MATH SCAND. 47 (1980), 5 20

ON THE DEVIATIONS OF A LOCAL RING

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Introduction.

Let R denote a local ring (always commutative, noetherian with residue field $k=R/\mathfrak{m}$) and let P_R be the Poincaré series of R, i.e. the power series

$$P_R(t) = \sum_q \dim_k \operatorname{Tor}_q^R(k,k)t^q .$$

It is well known that P_R may be written as a product of the form

$$(1) \qquad P_R(t) = \prod_{i=0}^{\infty} \frac{(1+t^{2i+1})^{\varepsilon_{2i}}}{(1-t^{2i+2})^{\varepsilon_{2i+1}}}$$

where $\varepsilon_q(R)=\varepsilon_q$ $(q=0,1,\ldots)$ are non-negative integers. These integers are called the *deviations* of R. ε_0 equals the embedding dimension of R. It is well known that R is regular if and only if $\varepsilon_1=0$ and that R is a complete intersection if and only if one of the following equivalent conditions are satisfied

(i) $\varepsilon_2=0$
(ii) $\varepsilon_3=0$
(iii) $\varepsilon_q=0$ for $q\geqq 2$
(iv) $\varepsilon_q=0$ for all q sufficiently large. Cf. [4].

We recall that R is said to be a complete intersection if the $\mathfrak{m}$-adic completion $\hat{R}$ is isomorphic to a quotientring of a regular local ring by an ideal generated by a regular sequence.

In section 1 of the present paper we shall prove a technical result 1.5 that makes it possible to replace (iv) above with the following condition

(iv′) $\varepsilon_q=0$ for all *odd* q sufficiently large.

This result is used in section 2 to yield a characterization of local complete intersections which is similar to the characterization of regular local rings in terms of projective dimension. We replace the class of modules of finite projective dimension by the class of modules whose Betti numbers are bounded

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Received June 15, 1979.

by a polynomial. The characterization yields a new proof of the fact that the class of local complete intersections is closed with respect to localizations.

In section 3 we investigate how the alternating sums

$$\chi_r(R) = \sum_{i=1}^{r} (-1)^i \varepsilon_i(R) \qquad (r \geqq 0)$$

behave with respect to localization. Under "mild" conditions on R, cf. 3.4, we obtain

$$\chi_r(R) \leqq \dim R \qquad \text{for } r \text{ odd}$$

and

$$\operatorname{depth} R \leqq \chi_r(R) \qquad \text{for } r \text{ even, } r \geqq 0$$

Moreover, if $\varepsilon_{r+1}(R)=0$ for some $r \geqq 0$ then

$$\operatorname{depth} R \leqq \chi_r(R) \leqq \dim R .$$

It is easily seen that P_R can be written in the form

$$P_R(t) = \frac{(1+t)^{\varepsilon_0}}{\prod_{j\geqq 2}(1-t^j)^{f_j}} \tag{2}$$

where $f_j(R)=f_j$ $(j=2,3,\ldots)$ are integers. André [1] has shown that if k has characteristic 2, then $f_j \geqq 0$ for all $j \geqq 2$; in other words

$$\varepsilon_{2q-1} \geqq \varepsilon_{q-1} \qquad \text{for all odd } q+1$$

as seen by comparing with (1). In section 1 of the present paper we have included a new proof of these inequalities—still assuming k to have characteristic 2. However, one may ask if they are valid in general. In section 4 we shall show that several classes of rings satisfy the following even stronger property:

(*) If R is not a complete intersection, then $f_q(R)>0$ for all $q \geqq 2$.

0. Preliminaries.

Concerning notation and terminology we will use [6] as a reference. In addition to the notation ε_q $(q \geqq 0)$ for the deviation we will also be using the notation e_i $(i \geqq 1)$ as in [1]. The translation is $e_i = \varepsilon_{i-1}$ for $i \geqq 1$.

$$X = R\langle \ldots S_i \ldots ; dS_i = s_i \rangle$$

denotes the R-algebra obtained from R by Tate's method of adjoining variables to kill cycles. X has a structure of a strictly skew-commutative DG-algebra over R with a system of divided powers compatible with the differential. In particular, for a homogeneous element x of degree $q>0$ we have for q even

$$d(x^{(j)}) = d(x)x^{(j-1)}$$

and

$$x^j = j!\,x^{(j)}\,.$$

For q odd we have $x^2=0$.

