \{\mathcal{H}_t\}, L(P) \rangle$ , and such that  $X^\theta \in \Lambda(M^\theta)$ . The localizing sequence of stopping times for  $X^\theta$  and  $M^\theta$  will be adapted to  $\langle \Omega, \{\mathcal{F}_t'\}, L(P) \rangle$ , and each  $M_t^\theta$  is  $\mathcal{F}_t''$ -measurable. Moreover,*

$$\left( \int X dM \right)^\theta = \int X^\theta dM^\theta.$$

Some extension of the algebras  $\mathcal{F}_t'$  (like  $\mathcal{F}_t''$ ) is necessary since the stopping times will not be adapted to  $\{\mathcal{F}_t'\}$ . We need not, however, use the whole of  $\mathcal{F}_t''$ ; it is enough to add its null-sets to  $\mathcal{F}_t'$ .

### 3. $L^2$ -martingales as right standard parts.

In the last section we saw how we could replace a given martingale by a martingale adapted to a basis of Loeb-algebras having the same properties of stochastic integration. In this section we shall go one step further by showing that each martingale of the latter type may be considered as the right standard part of a hyperfinite martingale.

Before proceeding, the reader should recall the comments on the family  $\langle \Omega, \{\mathcal{G}_t\}, P \rangle$  preceding Lemma 6.

**THEOREM 10.** *Let  $\langle \Omega, \{\mathcal{G}_t\}_{t \in \mathbb{R}_+}, P \rangle$  be an increasing family of hyperfinite probability spaces. Let  $N: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$  be an  $L^2$ -martingale with respect to  $\langle \Omega, \{\mathcal{H}_t\}, L(P) \rangle$  and assume that  $N_0$  is  $\sigma(L(\mathcal{G}_0) \cup \mathcal{N})$ -measurable. Assume further that for each  $t \in \mathbb{R}_+$  and each sequence  $\{t_n\}_{n \in \mathbb{N}}$  decreasing to  $t$  we have  $N(t_n) \rightarrow N(t)$   $L(P)$ -a.e.*



Then there exist a hyperfinite time-line  $T$ , an internal basis  $\langle \Omega, \{\mathcal{G}_t\}_{t \in T}, P \rangle$  extending  $\langle \Omega, \{\mathcal{G}_t\}_{t \in \mathbb{R}_+}, P \rangle$ , and an  $SL^2$ -martingale  $M: T \times \Omega \rightarrow {}^*\mathbb{R}$  adapted to  $\langle \Omega, \{\mathcal{G}_t\}_{t \in T}, P \rangle$  such that for each  $t \in \mathbb{R}_+$

$${}^\circ M^+(t, \omega) = N(t, \omega) \quad \text{for } L(P)\text{-almost all } \omega .$$

Moreover, we may take  $M$  to be well-behaved,  $S$ -right-continuous at  $0$ , and such that for all  $t \in \mathbb{R}_+$

$${}^\circ[M]^+(t, \omega) = [N](t, \omega) \quad \text{for } L(P)\text{-almost all } \omega .$$

PROOF. By saturation we may extend the family  $\{\mathcal{G}_t\}_{t \in \mathbb{R}_+}$  to an increasing internal family  $\{\mathcal{G}_t\}_{t \in S}$ , where  $S$  is a hyperfinite time-line.

For each  $t \in \mathbb{R}_+$ , the random variable  $N_t: \Omega \rightarrow \mathbb{R}$  is  $\mathcal{H}_t$ -measurable and thus it is  $\sigma(L(\mathcal{G}_s) \cup \mathcal{N})$ -measurable for all  $s > t$ ,  $s \in \mathbb{R}_+$ . Consequently there is an  $L_t^{(s)} \in SL^2(\Omega, \mathcal{G}_s, P)$  such that  ${}^\circ L_t^{(s)} = N_t$   $L(P)$ -a.e. We may extend the sequence  $\{L_t^{(t+1/n)}\}_{n \in \mathbb{N}}$  by saturation to an internal sequence  $\{L_t^{(t+1/\gamma)}\}_{\gamma \leq \eta}$ ,  $\eta \in {}^*\mathbb{N} \setminus \mathbb{N}$ , where  $L_t^{(t+1/\gamma)}$  is  $\mathcal{G}_{(t+1/\bar{\gamma})}$ -measurable (recall that  $\bar{r}$  is the least element in  $S$  larger than  $r$ ). There must be an internal initial segment  $\{L_t^{(t+1/\gamma)}\}_{\gamma \leq \xi}$  such that

