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DIMENSION IN RINGS WITH SOLVABLE ALGEBRAIC GROUP ACTION

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In [3], Fossum and Foxby have shown that a number of properties of a graded commutative ring and its category of modules can be deduced from the corresponding properties for graded ideals and graded modules. If the $\mathbf{Z}$-graded ring $R = \oplus R_i$ is an algebra over the field k, then the algebraic group $G = \mathrm{GL}_1(k)$ can be made to act on R: t in G acts on R_i by multiplication by t^i, and then a graded R-module is a simultaneous R and G-module. This suggests that for a ring R on which an algebraic group G acts, various properties of R can be deduced from the corresponding properties for simultaneous R and G modules. We show in this paper that this is indeed the case when the algebraic group G is linear and solvable.

We fix the following notation: k is an algebraically closed field and G is a solvable linear algebraic group over k. A finite dimensional vector space V with G-action is a G-module if the induced homomorphism $G \to \mathrm{GL}(V)$ is a homomorphism of algebraic groups over k. A vector space W with G-action is a rational G-module if W is a union of finite dimensional G-modules in the above sense. A *ring is a commutative Noetherian k-algebra which is a rational G-module such that G acts by k-algebra automorphisms. A *module M over a *ring R is an R-module and a rational G-module such that $g(rm) = g(r)g(m)$ for g in G, r in R, and m in M. A *homomorphism of *modules over a *ring R is an R-module homomorphism preserving G-actions. A semi-invariant of weight X in a rational G-module V is a non-zero vector v in V such that $g(v) = X(g)v$ for some algebraic character $X\colon G \to \mathrm{GL}_1(k)$ of G. Since G is solvable, every non-zero rational G-module contains a semi-invariant. An *ideal of a *ring R is an ideal which is a sub-*module. If P is a prime *ideal of the *ring R, *ht(P) is the length of the longest saturated chain of prime *ideals ending with P, and *dim R is the supremum of *ht (P) as P varies over prime *ideals of R. We call a *ring R *simple if the only *ideals of R are 0 and R. If R is a *ring and S is a multiplicatively closed set of semi-invariants in R, then $S^{-1}R$ is a *ring and a *module over R.

In these notations, the main results may be summarized as follows: our

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principal technical tool is the fact (Theorem 3) that a *simple *ring is regular of dimension at most $\dim G$. This implies that the Krull dimension of R, $\dim R$, is bounded by $^*\dim R + \dim G$ (Theorem 7), and that projective and injective dimension of a *module can be bounded by considering Ext's just ranging over *modules (Theorem 8 and Corollary 11). Finally, we show that a *ring R is regular (respectively Gorenstein) if all its localizations at *prime ideals are (Corollaries 12 and 13).

We restrict attention only to solvable groups to guarantee that a non-zero *ideal in a *ring contains a non-zero principal *ideal, which is necessary for induction arguments. Our solvable groups are not necessarily reductive, however, so semi-invariants in a *homomorphic image may not have semi-invariant preimages. It is this fact which prevents a good notion of "*localization", which means that a number of the techniques of [3] cannot be extended to our setting.

LEMMA 1. *Let R be a *ring.*

a) *R is *simple if and only if every semi-invariant is a unit.*

b) *If R is a domain and S is the set of semi-invariants of R then $S^{-1}R$ is *simple.*

PROOF. Part a) follows immediately from the fact that non-zero *ideals contain non-zero principal *ideals. Then to establish b), we need to show that every semi-invariant of $S^{-1}R$ is a unit. So let f/g in $S^{-1}R$ be semi-invariant of weight α, where $f \in R$ and $g \in S$, with g of weight β. For t in G,

$$\alpha(t)f/g \;=\; t(f/g) \;=\; t(f)/t(g) \;=\; t(f)/\beta(f)g\,,$$

so $t(f) = (\alpha\beta^{-1})(t)f$ and f is semi-invariant, so f/g is a unit in $S^{-1}R$.