Let IX denote the augmentation ideal in X with respect to the canonical augmentation $X \twoheadrightarrow k$.
Put

$$Q(X) = IX/I^{(2)}X$$

where $I^{(2)}X$ denotes the ideal in X generated by $(IX)^2$ and all divided powers of elements in IX of even degree. With the canonical differential d, $Q(X)$ becomes a complex of vectorspaces over k.

If $R \to \bar{R}$ is a ring homomorphism inducing an isomorphism $H_0(X) \cong \bar{R}$, and if $H_q(X)=0$ for $q \geqq 1$. Then X is called an *R-algebra resolution of $\bar{R}$*. X is called an *acyclic closure* of $R \to \bar{R}$ if in addition it satisfies a certain minimality condition that is equivalent to $d_q\colon Q(X)_q \to Q(X)_{q-1}$ being zero for $q \geqq 2$. Given $R \to \bar{R}$, such an acyclic closure always exists by [6; 1.9.3].

Put $\pi(R)=2p$ if k has characteristic $p>0$, and put $\pi(R)=\infty$ if k has characteristic 0.

0.1 LEMMA. *Let $\mathfrak{a}$ be an ideal in a regular ring $\tilde{R}$. Put $R=\tilde{R}/\mathfrak{a}$ and let Y be an $\tilde{R}$-algebra resolution of R. Then we have*

(i) $\dim Q(Y)_0 = \varepsilon_0(\tilde{R})$,
(ii) $\dim H_q(Q(Y)) = \varepsilon_q(R)$ *for* $q<\pi(R)-1$,
(iii) *If $\tilde{R}$ and R have the same embedding dimension and Y is an acyclic closure of $\tilde{R} \to R$, then*

$$\dim Q(Y)_q = \varepsilon_q(R) \quad \textit{for } q<\pi(R)-1\,.$$

PROOF. (i) is obvious from the definition of $Q(Y)$ since $\varepsilon_0(\tilde{R})$ is the embedding dimension of $\tilde{R}$.

(ii). In [6; 3.1.1] we introduced k-vector spaces $V_q(\tilde{R},R)$ with dimension $v_q(\tilde{R},R)$ $(q>0)$. By [6; 3.2.3] we have isomorphisms of vector spaces

$$V_q(\tilde{R},R) \cong H_q(Q(Y)) \quad \text{for } 1<q<\pi(R)$$
$$V_1(\tilde{R},R) \cong \operatorname{coker}(d_2\colon Q(Y)_2 \to Q(Y)_1)\,.$$

Indeed, the complex $Y/D(Y)$ in [6; 3.2.3] coincides with the complex $Q(Y)$ in positive degrees, but is zero in degree zero. Now by [6; 3.3.2] we have

$$\varepsilon_q(R) = \nu_q(\tilde{R}, R) \quad \text{for } 1<q<\pi(R)-1 .$$

So to establish (ii) it remains to consider the cases $q=0,1$.

In low degrees $Q(Y)$ look like

$$\to Y_2/\tilde{\mathfrak{m}} Y_2 + Y_1^2 \xrightarrow{d_2} Y_1/\tilde{\mathfrak{m}} Y_1 \xrightarrow{d_1} \tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2 \to 0$$

where $\tilde{\mathfrak{m}}$ is the maximal ideal in $\tilde{R}$, and where

$$\operatorname{im} d_1 = \mathfrak{a} + \tilde{\mathfrak{m}}^2/\tilde{\mathfrak{m}}^2$$

so

$$\dim \operatorname{im} d_1 = \varepsilon_0(\tilde{R}) - \varepsilon_0(R) . \tag{3}$$

On the other hand, by [6; 3.3.2(iii)] we have

$$\dim \operatorname{coker} d_2 = \nu_1(\tilde{R}, R) = \varepsilon_1(R) - \varepsilon_0(R) + \varepsilon_0(\tilde{R}) . \tag{4}$$

From (3) and (4) we get

$$\dim H_1(Q(Y)) = \varepsilon_1(R) .$$

From (i) and (3) we get

$$\dim H_0(Q(Y)) = \varepsilon_0(R) .$$

It remains to prove (iii), but this is an immediate consequence of (ii), for if $\tilde{R}$ and R have the same embedding dimension, and Y is an acyclic closure, then the differential on $Q(Y)$ is zero.