$$\|L_t^{(t+1/\gamma)} - L_t^{(t+1)}\|_2 \leq 1/\gamma ,$$

for all  $\gamma \leq \xi$ . By construction  $\xi$  must be infinite and it follows that  ${}^\circ L_t^{(t+1/\xi)} = N_t$   $L(P)$ -a.e.;  $L_t^{(t+1/\xi)} \in SL^2(\Omega, \mathcal{G}_{(t+1/\bar{\xi})}, P)$  and  $t + 1/\bar{\xi} \approx t$ .

We denote  $(t+1/\bar{\xi})$  by  $\hat{t}$  and choose  $\hat{0} = 0$ ; this is possible since we have assumed that  $N_0$  is  $\sigma(L(\mathcal{G}_0) \cup \mathcal{N})$ -measurable.

Given a finite set  $\hat{S} = \{\hat{t}_1, \dots, \hat{t}_n\}$  of such elements we show how to turn the process  $L_t^{(\hat{t}_i)}$  into an  $SL^2$ -martingale with respect to the basis  $\langle \Omega, \{\mathcal{G}_t\}_{t \in \hat{S}}, P \rangle$ ; i.e. we construct an  $SL^2$ -martingale  ${}^S M: \hat{S} \times \Omega \rightarrow \mathbb{R}$  adapted to  $\langle \Omega, \{\mathcal{G}_t\}_{t \in \hat{S}}, P \rangle$  such that

$$\|{}^S M(\hat{t}_i) - L_t^{(\hat{t}_i)}\|_2 \approx 0 \quad \text{for each } \hat{t}_i \in \hat{S}:$$

Assume that  $\hat{t}_1 < \hat{t}_2 < \dots < \hat{t}_n$ ; we define  ${}^S M(\hat{t}_1) = L_t^{(\hat{t}_1)}$ . If we have constructed  ${}^S M(\hat{t}_j)$  for  $j \leq i$ , we define  ${}^S M(\hat{t}_{i+1})$  in the following way: For each  $\omega \in \Omega$ , let

$$[\omega]_i = \bigcap \{A \in \mathcal{G}_{\hat{t}_i} : \omega \in A\} ,$$

and define  $B$  by:

$$B = \left\{ \gamma \in {}^*\mathbb{N} : P \left\{ \omega : \left| \int_{[\omega]_i} (L_{\hat{t}_{i+1}}^{(\hat{t}_{i+1})} - {}^S M(\hat{t}_i)) dP \right| > 1/\gamma P([\omega]_i) \right\} < 1/\gamma \right\} .$$

Then  $B$  is an internal set and we prove that  $\mathbb{N} \subset B$ :

Assume  $n \in \mathbb{N}$ , but  $n \notin B$ : Then either

$$P\left\{\omega : \int_{[\omega]_i} (L_{t_{i+1}}^{\hat{t}_{i+1}} - {}^S M(\hat{t}_i)) dP > 1/nP([\omega]_i)\right\} \geq 1/2n$$

or

$$P\left\{\omega : \int_{[\omega]_i} (L_{t_{i+1}}^{\hat{t}_{i+1}} - {}^S M(\hat{t}_i)) dP < -1/nP([\omega]_i)\right\} \geq 1/2n .$$

Assume the first; the argument in the second case is similar. The set

$$C = \left\{\omega : \int_{[\omega]_i} (L_{t_{i+1}}^{\hat{t}_{i+1}} - {}^S M(\hat{t}_i)) dP > 1/nP[\omega]_i\right\}.$$

is in  $\mathcal{G}_{\hat{t}_i}$  and  $P(C) \geq 1/2n$ . We have

$$\begin{aligned} 1/2n^2 &\leq \int_C (L_{t_{i+1}}^{\hat{t}_{i+1}} - {}^S M(\hat{t}_i)) dP = \int_C ({}^\circ L_{t_{i+1}}^{\hat{t}_{i+1}} - {}^\circ {}^S M(\hat{t}_i)) dL(P) \\ &= \int_C (N_{t_{i+1}} - N_{t_i}) dL(P) = 0, \end{aligned}$$

which is impossible. Hence  $N \subset B$ , and since  $B$  is internal we may find an infinite  $\gamma \in B$ .