LEMMA 2. *Let R be a *ring which is an affine domain and let S be the set of semi-invariants of R. Then $S^{-1}R$ is a regular ring of dimension at most the dimension of G.*

PROOF. We first bound the dimension of $S^{-1}R$. We can regard R as the coordinate ring of an affine variety V with G-action. Let Q be a prime of $S^{-1}R$ and let $P = Q \cap R$. Then $\mathrm{ht}\,(Q) = \mathrm{ht}\,(P)$ and $P \cap S = \varnothing$. The zero set W of P in V is a closed subvariety with $\mathrm{ht}\,(P) = \mathrm{codim}\,(W)$. Let $I = \{f \in R \mid f(G \cdot W) = 0\}$. Then I is an *ideal of R contained in P. If $I \neq 0$, then I contains an element of S. Since $P \cap S = \varnothing$, $I = 0$, so $G \cdot W$ is dense in V. Then $G \cdot W$ contains a non-empty open subset U of V. We may assume U is G-stable. Moreover, V contains a G-stable open subset U' such that a geometric quotient U'/G exists [6]. Let $U_0 = U \cap U'$ and let $U_1 = U_0 \cap W$. U_1 is open in W and closed in U_0, and U_0

$=G\cdot U_1$. Moreover, $\operatorname{codim}_{U_0} U_1 = \operatorname{codim}_V W$, so we may assume that $G\cdot W = V$ and that the geometric quotient V/G exists. Let $p\colon V \to V/G$ be the quotient map. The fibres of p are orbits and hence have dimension at most $\dim(G)$, so $\dim(V/G) \geqq \dim V - \dim G$. Also, $p\colon W \to V/G$ is onto, so $\dim W \geqq \dim(V/G)$, and hence

$$\dim G \geqq \dim V - \dim W = \operatorname{codim}_V(W) = \operatorname{ht}(P).$$

To see that $S^{-1}R$ is regular, we note that since $S^{-1}R$ is a localization of an affine algebra, the singular locus is closed in $\operatorname{Spec}(S^{-1}R)$ [5, Thm. 73, p. 247]. Since the singular locus is G-stable, its defining radical ideal is an *ideal. By Lemma 1(b), $S^{-1}R$ is *simple, so the singular locus is empty.

THEOREM 3. *A simple *ring is a regular integral domain of dimension at most the dimension of G.*

PROOF. Let R be a simple *ring. The minimal primes of R are *ideals and hence R is a domain. Write $R = \operatorname{dir\,lim} R_i$, where R_i are sub-*rings of R which are affine, and let S_i be the set of semi-invariants of R_i. By Lemma 1(a), $S_i^{-1}R_i \subseteqq R$ so $R = \operatorname{dir\,lim} S_i^{-1}R_i$. Let $T_i = S_i^{-1}R_i$. Suppose $T_i \subseteqq T_j$, and let $U = \{P \in \operatorname{Spec}(T_i) \mid (T_j)_P \text{ is } T_i\text{-flat}\}$. Then U is open and non-empty in $\operatorname{Spec}(T_i)$ by generic flatness [5, Lemma 1, p. 156] and U is clearly G-stable. Then the defining radical ideal of the complement of U is an *ideal; since T_i is *simple by Lemma 1(b), the complement is empty and T_j is T_i-flat. Since $R = \operatorname{dir\,lim} T_i$, and the inclusions $T_i \subseteqq T_j$ are flat morphisms, R is regular of dimension at most $\dim G$ since each T_i is.

We now turn to computations of heights of prime ideals in a *ring. We will use the following construction:

LEMMA 4. *Let P be a prime ideal of the *ring R. The sum $*P$ of all the *ideals contained in P is a prime *ideal.*

PROOF. P contains a minimal prime P_0 over $*P$. P_0 is necessarily an *ideal so $P_0 = *P$ and hence $*P$ is prime.