1. On the existence of non-vanishing deviations, and a new proof of the André inequalities.

Recall that there exists a minimal R-algebra resolution X^* of k obtained from R by adjoining variables killing cycles. Also recall that the deviation e_j is the number of adjoined variables of degree j. For $q \geqq 1$ let X^q be the subalgebra of X^* obtained by adjoining all the variables of degree $\leqq q$. X^q corresponds to $F_q X^*$ in the notation of [6]. Recall that for $q>1$ we have

$$e_q = \varepsilon_{q-1} = \dim_k H_{q-1}(X^{q-1}) . \tag{5}$$

1.1 Lemma. *Let $q \geqq 1$ and let σ be an homology class in $H(X^q)$ of positive degree. Put*

$$p = \inf\left(\tfrac{1}{2}\pi(R), \dim_k \mathfrak{m}/\mathfrak{m}^2 + 1\right) .$$

Then we have $\sigma^p = 0$.

PROOF. If $p = \dim_k \mathfrak{m}/\mathfrak{m}^2 + 1$, then we have $\sigma^p = 0$ by [4, Lemma 1]. Now assume that k has characteristic $p > 0$. Let z be a cycle in $Z(X^q)$ representing σ. We may assume that z has even degree. Let $z^{(p)}$ denote the pth divided power. Then we have

$$z^p = p!\, z^{(p)} \in B_0(X^q) Z(X^q) \subset B(X^q) .$$

Indeed, $p! \in \mathfrak{m} = B_0(X^q)$ since $q \geqq 1$.

1.2. THEOREM (André [1]). *Assume that* $k = R/\mathfrak{m}$ *has characteristic* 2 *and let* q *be an odd integer,* $q > 1$. *Then we have*

$$e_{2q} \geqq e_q .$$

PROOF. Because of (5) the theorem follows immediately from the following two lemmas.

1.3. LEMMA. *The inclusion* $X^q \subset X^{2q-1}$ *induces an isomorphism of vector spaces*

$$H_{2q-1}(X^q) \xrightarrow{\cong} H_{2q-1}(X^{2q-1}) .$$

PROOF. We have

$$X^{2q-1} = X^q \langle S_1, \ldots, S_r;\, dS_j = s_j \rangle$$

where $S_1, \ldots, S_r$ are variables satisfying

$$q < \deg S_j \leqq 2q-1 \qquad \text{for } j = 1, \ldots, r .$$

Put $Y^0 = X^q$ and $Y^i = Y^{i-1} \langle S_j;\, dS_j = s_j \rangle$ for $i = 1, \ldots, r$. It suffices to show that each of the inclusions $Y^{i-1} \subset Y^i$ induce an isomorphism

$$H_{2q-1}(Y^{i-1}) \cong H_{2q-1}(Y^i) .$$

However, this is easily seen from the homology sequence associated with the extension $Y^{i-1} \subset Y^i$ cf. [6; 1.3].

REMARK. The proof of 1.3 makes no use of the conditions on q and the characteristic of k. In 1.4 however, these conditions are essential.

1.4 LEMMA. *There exists an epimorphism of vector spaces*

$$H_{2q-1}(X^q) \twoheadrightarrow H_{q-1}(X^{q-1}) .$$

PROOF. Put $X^q = X^{q-1}\langle V_1,\ldots,V_m;\, dV_j = v_j\rangle$ where V_j are the adjoined variables of degree q, and consequently $m = e_q$. Put

$$L^i = X^{q-1}\langle V_1,\ldots,\hat{V}_i,\ldots,V_m;\, dV_j = v_j\rangle \qquad i = 1,\ldots,m$$

where $\hat{V}_i$ means that V_i has been omitted. Then we have

$$X^q = L^i\langle V_i;\, dV_i = v_i\rangle .$$

The cycles $v_1,\ldots,v_m$ represent a basis for the k-vector space $H_{q-1}(X^{q-1})$ and the cycle v_i represent a basis for $H_{q-1}(L^i)$. Hence the inclusions $X^{q-1} \subset L^i$, $i = 1,\ldots,m$, induce an isomorphism of vector spaces

$$H_{q-1}(X^{q-1}) \cong \coprod_i H_{q-1}(L^i) . \tag{6}$$

Consider the exact sequence

$$0 \to L^i \to X^q \xrightarrow{J^i} L^i \to 0$$

associated with the adjunction of V_i to L^i. Recall that

$$J^i(y_1 V_i + y_2) = y_1$$

for $y_1, y_2 \in L^i$. Since v_i has degree $q-1$, we have $v_i \in X^{q-1}$ for all i. Since the characteristic of k equals 2, 1.1 shows that we have $v_i^2 = dx_i$ for some $x_i \in X^{q-1}_{2q-1}$.