Let  $D$  be the set of internal measure less than  $1/\gamma$  such that

$$\left| \int_{[\omega]_i} (L_{t_{i+1}}^{\hat{t}_{i+1}} - {}^S M(\hat{t}_i)) dP \right| \geq 1/\gamma P[\omega]_i \quad \text{for } \omega \in D ;$$

then  $D \in \mathcal{G}_{\hat{t}_i}$ . For  $\omega \in D$ , we define  ${}^S M(\hat{t}_{i+1}) = {}^S M(\hat{t}_i)$ . For  $\omega \in \Omega \setminus D$ , let

$${}^S M(\hat{t}_{i+1}, \omega) = L_{t_{i+1}}^{\hat{t}_{i+1}}(\omega) - \int_{[\omega]_i} (L_{t_{i+1}}^{\hat{t}_{i+1}} - {}^S M(\hat{t}_i)) dP / P[\omega]_i .$$

It follows immediately that for each  $\omega \in \Omega$ ,

$$\int_{[\omega]_i} ({}^S M(\hat{t}_{i+1}) - {}^S M(\hat{t}_i)) dP = 0 ,$$

and hence  ${}^S M$  is a martingale. Since  ${}^S M(\hat{t}_i) \in SL^2(\Omega, \mathcal{G}_{\hat{t}_i}, P)$  by induction hypothesis, we have

$$\int_D {}^S M(\hat{t}_{i+1})^2 dP = \int_D {}^S M(\hat{t}_i)^2 dP \approx 0 .$$

We also have

$$\int_{\Omega \setminus D} |{}^S M(\hat{t}_{i+1}) - L_{t_{i+1}}^{\hat{t}_{i+1}}|^2 dP \leq 1/\gamma^2 ,$$

and thus  $\hat{S}M(\hat{t}_{i+1}) \in SL^2(\Omega, \mathcal{G}_{\hat{t}_{i+1}}, P)$ . Since obviously

$${}^\circ\hat{S}M(\hat{t}_{i+1}) = {}^\circ L_{\hat{t}_{i+1}}^{\hat{t}_{i+1}} = N_{\hat{t}_{i+1}} \quad \text{a.e. ,}$$

we have constructed the martingale  $\hat{S}M$  by induction.

We now turn to the next step of the proof: So far we have only constructed approximating martingales on finite time-lines, we must extend this to hyperfinite time-lines: Let  $\{q_n\}_{n \in \mathbf{N}}$  be an enumeration of the non-negative rationals, and let  $\hat{S}_n = \{\hat{q}_1, \dots, \hat{q}_n\}$ . We may extend the sequence  $\{\hat{S}_n\}_{n \in \mathbf{N}}$  to an increasing internal sequence  $\{\hat{S}_n\}_{n \leq v}$  of hyperfinite subsets of  $S$ , and the sequence  $\{\hat{S}_n M\}_{n \in \mathbf{N}}$  to a sequence of internal martingales adapted to  $\langle \Omega, \{\mathcal{G}_t\}_{t \in \hat{S}_n}, P \rangle$ .

If  $m, n \in \mathbf{N}$ ,  $m < n$ , then for all  $\hat{q}_k$ ,  $k \leq m$ , we have:

$$\|\hat{S}_n M(\hat{q}_k) - \hat{S}_m M(\hat{q}_k)\|_2 < 1/m .$$

By saturation there exists an infinite  $\eta$  such that

$$\|\hat{S}_n M(\hat{q}_k) - \hat{S}_\eta M(\hat{q}_k)\|_2 < 1/\eta$$

for all  $n \in \mathbf{N}$ ,  $k \leq n$ . It follows that  $\hat{S}_\eta M(\hat{q}_k) \in SL^2(\Omega, \mathcal{G}_{\hat{q}_k}, P)$  and  ${}^\circ\hat{S}_\eta M(\hat{q}_k) = N(q_k)$ ,  $L(P)$ -a.e., for  $q_k \in \mathbf{Q}$ .