PROPOSITION 5. *Let R be a *ring and let P be a prime ideal of R*

a) *Some ranking chain of P contains $*P$.*

b) *If $P = *P$, $\operatorname{ht}(P) = *\operatorname{ht}(P)$.*

PROOF. We prove both parts simultaneously by induction on $n = \operatorname{ht}(P)$. If $n = 0$, then $P = *P$ and both follow. To apply induction, we observe that if I is an *ideal contained in P, then, by Lemma 4, $I \subseteqq {}^*P$ and $({}^*P)/I = {}^*(P/I)$. Now let $P_0 \subseteqq P_1 \subseteqq \dots \subseteqq P_n = P$ be a saturated chain of length $n = \text{ht}(P) > 0$. Then P_0 is a minimal prime of R and hence an *ideal. Moreover, in R/P_0, $\text{ht}(P/P_0) = \text{ht}(P)$. If a) and b) hold for P/P_0 in R/P_0, we can pull back the relevant primes to see that a) and b) hold for P in R. Thus we may assume R is a domain. If ${}^*P = 0$, a) holds. If not, P contains a semi-invariant x. Let $I = Rx$. I is an *ideal, and in the ring R/I, $\text{ht}(P/I) = n-1$ [4, Thm. 155, p. 113]. Let $Q_1 \subseteqq Q_2 \subseteqq \dots \subseteqq Q_n = P/I$ be a ranking chain for P/I in R/I. By induction for part a), some $Q_j = {}^*(P/I)$. If Q_i' is the inverse image of Q_i in R, then $0 \subseteqq Q_1' \subseteqq \dots \subseteqq Q_n' = P$ is a ranking chain for P in R, and $Q_j' = {}^*P$. By induction for part b), we may assume each Q_i is an *ideal. With Q_i' as inverse images, then, $0 \subseteqq Q_1' \subseteqq \dots \subseteqq Q_n' = P$ is a ranking chain for P composed of *ideals. Thus $\text{ht}(P) = {}^*\text{ht}(P)$.

Corollary 6. *Let R be a *ring and let P be a prime ideal of R. Then* $\text{ht}(P) = {}^*\text{ht}({}^*P) + \text{ht}(P/{}^*P)$.

Proof. Let $n = \text{ht}(P)$ and let $P_0 \subseteqq P_1 \subseteqq \dots \subseteqq P_n = P$ be a ranking chain for P. For each i, we have $\text{ht}(P) = \text{ht}(P_i) + \text{ht}(P/P_i)$. By Proposition 5 a), ${}^*P = P_i$ for some i, and by Proposition 5 b), $\text{ht}({}^*P) = {}^*\text{ht}({}^*P)$.

Using Theorem 3, we can bound the term $\text{ht}(P/P^*)$ of Corollary 6, and hence determine the relation between dimension and *dimension for a *ring.

Theorem 7. *Let R be a *ring and P a prime ideal of R.*

a) $\text{ht}(P) \leqq {}^*\text{ht}({}^*P) + \dim G$.

b) $\dim R \leqq {}^*\dim R + \dim G$.

c) $\text{ht}(P) = {}^*\text{ht}({}^*P) + n$, *where n is the minimal number of generators of $PR_P/{}^*PR_P$ in $R_P/{}^*PR_P$.*

Proof. Let $\bar{R} = R/{}^*P$, let $\bar{S}$ be the set of semi-invariants in $\bar{R}$ and let $T = \bar{S}^{-1}\bar{R}$. Since ${}^*(P/{}^*P) = 0$, $(P/{}^*P) \cap \bar{S} = 0$. Thus $\text{ht}(P/{}^*P) = \text{ht}(P/{}^*PT)$. By Lemma 1 b) T is simple and by Theorem 3, T is regular of dimension at most $\dim G$. Since $\dim T \leqq \dim G$, $\text{ht}(P/{}^*P) \leqq \dim G$, and parts a) and b) follow from Corollary 6. Next, we let $Q = P/{}^*PT$. Since T is regular, $\text{ht}(Q) = \text{ht}(QT_Q) = n$, where n is the minimal number of generators of QT_Q. But $T_Q = R_P/{}^*PR_P$ and $QT_Q = PR_P/{}^*PR_P$ so c) follows.

We next turn to calculations of homological dimension.