Put

$$w_i = v_i V_i - x_i .$$

Then w_i is a cycle in X^q_{2q-1}. We have

$$J^j(w_i) = v_i \delta_{ij} \qquad (\delta_{ij} \text{ denotes Kronecker delta}) .$$

This shows that the map

$$H_{2q-1}(X^q) \to \coprod_i H_{q-1}(L^i) \tag{7}$$

induced by the maps J^i, $i = 1,\ldots,m$, is surjective. From (6) and (7) we get the wanted epimorphism.

1.5. PROPOSITION. *Let $q > 1$ be an odd integer, and assume that $e_q \neq 0$. Put*

$$p = \inf\left(\tfrac{1}{2}\pi(R), \dim \mathfrak{m}/\mathfrak{m}^2 + 1\right) .$$

Then there exists an even integer i with $e_i \neq 0$ satisfying

$$q < i \leqq pq - p + 2 .$$

PROOF. Assume to the contrary that $e_i=0$ for all even i satisfying $q<i\leqq Q$, where $Q=pq-p+2$. Let X^q and L^i be as in the previous proof. By the assumption on the e_i we may assume that

$$X^Q = L^1\langle U_1,\ldots,U_s;\ dU_j=u_j\rangle$$

where $U_1,\ldots,U_s$ are variables of odd degree.

Put $K^0=L^1$ and inductively

$$K^j = K^{j-1}\langle U_j;\ dU_j=u_j\rangle \qquad j=1,\ldots,s\,.$$

By descending induction on j we will prove the following statement:

$$\mathrm{A}(j):\qquad H_i(K^j) = 0 \quad \text{for } i \text{ odd, } i<Q\,.$$

For $j=s$ the statement is true since all cycles have been killed up to (and including) degree $Q-1$. To establish the step from j to $j-1$, consider the exact sequence associated with the adjunction of U_j:

$$0 \to K^{j-1} \to K^j \to K^{j-1} \to 0\,.$$

From the corresponding homology sequence we get the exact sequence

$$H_{i+1}(K^j) \to H_{i-\mu}(K^{j-1}) \xrightarrow{\ \delta\ } H_i(K^{j-1}) \to H_i(K^j) \tag{8}$$

where μ is the degree of dU_j and hence is even. So $i-\mu$ is odd iff i is odd. Assuming A(j), we obtain from (8) an epimorphism

$$H_{i-\mu}(K^{j-1}) \to H_i(K^{j-1})$$

for all odd $i<Q$. Hence for all natural numbers a and all odd $i<Q$ we have an epimorphism

$$H_{i-a\mu}(K^{j-1}) \to H_i(K^{j-1})\,.$$

By choosing a so large that $i-a\mu<0$ we see that $H_i(K^{j-1})=0$. Using now the statement A(j), (8) yields an injective map

$$\delta\colon H_{i-\mu}(K^{j-1}) \to H_i(K^{j-1}) \tag{9}$$

for all even $i<Q-1$.

Let us now put $j=1$. Then $K^{j-1}=L^1$, and we may assume that u_j equals v_1 and has degree $\mu=q-1$. By composition, (9) yields a map

$$\delta^p\colon H_0(L^1) \to H_{p\mu}(L^1)$$

which is injective since μ is even and $p\mu<Q-1$. On the other hand δ is the connecting homomorphism so it is multiplication by the homology class of the cycle u_j. According to 1.1 it follows that $\delta^p=0$. Hence $H_0(L^1)=0$ which is a contradiction.

2. A characterization of local complete intersections in terms of growth of the Betti numbers.

In this section modules are assumed to be finitely generated. We start by introducing a class of modules which is a natural extension of the class of modules of finite projective dimension.

2.1. Definition. Let M be an R-module. We will say that M has *polynomial growth* if there exists a polynomial f with integral coefficients such that

$$b_p^R(M) \leqq f(p) \qquad \text{for all } p$$

where $b_p^R(M) = \dim_k \operatorname{Tor}_p^R(M,k)$ are the Betti numbers of M.