Using this and the assumption that  $N$  is right continuous we have  ${}^\circ(\hat{S}_\eta M)^+ = N$  a.e.  $\hat{S}_\eta M$  is right continuous at 0, since  $\hat{0} = 0$ .

By letting  $T$  be a suitable sub-line of  $\hat{S}_\eta$ , we can make the restriction of  $\hat{S}_\eta M$  to  $T$  well-behaved by [8, Theorem 23]. Let  $M$  be this restriction, then by [8, Theorem 21]  ${}^\circ[M]^+ = [N]$ . This proves the theorem.

The assumption that  $N_0$  is  $\sigma(L(\mathcal{G}_0) \cup \mathcal{N})$ -measurable was made in Theorem 10 to ensure us that we could choose  $M$  right-continuous at 0. We saw in [8, Theorem 17] that this was a necessary condition when we compared stochastic integration with respect to  $M$  with stochastic integration with respect to  ${}^\circ M^+$ .

Using Theorem 10 one may prove the following analogous statement for local  $L^2$ -martingales:

**THEOREM 11.** *Let  $\langle \Omega, \{\mathcal{G}_t\}_{t \in \mathbf{R}_+}, P \rangle$  be an increasing sequence of hyperfinite probability spaces. Let  $N: \mathbf{R}_+ \times \Omega \rightarrow \mathbf{R}$  be a right continuous local  $L^2$ -martingale with respect to  $\langle \Omega, \{\mathcal{H}_t\}, L(P) \rangle$ , such that  $N_0$  is  $\sigma(L(\mathcal{G}_0) \cup \mathcal{N})$ -measurable. Then there exists a local  $SL^2$ -martingale  $M: T \times \Omega \rightarrow {}^*\mathbf{R}$  adapted to an extension  $\langle \Omega, \{\mathcal{G}_t\}_{t \in T}, P \rangle$  of  $\langle \Omega, \{\mathcal{G}_t\}_{t \in \mathbf{R}_+}, P \rangle$  such that for all  $t \in \mathbf{R}_+$ :*

$${}^\circ M^+(t, \omega) = N(t, \omega) \quad \text{for } L(P)\text{-almost all } \omega .$$

Moreover, we may assume  $M$  to be right continuous at 0 and such that for all  $t \in \mathbf{R}_+$ :

$${}^\circ[M]^+(t, \omega) = [N](t, \omega) .$$

I don't know if we can make the martingale  $M$  in Theorem 11 well-behaved; application of [8, Theorem 23] as in the proof of Theorem 10 is not possible since the restriction may fail to be a local  $SL^2$ -martingale (see [8, Example 10 and the comments following the proof of Theorem 23]). The equality  ${}^\circ[M]^+ = [N]$  can be proved directly without using [8, Theorem 21].

**4. Hyperfinite representation of martingales.**

Time has come to gather our results. In Section 2 we found a representation of arbitrary  $L^2$ -martingales as  $L^2$ -martingales on Loeb-spaces, and in the last section we saw how we could represent martingales of the latter type as right standard parts of  $SL^2$ -martingales. From [8, Theorem 17] we know the relationship between stochastic integration with respect to an  $SL^2$ -martingale and stochastic integration with respect to its right standard part. The following definition seems natural:

**DEFINITION 12.** By a *weak hyperfinite representation* of a martingale  $N$  we mean a weak Loeb-space representation  $\{\langle \Omega, \{\mathcal{G}_t\}, P \rangle, \{\mathcal{F}'_t\}, \theta, N^\theta\}$  and a hyperfinite martingale  $M$  adapted to some extension  $\langle \Omega, \{\mathcal{G}_t\}_{t \in T}, P \rangle$  of  $\langle \Omega, \{\mathcal{G}_t\}_{t \in R_+}, P \rangle$  such that  ${}^\circ M^+$  is equivalent to  $N^\theta$ .