Definition. Let R be a *ring and M a *module. The *projective dimension of M is the least integer n such that $\text{Ext}^i_R(M, N) = 0$ for all $i > n$ and all *modules N and is denoted $^*\mathrm{pd}_R M$. The *injective dimension of M is the least integer n such that $\mathrm{Ext}_R^i(N,M)=0$ for all $i>n$ and all *modules N and is denoted $^*\mathrm{id}_R M$. The global *dimension of R is the least integer n such that $\mathrm{Ext}_R^i(P,Q)=0$ for all $i>n$ and all *modules P and Q, and is denoted $^*\mathrm{gl\,dim}\,R$.

We have the obvious inequalities $^*\mathrm{pd}_R M \leqq \mathrm{pd}_R M$, $^*\mathrm{id}_R M \leqq \mathrm{id}_R M$ and $^*\mathrm{gl\,dim}\,R \leqq \mathrm{gl\,dim}\,R$. We next establish upper bounds for the relevant dimensions.

THEOREM 8. *Let R be a *ring and M a *module. Then* $^*\mathrm{pd}_R M = \mathrm{pd}_R M$.

PROOF. We observe that if V is a rational G-module, then $R \otimes_k V$ is an R-projective *module. If M is a *module, then we can regard M as a rational G-module, and then the natural surjection $R \otimes_k M \to M$ is a *homomorphism of *modules. By standard dimension shift arguments, it thus suffices to prove the theorem in case $^*\mathrm{pd}_R M=0$. Let K be the kernel of $\otimes_k M \to M$. Then in the exact sequence

$$\mathrm{Hom}_R(M, R\otimes_k M) \to \mathrm{Hom}_R(M,M) \to \mathrm{Ext}_R^1(M,K)\,,$$

the last term is zero since K is a *module and $^*\mathrm{pd}_R M=0$, so M is a direct summand of $R\otimes_k M$ and hence is projective so $\mathrm{pd}_R M=0$.

THEOREM 9. *Let R be a *ring. Then* $^*\mathrm{gl\,dim}\,R \leqq \mathrm{gl\,dim}\,R \leqq {}^*\mathrm{gl\,dim}\,R+\dim G$.

PROOF. We recall that

$$\mathrm{gl\,dim}\,R \;=\; \sup\{\mathrm{pd}_R(R/M) \mid M \text{ a maximal ideal}\}\,.$$

Let M be a maximal ideal of R, let S be the set of semi-invariants in $R/{}^*M$, and let $T=S^{-1}(R/{}^*M)$. Then R/M can be regarded as a T-module, and hence $\mathrm{pd}_R(R/M) \leqq \mathrm{pd}_T(R/M)+\mathrm{pd}_R(T)$. By Lemma 1 and Theorem 3, T is regular of dimension at most $\dim G$, so $\mathrm{pd}_T(R/M) \leqq \dim G$, and since T is a *module, $\mathrm{pd}_R(T)={}^*\mathrm{pd}_R(T)$ by Theorem 8. Thus

$$\mathrm{pd}_R(R/M) \;\leqq\; \dim G + {}^*\mathrm{pd}_R(T) \;\leqq\; \dim G + {}^*\mathrm{gl\,dim}\,G\,,$$

and the upper bound obtains. The lower bound was previously noted.

To compute injective dimension, we need a result on the Bass numbers defined in [1, p. 11].

PROPOSITION 10. *Let R be a *ring, M an R-module, and P a prime ideal of R. Suppose $\mu_i({}^*P,M)=0$ for $i>r$. Then $\mu_i(P,M)=0$ for $i>r+\dim G$.*

PROOF. By assumption, $\mathrm{Ext}^q_R(R/{}^*P, M)=0$ for $q<r$. Let $T=R_P/{}^*PR_P$. Then $\mathrm{Ext}^q_{R_P}(T, M_P)=0$ if $q>r$. If S is the set of semi-invariants in $R/{}^*P$, $S\cap P/{}^*P=\varnothing$ so T is a localization of $S^{-1}(R/{}^*P)$, and hence T is regular of dimension at most $\dim G$ by Lemma 2. Since $\mu_i(P, M)=\mu_i(PR_P, M_P)$, we may replace R by R_P, and extend P, ${}^*P$, and M to R_P. By [2, p. 348] there is a spectral sequence $\mathrm{Ext}^p_T(R/P, \mathrm{Ext}^q_R(T, M) \Rightarrow \mathrm{Ext}^n_R(R/P, M))$. Now $\mathrm{Ext}^p_T(\cdot,\cdot)=0$ for $p>\dim G$ and $\mathrm{Ext}^q_R(T, M)=0$ for $q>r$, so in the spectral sequence $E_2^{p,q}=0$ for $p+q>r+\dim G$. Thus $\mathrm{Ext}^n_R(R/P, M)=0$ for $n>r+\dim G$, so $\mu_i(P, M)=0$ for $i>r+\dim G$.