2.2. Lemma. *Let x be a non-zerodivisor contained in $\mathfrak{m}$. Put $\bar{R} = R/xR$. Let M be an $\bar{R}$-module. Then M has polynomial growth as an $\bar{R}$-module if and only if it has polynomial growth as an R-module.*

Proof. From the standard change-of-ring spectral sequence

$$\operatorname{Tor}_p^{\bar{R}}(M, \operatorname{Tor}_q^R(k, \bar{R})) \Rightarrow \operatorname{Tor}_{p+q}^R(M,k)$$

we obtain an exact sequence

$$\ldots \to \operatorname{Tor}_{i-1}^{\bar{R}}(M,k) \to \operatorname{Tor}_i^R(M,k) \to \operatorname{Tor}_i^{\bar{R}}(M,k) \to \operatorname{Tor}_{i-2}^{\bar{R}}(M,k) \to \ldots$$

from which we deduce

$$b_i^R(M) \leqq b_i^{\bar{R}}(M) + b_{i-1}^{\bar{R}}(M) \tag{10}$$

and

$$b_i^{\bar{R}}(M) \leqq b_i^R(M) + b_{i-2}^R(M) + b^R{}_{i-4}(M) + \ldots \tag{11}$$

for all i. Now the lemma follows easily from (10) and (11).

2.3. Theorem. *The following statements are equivalent*

(i) *R is a local complete intersection,*
(ii) *all finitely generated R-modules have polynomial growth,*
(iii) *$R/\mathfrak{m}$ has polynomial growth.*

Proof. There is no loss of generality in assuming that R is complete in the $\mathfrak{m}$-adic topology. Hence we may assume that $R = \tilde{R}/\mathfrak{a}$ where $\tilde{R}$ is a regular local ring. Let M be a finitely generated R-module. Now assume (i). Then $\mathfrak{a}$ is generated by a regular sequence $a_1, \ldots, a_r$. Clearly M has polynomial growth as an $\tilde{R}$-module. By successive use of 2.2 we find that M has polynomial growth over $\tilde{R}/\mathfrak{a}$. Hence (i) implies (ii). Since (iii) follows trivially from (ii), it remains to prove that (i) follows from (iii).

Assume that $R/\mathfrak{m}$ has polynomial growth. Let f be a polynomial of degree $d-1$ such that $b_p^R(k) \leqq f(p)$ for all p. We shall first show that in the product formula

$$P_R(t) = \sum_{p \geqq 0} b_p^R(k) t^p = \prod_{q \geqq 0} \frac{(1+t^{2q+1})^{e_{2q+1}}}{(1-t^{2q+2})^{e_{2q+2}}}$$

there are at most d factors in the denominator.

Assume to the contrary that there are at least $d+1$ factors. Then pick $d+1$ such factors

$$1-t^{2q_1}, \ldots, 1-t^{2q_{d+1}} .$$

Let N be the least common multiple of $2q_1, \ldots, 2q_{d+1}$. Then we have

$$P_R(t) \gg \frac{1}{(1-t^N)^{d+1}} = \sum_{q \geqq 0} \binom{q+d}{d} t^{qN}$$

where $\gg$ means coefficientwise comparison. So

$$b_{qN}^R(k) \geqq \binom{q+d}{d} \qquad \text{for all } q \geqq 0 .$$

This is a contradiction since $\binom{q+d}{d}$ is a polynomial in q of degree d.

It follows from what we have shown that $e_{2q}=0$ for all q sufficiently large. Hence from 1.5 we obtain $e_{2q+1}=0$ for all q sufficiently large. It now follows from the theorem in [4] that R is a complete intersection.

2.4. Corollary (Avramov [2]). *Let $\mathfrak{p}$ be a prime ideal in a local complete intersection R. Then $R_{\mathfrak{p}}$ is a complete intersection.*

Proof. By 2.3 $R/\mathfrak{p}$ has polynomial growth over R. Hence $(R/\mathfrak{p})_{\mathfrak{p}}$ has polynomial growth over $R_{\mathfrak{p}}$. By 2.3 $R_{\mathfrak{p}}$ is a complete intersection.

2.5. Corollary. *Assume that $P_R(t)$ is the power series of a rational function and that the radius of convergence is at least 1. Then R is a complete intersection.*

Proof. As in the proof of 2.3 it suffices to show that $e_{2q}=0$ for all q sufficiently large. Assume to the contrary that $e_{2q} \neq 0$ for infinitely many indices

$$2q_1 < 2q_2 < \ldots .$$

Put

$$f(t) = P_R(t) \prod_{j=1}^{\alpha+1} (1-t^{2q_j})$$

where α is the order of the pole of $P_R(t)$ at $t=1$. Then $f(t)$ has non-negative integral coefficients and infinitely many of them are positive. On the other hand $\lim_{t\to 1} f(t)=0$, which is a contradiction.

3. Alternating sums of deviations.

In this section we assume that R is a homomorphic image of a regular local ring $\tilde{R}$. $\dim R$ denotes the Krull dimension of R and depth R is the length of a maximal regular sequence in $\mathfrak{m}$. Recall that $\pi(R)=2p$ if $k=R/\mathfrak{m}$ has characteristic $p\neq 0$, and that $\pi(R)=\infty$ otherwise.