Combining the results mentioned above, we get:

**THEOREM 13.** *Let  $N$  be a right continuous  $L^2$ -martingale. Then  $N$  has a weak hyperfinite representation  $M$  which is a well-behaved  $SL^2$ -martingale  $S$ -right continuous at 0, such that  ${}^\circ[M]^+ = [N]^\theta$ . If  $X \in \Lambda^2(N)$  then there exists a  $Y \in SL^2(M)$  such that  $\int Y dM$  is a hyperfinite representation of  $\int X dN$ .*

**PROOF.** By Theorem 7,  $N$  has a weak Loeb-space representation  $\{\langle \Omega, \{\mathcal{G}_t\}, P \rangle, \{\mathcal{F}'_t\}, \theta, N^\theta\}$ . By Lemma 6,  $N^\theta$  satisfies the conditions of Theorem 10, and by that theorem we obtain a weak hyperfinite representation  $M$  of  $N$ , which is well-behaved,  $S$ -right continuous at 0, and such that  ${}^\circ[M]^+ = [N^\theta] = [N]^\theta$ .

If  $X \in \Lambda^2(N)$ , then  $X^\theta \in \Lambda^2(N^\theta)$  and  $(\int X dN)^\theta = \int X^\theta dN^\theta$  by Proposition 8. By [8, Lemma 15 and Theorem 17],  $X^\theta$  has a 2-lifting  $Y$  with respect to  $M$  such that

$${}^\circ \left( \int Y dM \right)^+ = \int X^\theta d{}^\circ M^+ = \int X^\theta dN^\theta = \left( \int X dN \right)^\theta.$$

Thus  $\int Y dM$  is a hyperfinite representation of  $\int X dN$ , and the theorem is proved.

Using [8, Corollary 18] and Theorem 11, one may prove the corresponding result for local  $L^2$ -martingales:

**THEOREM 14.** *Let  $N$  be a right continuous local  $L^2$ -martingale and let  $X \in \mathcal{A}(N)$ . Then there exists a weak hyperfinite representation  $M$  of  $N$  which is a local  $SL^2$ -martingale,  $S$ -right continuous at 0, and with  ${}^\circ[M]^+ = [N]^{\theta}$ . Moreover, there is a  $Y \in SL(M)$  such that  $\int Y dM$  is a hyperfinite representations of  $\int X dN$ .*

Our purpose in this paper has been to show that everything that can be done by the standard theory of stochastic integration, can also be done by the nonstandard theory. Theorems 13 and 14 show this, since by using them we can define stochastic integration with respect to any right-continuous local  $L^2$ -martingale from the nonstandard stochastic integral with respect to a closely related local  $SL^2$ -martingale. Thus by using nonstandard methods we do not condemn our theory to less generality than what we could obtain by sticking to the standard methods.

**5. The transformation formula revisited.**

As an application of the theory for weak hyperfinite representations, we shall use the nonstandard version of the transformation formula ([7, Theorem 22]) to prove the standard version. Let us first prove it for the right standard part  ${}^\circ M^+$  of a well-behaved  $SL^2$ -martingale  $M$  which is  $S$ -right continuous in 0 and such that  ${}^\circ[M]^+ = [{}^\circ M^+]$ , and then use Theorem 13 to prove it in general: If  $\varphi: \mathbb{R} \rightarrow \mathbb{R}$  is in  $\mathcal{C}^2(\mathbb{R})$ , we get from [7, Theorem 22]:

$$\begin{aligned} \varphi({}^\circ M^+(t)) - \varphi({}^\circ M^+(0)) &= \lim_{n \rightarrow \infty} ({}^*\varphi(M(t+1/n)) - {}^*\varphi(M_0)) \\ &= \lim_{n \rightarrow \infty} \left( \int_0^{t+1/n} {}^*\varphi'(M) dM + \frac{1}{2} \int_0^{t+1/n} {}^*\varphi''(M) d[M] \right. \\ &\quad + \sum_{s \leq t+1/n} ({}^*\varphi(M_s) - {}^*\varphi(M_{s-}) - {}^*\varphi'(M_{s-})(M_s - M_{s-}) \\ &\quad \left. - \frac{1}{2} {}^*\varphi''(M_{s-})(M_s - M_{s-})^2) \right). \end{aligned}$$

By [8, Lemma 20],  ${}^*\varphi'(M)$  is a lifting of  $\varphi'({}^\circ M^-)$  (where  ${}^\circ M^-$  is the left standard part of  $M$ ), and consequently

$$\lim_{n \rightarrow \infty} \int_0^{t+1/n} {}^*\varphi'(M) dM = \int_0^t \varphi'({}^\circ M^-) d{}^\circ M^+.$$