COROLLARY 11. *Let R be a *ring and M a *module. Then* ${}^*\mathrm{id}_R(M)\leqq \mathrm{id}_R(M)\leqq {}^*\mathrm{id}_R(M)+\dim G$.

PROOF. The first inequality is clear. For the second, we need to show that $\mu_i(P, M)=0$ for all $i\geqq {}^*\mathrm{id}_R(M)+\dim G$ for every prime ideal P of R. By assumption, $\mu_i({}^*P, M)=0$ for $i>{}^*\mathrm{id}_R(M)$, so the result follows from Proposition 10.

COROLLARY 12. *Let R be a *ring. Suppose R_P is regular for every prime *ideal P of R. Then R is regular.*

PROOF. Let M be an R module, P a prime ideal of R, and $Q\subseteqq P$ a prime. Then by hypothesis $\mu_i({}^*Q, M)=0$ for $i>\mathrm{ht}({}^*Q)$. By Proposition 10, $\mu_i(Q, M)=0$ for $i>\mathrm{ht}({}^*Q)+\dim G$, so by Theorem 7, $\mu_i(Q, M)=0$ for $i>\mathrm{ht}(Q)$. Thus $\mu_i(Q, M)=0$ for $i>\mathrm{ht}(P)$. It follows that $\mathrm{id}_{R_P} M_P$ is finite, so R_P is regular.

COROLLARY 13. *Let R be a *ring. Suppose R_P is Gorenstein for every prime *ideal P of R. Then R is Gorenstein.*

PROOF. The argument of Corollary 12 shows that for any prime P of R, $\mathrm{id}_{R_P} R_P$ is finite, so R is Gorenstein.

To establish the analogue of Corollaries 12 and 13 for the Cohen–Macaulay property, we need a stronger version of Proposition 10, which holds when the module M of that proposition is a *module. This, in turn, requires the following preliminary results.

LEMMA 14. *Let R be a *simple *ring. Then every finitely generated *module is free as an R-module.*

PROOF. Since every finitely-generated *module has a composition series in which the factors are cyclic and generated by semi-invariants, it suffices to prove the lemma for modules of this type. Thus let M be a *module generated by the semi-invariant m of weight α. Let $R(\alpha)$ be R, as an R-module, but with G-action $g\cdot r=\alpha(g)g(r)$. Then $R(\alpha)$ is a *module, and $R(\alpha)\to M$ by $r\mapsto rm$ is a surjective *homomorphism. If $R(\alpha)$ is *simple, then M is R-free. But $R(\alpha)\otimes_R R(\alpha^{-1})\to R$ by $a\otimes b\mapsto ab$ is an *isomorphism, and it follows that $R(\alpha)$ is *simple since R is.

Lemma 15. *Let A be a commutative ring, I an ideal of A, and M an A-module such that* $\operatorname{Ext}_A^i(A/I,M)$ *is a free A/I-module for all i, and* $\operatorname{Ext}_A^i(A/I,M)=0$ *for $i<d$. Let x be a non-zero divisor in A/I, and let $I=I+Ax$. Then* $\operatorname{Ext}_A^i(A/I',M)$ *is a free A/I'-module for all i and* $\operatorname{Ext}_A^i(A/I',M)=0$ *for $i<d$.*