3.1. Definition. For $r\geqq 0$ we define

$$\chi_r(R) = \sum_{i=0}^{r} (-1)^i \varepsilon_i(R) .$$

3.2. Remark. If R is a complete intersection, then we have $\chi_r(R)=\dim R$ for $r>0$. Cf. [6; 3.4.3].

3.3. Theorem. *Let $\mathfrak{p}$ be a prime ideal in R. Then for all r such that $0\leqq r<\pi(R)-1$ we have*

$$(-1)^r\chi_r(R) \geqq (-1)^r\chi_r(R_\mathfrak{p})+(-1)^r\dim R/\mathfrak{p} .$$

Equality holds if $\varepsilon_r(R)=0$. Equality also holds if $\varepsilon_{r+1}(R)=0$ provided that $r<\pi(R)-2$.

Proof. Let $\tilde{R} \to R$ be a surjective ring homomorphism, where $\tilde{R}$ is a regular local ring which may be assumed to have the same embedding as R. Let Y be an acyclic closure of $\tilde{R} \to R$. Let P be the inverse image of $\mathfrak{p}$ in $\tilde{R}$. Then $Y\otimes_{\tilde{R}} \tilde{R}_P$ is an $\tilde{R}_P$-algebra resolution of $R_\mathfrak{p}$.

Put

$$L = Q(Y\otimes_{\tilde{R}} \tilde{R}_P) .$$

Observe that L is a complex of vector spaces over $k(\mathfrak{p})=R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Then by 0.1 we have

$$\dim_{k(\mathfrak{p})} H_q(L) = \varepsilon_q(R_\mathfrak{p}) \quad \text{for } q<\pi(R)-1 . \tag{12}$$

It is easily seen that

$$\dim_{k(\mathfrak{p})} L_q = \dim_k Q(Y)_q \quad \text{for } 1\leqq q$$

so by 0.1 we have

(13) $$\dim_{k(\mathfrak{p})} L_q = \varepsilon_q(R) \quad \text{for } 1 \leqq q < \pi(R)-1$$

and

(14) $$\dim_{k(\mathfrak{p})} L_0 = \varepsilon_0(\tilde{R}_P) .$$

Now let r be an integer such that $0 \leqq r < \pi(R)-1$. Consider the following complex of vector spaces over $k(\mathfrak{p})$

$$0 \to d(L_{r+1}) \to L_r \to L_{r-1} \to \ldots \to L_0 \to 0 .$$

Comparing the alternating sum of dimensions of this complex and of its homology we obtain by (12), (13) and (14)

$$\varepsilon_0(\tilde{R}_p) + \sum_{i=1}^{r} (-1)^i \varepsilon_i(R) - (-1)^r \dim_{k(\mathfrak{p})} d(L_{r+1}) = \chi_r(R_{\mathfrak{p}}) .$$

Since

$$\varepsilon_0(\tilde{R}_P) = \dim \tilde{R}_P = \dim \tilde{R} - \dim \tilde{R}/P = \varepsilon_0(R) - \dim R/\mathfrak{p}$$

we have

$$\chi_r(R) - \dim R/\mathfrak{p} - (-1)^r \dim_{k(\mathfrak{p})} d(L_{r+1}) = \chi_r(R_{\mathfrak{p}})$$

from which the desired inequality follows. Clearly the inequality becomes equality if and only if $d(L_{r+1})=0$, so the last statement of the theorem follows using (13).

3.4. Corollary. *Assume that* $\mathfrak{p}$ *is a prime ideal such that* $R_{\mathfrak{p}}$ *is a complete intersection. Let* m_0 *be an even- and* m_1 *an odd non-negative integer satisfying* $m_0, m_1 < \pi(R)-1$. *Then we have*

(i) $\chi_{m_1}(R) \leqq \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p} \leqq \chi_{m_0}(R)$,
(ii) $\chi_{m_1}(R) \leqq \dim R$,
(iii) $\chi_{m_0}(R) \geqq \operatorname{depth} R$.

Moreover if r *is a non-negative integer such that* $\varepsilon_{r+1}(R)=0$ *and* $r < \pi(R)-2$, *then*

(iv) $\chi_r(R) = \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p}$.

In particular

(v) $\operatorname{depth} R \leqq \chi_r(R) \leqq \dim R$.