Using that  $M$  is well-behaved,  $S$ -right continuous at 0 and such that  ${}^\circ[M]^+ = [{}^\circ M^+]$ , it is not difficult to see that

$$\lim_{n \rightarrow \infty} \int_0^{t+1/n} \circ * \varphi''(M) d[M] = \int_0^t \varphi''(\circ M^-) d[\circ M]^+ .$$

Since  $M$  is well-behaved, the jumps of  $\circ M^+$  are exactly the noninfinitesimal jumps of  $M$ , and hence

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{s \leq t+1/n} \circ (*\varphi(M_s) - *\varphi(M_{s-}) - *\varphi'(M_{s-})(M_s - M_{s-}) \\ & \quad - \frac{1}{2} *\varphi''(M_{s-})(M_s - M_{s-})^2) \\ &= \sum_{s \leq t} (\varphi(\circ M_s^+) - \varphi(\circ M_s^-) - \varphi'(\circ M_s^-)(\circ M_s^+ - \circ M_s^-) \\ & \quad - \frac{1}{2} \varphi''(\circ M_s^-)(\circ M_s^+ - \circ M_s^-)^2) . \end{aligned}$$

Combining these results we have:

$$\begin{aligned} \varphi(\circ M^+(t)) - \varphi(\circ M^+(0)) &= \int_0^t \varphi'(\circ M^-) d\circ M^+ + \frac{1}{2} \int_0^t \varphi''(\circ M^-) d[\circ M^+] \\ &+ \sum_{s \leq t} (\varphi(\circ M_s^+) - \varphi(\circ M_s^-) - \varphi'(\circ M_s^-)(\circ M_s^+ - \circ M_s^-) - \frac{1}{2} \varphi''(\circ M_s^-)(\circ M_s^+ - \circ M_s^-)^2) \end{aligned}$$

which is the transformation formula for  $\circ M^+$ .

Let now  $N$  be a right-continuous  $L^2$ -martingale with left limits. To prove the transformation formula for  $N$ , we find a weak hyperfinite representation  $M$  of  $N$  as in Theorem 13. If  $N^-$  denotes the left limit of  $N$ ,  $N^-$  is predictable, and we know from Theorem 13 that  $(\int_0^t \varphi'(\circ M^-) d\circ M^+)_{\theta} = \int_0^t \varphi'(N^-) dN$ .

It is easy to see that  $(\int_0^t \varphi''(\circ M^-) d[\circ M^+]_{\theta} = \int_0^t \varphi''(N^-) d[N]$  by approximating  $\varphi''(\circ M^-)$  by the predictable processes

$$X_n = \sum_{k \geq 0} 1_{\langle k/2^n, k+1/2^n \rangle} \varphi''(\circ M_{k/2^n}^-) ,$$

and  $\varphi''(N^-)$  by the processes

$$Y_n = \sum_{k \geq 0} 1_{\langle k/2^n, k+1/2^n \rangle} \varphi''(N_{k/2^n}^-) :$$

Since  $\varphi''(\circ M^-)$  is left-continuous,  $X_n \rightarrow \varphi''(\circ M^-)$  everywhere, and similarly  $Y_n \rightarrow \varphi''(N^-)$ . Now

$$\begin{aligned} \left( \int_0^t \varphi''(\circ M^-) d[\circ M^+] \right)_{\theta} &= \left( \lim_{n \rightarrow \infty} \int_0^t X_n d[\circ M^+] \right)_{\theta} = \lim_{n \rightarrow \infty} \left( \int_0^t X_n d[\circ M^+] \right)_{\theta} \\ &= \lim_{n \rightarrow \infty} \left( \int_0^t Y_n d[N] \right) = \int_0^t \varphi''(N^-) d[N] . \end{aligned}$$

It remains to show that

$$\left( \sum_{s \leq t} (\varphi(\circ M_s^+) - \varphi(\circ M_s^-) - \varphi'(\circ M_s^-)(\circ M_s^+ - \circ M_s^-) - \frac{1}{2} \varphi''(\circ M_s^-)(\circ M_s^+ - \circ M_s^-)^2) \right)_\theta$$

$$= \sum_{s \leq t} (\varphi(N_s) - \varphi(N_s^-) - \varphi'(N_s^-)(N_s - N_s^-) - \frac{1}{2} \varphi''(N_s^-)(N_s - N_s^-)^2).$$