Proof. The exact sequence $0\to A/I\xrightarrow{x}A/I\to A/I'\to 0$ yields the long exact sequence

$$\operatorname{Ext}_A^i(A/I,M)\to\operatorname{Ext}_A^i(A/I,M)\to\operatorname{Ext}_A^i(A/I',M)$$
$$\to\operatorname{Ext}_A^{i+1}(A/I,M)\to\operatorname{Ext}_A^{i+1}(A/I,M)\,,$$

where the first and last maps are multiplication by x. Since $\operatorname{Ext}_A^i(A/I,M)$ and $\operatorname{Ext}_A^{i+1}(A/I,M)$ are A/I-free, we have a short exact sequence $0\to\operatorname{Ext}_A^i(A/I,M)\xrightarrow{x}\operatorname{Ext}_A^i(A/I,M)\to\operatorname{Ext}_A^i(A/I,M)\to 0$ for each $i\geqq 0$. This shows that $\operatorname{Ext}_A^i(A/I',M)$ has the asserted properties.

Theorem 16. *Let R be a *ring, M a finitely generated *module, and P a prime ideal of R. Let $s=\operatorname{ht}(P)-\operatorname{ht}({}^*P)$. Suppose $\mu_i({}^*P,M)=0$ for $i<r$. Then $\mu_i(P,M)=0$ for $i<r+s$.*

Proof. Let S be the set of semi-invariants in $R/{}^*P$ and let $B=S^{-1}(R/{}^*P)$. Then B is *simple, and $\operatorname{Ext}_R^q(B,M)$ is a B-*module, so $\operatorname{Ext}_R^q(B,M)$ is a free B-module by Lemma 14, and hence remains free when we further localize at P. Replace R by R_P and extend P, ${}^*P$, and M to R_P. Let $T=R/{}^*P$. Then $\operatorname{Ext}_R^q(T,M)$ is a free T-module and, as in Proposition 10, we have a spectral sequence $\operatorname{Ext}_T^p(R/P,\operatorname{Ext}_R^q(T,M))\Rightarrow\operatorname{Ext}_R^n(R/P,M)$. Since T is regular, $\operatorname{Ext}_T^p(R/P,T)=0$ for $p\neq\dim(T)$ by [1, Prop. 3.6, p. 14]. By Corollary 6, $\dim(T)=\operatorname{ht}(P/{}^*P)=s$. Thus the above spectral sequence collapses and we have $\operatorname{Ext}_T^s(R/P,\operatorname{Ext}_R^q(T,M))=\operatorname{Ext}_R^{s+q}(R/P,M)$. Thus $\mu_q({}^*P,M)=0$ implies $\mu_{s+q}(P,M)=0$ for all q. In particular, if $\mu_i({}^*P,M)=0$ for $i<r$, $\mu_i(P,M)=0$ for $s\leqq i<r+s$. Since T is regular local, $P/{}^*P$ is generated by a T-sequence $x_1,\ldots,x_s$. Let $I_j={}^*P+Rx_1+\ldots+Rx_j$, so $I_0={}^*P$ and $I_s=P$. Now $\operatorname{Ext}_R^i(R/I_0,M)=\operatorname{Ext}_R^i(T,M)$ is a free R/I_0-module for all i, and $\operatorname{Ext}_R^i(R/I_0,M)$

$=0$ for $i<r$ by hypothesis. By induction and Lemma 15, $\mathrm{Ext}_R^i(R/I_s, M) = \mathrm{Ext}_R^i(R/P, M) = 0$ for $i<s$, so $\mu_i(P, M) = 0$ for $i<s$, also.

COROLLARY 17. *Let R be a *ring, and suppose R_P is Cohen–Macaulay for every prime *ideal P of R. Then R is Cohen–Macaulay.*

PROOF. Let P be a prime ideal of R, and let $Q = {}^*P$. By assumption, R_Q is Cohen–Macaulay, so by [1, 3.7, p. 14], $\mu_i(Q, R) = 0$ for all $i < \mathrm{ht}(Q)$. By Theorem 16, $\mu_i(P, R) = 0$ for $i < \mathrm{ht}(Q) + (\mathrm{ht}(P) - \mathrm{ht}(Q)) = \mathrm{ht}(P)$; by [1, 3.7, p. 14] again, R_P is Cohen–Macaulay. This holds for all P, so R is Cohen–Macaulay.

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