Proof. (i) is an immediate consequence of 3.2 and 3.3. (ii) follows from (i). To prove (iii) we let $\mathfrak{p}_0$ be a minimal prime ideal contained in $\mathfrak{p}$. Since $R_{\mathfrak{p}_0}$ is a localization of $R_{\mathfrak{p}}$ it follows from 2.4 that $R_{\mathfrak{p}_0}$ is a complete intersection. Hence from (i) we obtain

$$\chi_{m_0}(R) \geqq \dim R/\mathfrak{p}_0 .$$

Since $\mathfrak{p}_0 \in \operatorname{Ass} R$ we have $\dim R/\mathfrak{p}_0 \geqq \operatorname{depth} R$, so (iii) follows.

(iv) follows from (i), for putting $\{m_0, m_1\} = \{r, r+1\}$ yields

$$\chi_{m_0}(R) = \chi_{m_1}(R) .$$

Since in this case we also have $\chi_r(R) = \chi_{m_0}(R)$, we see that (v) follows from (ii) and (iii).

4. On the positivity of the exponents $f_j(R)$.

In this section we shall demonstrate a simple method for finding lower bounds for the invariants $f_j(R)$ defined in the introduction, formula (2).

One may ask if the following property (*) holds for all local rings R:

(*) If R is not a complete intersection then $f_q(R) > 0$ for all $q \geqq 2$.

We do not know any counterexample to (*). In the theorem below we list a few classes of rings satisfying (*). We have not tried to make the list as complete as possible, it can easily be enlarged.

4.1. THEOREM. *The following rings R satisfy* (*).

(i) *Golod rings.*
(ii) $R = A/0:\mathfrak{m}$ *where A is an artinian local complete intersection.*
(iii) *Gorenstein rings of embedding dimension* $\leqq 3$.
(iv) $R = \mathbf{Q}[X_1, \ldots, X_5]/(X_1^2, \ldots, X_5^2, X_1(X_2 + \ldots + X_5), X_2X_3, X_4X_5)$
(This is an example, due to Roos, of a ring where $\operatorname{Ext}_R(\mathbf{Q}, \mathbf{Q})$ *with the Yoneda product is not a finitely generated algebra over* $\mathbf{Q}$.)
(v) $R = k[[X_1, \ldots, X_n]]/\mathfrak{a}$ *where k is a field, and $\mathfrak{a}$ is an ideal generated by four monomials in* $X_1, \ldots, X_n$.

Before proving the theorem we will make some remarks concerning a certain product representation of power series.

Let F be a power series in $1 + t\mathbf{Z}[[t]]$, i.e. F has constant term equal to 1. Then there exist uniquely determined integers $\varphi_q(F)$ for $q > 0$, such that

$$F(t) = \prod_{q>0} (1 - t^q)^{\varphi_q(F)} .$$

Clearly the map

$$\varphi\colon 1 + t\mathbf{Z}[[t]] \to t\mathbf{Z}[[t]]$$

sending F to $\sum \varphi_q(F)t^q$ is an isomorphism from the multiplicative group $1+t\mathbf{Z}[[t]]$ to the additive group $t\mathbf{Z}[[t]]$.

4.2. LEMMA. *Let H and K be a power series in $t\mathbf{Z}[[t]]$ with non-negative coefficients. Then we have*

(i) $\varphi(1-H) \gg H$,

(ii) *If $K \gg H$ then $\varphi(1-K)-\varphi(1-H) \gg K-H$.*

($\gg$ *denotes coefficientwise comparison.*)

PROOF. (i) We will first prove that $\varphi(1-H) \gg 0$. We define a sequence of power series $\{H^{(i)}\}_{i \geqq 0}$ as follows.

Put $H^{(0)}=H$. For $j \geqq 0$, put $H^{(j+1)}=H^{(j)}$ if $H^{(j)}=0$. If $H^{(j)} \neq 0$, put

$$H^{(j+1)} = (H^{(j)}-t^w)(1-t^w)^{-1} \tag{15}$$

where w is the leading degree of $H^{(j)}$. We see that $H^{(j)} \gg 0$ implies $H^{(j+1)} \gg 0$. Hence by induction we have $H^{(i)} \gg 0$ for all $i \geqq 0$.