This we shall leave to the reader, only warning him that it includes checking of some rather irritating details. Hence we have proved:

**THEOREM 15. (The transformation formula).** *Let  $N$  be a right continuous local  $L^2$ -martingale with left limits, and let  $\varphi: \mathbb{R} \rightarrow \mathbb{R}$  be in  $\mathcal{C}^2(\mathbb{R})$ . Then for all  $t \in \mathbb{R}_+$ :*

$$\varphi(N_t) - \varphi(N_0) = \int_0^t \varphi'(N^-) dN + \frac{1}{2} \int_0^t \varphi''(N^-) d[N] +$$

$$+ \sum_{s \leq t} (\varphi(N_s) - \varphi(N_s^-) - \varphi'(N_s^-)(N_s - N_s^-) - \frac{1}{2} \varphi''(N_s^-)(N_s - N_s^-)^2) \quad \text{a.e.};$$

and where the sum is absolutely convergent.

**COROLLARY 16. (Itô's formula).** *Let  $N$  be a continuous local  $L^2$ -martingale, and let  $\varphi: \mathbb{R} \rightarrow \mathbb{R}$  be in  $\mathcal{C}^2(\mathbb{R})$ . Then:*

$$\varphi(N_t) - \varphi(N_0) = \int_0^t \varphi'(N) dN + \frac{1}{2} \int_0^t \varphi''(N) d[N].$$

We may now prove the result we used in the proof of [8, Theorem 21]:

**COROLLARY 17.** *Let  $N$  be a right continuous local  $L^2$ -martingale with left limits. Then for  $t \in \mathbb{R}_+$ :*

$$[N](t) = N(t)^2 - N(0)^2 - 2 \int_0^t N^- dN \quad \text{a.e. .}$$

**PROOF.** Apply Theorem 15 with  $\varphi: x \rightarrow x^2$ :

$$N_t^2 - N_0^2 = \int_0^t 2N^- dN + \frac{1}{2} \int_0^t 2 d[N] + \sum_{s \leq t} (N_s^2 - N_s^{-2} - 2N_s^-(N_s - N_s^-)$$

$$- \frac{1}{2} \cdot 2(N_s - N_s^-)^2).$$

Performing the multiplications in each term inside the sum we get zero, and hence we are left with  $N_t^2 - N_0^2 = 2 \int_0^t N^- dN + [N](t)$ , which proves the corollary.

Notice that the proof we have given of Corollary 17 is circular; we used Corollary 17 to prove [8, Theorem 21] which we used to prove Theorem 10 which we used to prove Theorem 13 which we used to prove Theorem 15 and hence Corollary 17. This also makes the proof of the transformation formula circular, since Métivier used this formula to prove Corollary 17. Luckily, it is not difficult to avoid this circularity since we can prove Theorem 10 without using [8, Theorem 21]:

The argument would proceed as in the proof we have given up to the use of [8, Theorem 21]; we thus have a well-behaved  $SL^2$ -martingale  $M$  which is  $S$ -continuous at 0, such that  ${}^{\circ}M^+ = N$ . Using the definition of  $[N]$  as the  $L^1$ -limit of  $\sum_{t_i \in \pi} (N(t_{i+1}) - N(t_i))^2$  as the partition  $\pi$  gets finer, it is easy to find a restriction  $\tilde{M}$  of  $M$  to a subline such that  ${}^{\circ}[\tilde{M}]^+ = [N]$ . This martingale  $\tilde{M}$  would then satisfy Theorem 10.

As we have now proved Corollary 17 (and [8, Theorem 21]), the theory developed in these three papers should be reasonably self-contained. In addition to some simple facts about the paths of real-valued martingales, we have only used the most basic results from the standard theory for stochastic integration. We have therefore developed by purely nonstandard methods a theory for stochastic integration just as powerful as the standard theory. One may also hope that the discreteness of the hyperfinite timeline and the simplicity of the nonstandard definition of the stochastic integral will make the hyperfinite theory easier and more intuitive to work with; some evidence of this may already be seen in Keisler's theory for stochastic differential equations [5].

NOTE ADDED IN PROOF. Some of the results of this paper have been obtained independently by D. N. Hoover and E. Perkins.

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