Passing from $H^{(j)}$ to $H^{(j+1)}$ reduces the leading coefficient by 1 while all the preceeding coefficients remain equal to zero. If follows that the sequence $\{H^{(i)}\}$ converges coefficientwise to 0. An equivalent way of writing (15) is

$$1-H^{(j)} = (1-t^w)(1-H^{(j+1)}) .$$

Since $H(i)$ tends to 0 it follows that there exist non-negative integers c_w for $w>0$ such that

$$1-H = \prod_{w>0} (1-t^w)^{c_w} .$$

Hence

$$\varphi(1-H) \gg 0 .$$

Now let q be an arbitrary positive integer. Let ct^q be the term of degree q in H. Put

$$\bar{H} = (H-ct^q)(1-ct^q)^{-1} . \tag{16}$$

Then $\bar{H} \gg 0$, so by the previous argument we have $\varphi(1-\bar{H}) \gg 0$.

From (16) we obtain

$$1-H = (1-\bar{H})(1-ct^q)$$

so

$$\varphi_q(1-H) = \varphi_q(1-\bar{H})+\varphi_q(1-ct^q) \geqq \varphi_q(1-ct^q) = c .$$

Hence

$$\varphi(1-H) \gg H .$$

(ii) Now assume that $K \gg H \gg 0$. We apply φ to the identity

$$(1-H)(1-G)^{-1} = 1-(H-G)(1-G)^{-1} .$$

By (i) we obtain

$$\varphi(1-H)-\varphi(1-G) \gg (H-G)(1-G)^{-1} \gg H-G .$$

PROOF OF THE THEOREM. We write P_R in the form

$$P_R(t) = \frac{(1+t)^{\varepsilon_0}}{G(t)}$$

where $G(t) \in 1+t^2\mathbf{Z}[[t]]$. We assume that R is not a complete intersection, so to verify (*) is equivalent to showing $\varphi_q(G) \geqq 1$ for $q \geqq 2$, i.e. to showing $\varphi(G) \gg t^2(1-t)^{-1}$.

(i) In this case there exist non-negative integers $c_1,\ldots,c_n$ with $n \geqq 2$, $c_1 \geqq 2$, $c_2 \geqq 1$ such that $G=1-c_1t^2-\ldots-c_nt^{n+1}$ (cf. e.g. [6]).
By 4.2 (ii) we have

$$\varphi(G) \gg \varphi(1-2t^2-t^3) .$$

Writing

$$1-2t^2-t^3 = (1-t^2)(1-t^2(1-t)^{-1})$$

we obtain by 4.2 (i)

$$\varphi(1-2t^2-t^3) \gg \varphi(1-t^2(1-t)^{-1}) \gg t^2(1-t)^{-1} .$$

Hence $\varphi(G) \gg t^2(1-t)^{-1}$.

(ii) In this case we can write

$$G = (1-t^2)^n(1-t^2(1-t)^{-n}), \quad n \geqq 2. \quad \text{Cf. [5].}$$

So by 4.2 (i) we get

$$\varphi(G) \gg \varphi(1-t^2(1-t)^{-n}) \gg t^2(1-t)^{-n} \gg t^2(1-t)^{-1} .$$

(iii) According to Wiebe [8] we have

$$G = 1-rt^2-rt^3+t^5$$

where r is an integer $r \geqq 5$.
We have

$$G = (1-t^2)(1-t^3-(r-1)t^2(1-t)^{-1})$$

so in the same way as above we obtain

$$\varphi(G) \gg (r-1)t^2(1-t)^{-1} \gg 4t^2(1-t)^{-1}.$$

(iv) According to Roos [7] we have

$$G = (1+t)^5(1-t)^{-1}(1-6t+12t^2-9t^3).$$

Put $G=(1-t^2)^5(1-H)$. It is a matter of straightforward computation to show that $H \in t^2\mathbf{Z}[[t]]$ and $H \gg t^2(1-t)^{-1}$. As before we have $\varphi(G) \gg H$.

(v) According to a classification due to Frøberg [3] there are 22 different Poincaré series possible in this case. The Poincaré series are given explicitly in [3]. One can show that in all but one of the 22 cases there exists an integer m ($0 \leqq m \leqq 3$) and a power series $H \in t\mathbf{Z}[[t]]$ with $H \gg t^2(1-t)^{-1}$ such that

$$G = (1-t^2)^m(1-H).$$

The exceptional case is (h) in the classification where

$$G = 1-4t^2-2t^3+4t^4+4t^5+t^6.$$

This case can be taken care of by observing that we have

$$G = (1-2t^2-t^3)^2.$$

We showed in (i) that $\varphi(1-2t^2-t^3) \gg t^2(1-t)^{-1}$. Since φ is a group homomorphism, we obtain

$$\varphi(G) = 2\varphi(1-2t^2-t^3) \gg 2t^2(1-t)^{-1}.$$

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