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ON THE POSTAGE STAMP PROBLEM WITH THREE STAMP DENOMINATIONS

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1. Introduction.

Given stamps (in sufficient supply) of k different integral denominations

$$1 = a_1 < a_2 < \ldots < a_k . \tag{1.1}$$

The envelope does only give room for at most h stamps. What is the greatest consecutive range of postal rates from one unit upwards which can be formed under these conditions?

In mathematical formulation: Given a basis

$$A_k = \{a_1, a_2, \ldots, a_k\}$$

of integers satisfying (1.1). We form all linear combinations

$$\sum_{i=1}^{k} x_i a_i ; \qquad x_i \geqq 0, \ \sum_{i=1}^{k} x_i \leqq h , \tag{1.2}$$

and ask for the smallest integer $N_h(A_k)$ which is not represented by such a combination.

We shall operate instead with the bound

$$n_h(A_k) = N_h(A_k) - 1 ,$$

which in German is called the "Reichweite" of the basis A_k with respect to the maximal number h of — possibly repeated — addends. No established English name for this seems to exist, but words like "span", "width" and "breadth" occur. Personally, I prefer the more direct translation "range". To avoid risk of confusion with the use of this word in the meaning "range of a function", we shall use *h-range*.

The h-range of A_k is then defined as follows: All integers in the interval $[0, n_h(A_k)]$ have a representation of the form (1.2), while $n_h(A_k)+1$ has no such representation.

The postage stamp problem has been rather intensively studied, particularly by German number theoreticians. The main interest has been centered around

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the "*global*" aspect: Given h and k, find an *extremal basis* A_k^* to obtain the largest possible *extremal h-range*

$$n_h(k) = n_h(A_k^*) .$$

In this general form, the problem is very difficult from a theoretical point of view. As would be expected, the numerical determination of $n_h(k)$ and A_k^* has been a tempting target for several computer specialists.

To study the extremal h-range, it is of course necessary to know something about the "*local*" aspect: Determine $n_h(A_k)$ when h, k and a particular basis A_k are given. However, local problems play a much less dominant role in the literature.

The case $k=2$ is completely solved, and it is therefore natural to study what happens in the case of three stamp denominations ($k=3$). The pioneer here was Hofmeister, who solved [3, Satz 2] the global problem almost completely. (The "almost" can now be deleted, cf. Section 4 below.)

Later, some of Hofmeister's pupils have also made great advances with the local problem for $k=3$. Their results have been simplified and extended by Rödseth [8], [9], who gave a fairly simple procedure to determine $n_h(A_3)$. A priori, it is not possible to estimate the *number of steps* of the procedure necessary to give the final result. Strictly speaking, Rödseth's formulas are therefore not "explicit" for $n_h(A_3)$.

Truly explicit formulas for $n_h(A_3)$ had earlier been given by Salié [10], although under rather restrictive conditions on the basis elements a_2 and a_3. His arguments were based only on the original definition of $n_h(A_3)$.

One main purpose of the present paper is an application of Rödseth's method, in order to generalize the formulas of Salié for $n_h(A_3)$. The resulting formulas are explicit in the sense that at most two steps of a simple division algorithm are necessary to obtain the final result. Each of these divisions may represent several steps of Rödseth's general procedure. — Asymptotically, our formulas cover slightly more than 99% of all "admissible" bases A_3 (cf. Fig. 1 p. 59).

Both Hofmeister and Rödseth spent the Autumn term 1978 with me in Bergen. I want to thank them for cooperation and inspiration in a field of common interest to all of us.

My approach to the problem has been an "experimental" one, with numerical evidence entering heavily into all stages of the development. Hours of computing have been performed on the UNIVAC 1110 at the University of Bergen. I am in the fortunate position that the computer specialist at our institute, Svein Mossige, also has a solid background in number theory. I may safely say that this paper would never have been accomplished without

Mossige's programming assistance, and I thank him for his help and support over a period of several months.

2. Known results.

The most comprehensive report on the postage stamp problem, including references, is found in the lecture notes [4] by Hofmeister. He followed up with another set of notes [5], which deals mainly with the Frobenius "coin exchange" problem, but which also treats the connection with the stamp problem.

In this section, we confine ourselves to those parts of the established theory which form the necessary background for our new results.

We first note that we may always assume

$$1 < a_2 < h+2 , \tag{2.1}$$

since $a_2 \geqq h+2$ would mean that no representation (1.2) is possible for $h+1$. Under this condition, it is well known that

$$n_h(A_2) = n_h(1, a_2) = (h+3-a_2)a_2-2 . \tag{2.2}$$

For $k=3$, we must similarly assume that

$$a_2 < a_3 < n_h(A_2)+2 = (h+3-a_2)a_2 . \tag{2.3}$$

Under the conditions (2.1) and (2.3), we shall say that the basis $A_3 = \{1, a_2, a_3\}$ is *admissible*. In what follows, *only such bases* A_3 will usually be considered.

We shall use the following notation throughout:

$$\begin{cases} a_3 = fa_2+r, & 0 \leqq r < a_2 \\ a_3 = qa_2-s, & 0 \leqq s < a_2 . \end{cases} \tag{2.4}$$

If $r=s=0$, then $q=f$. Otherwise, we have $q=f+1$ and $s=a_2-r$.

It follows from (2.3) that $h \geqq a_2+f-2$. For given a_2 and a_3, we put

$$h_0 = a_2+f-2 = a_2+\left[\frac{a_3}{a_2}\right]-2 , \tag{2.5}$$

which is *the smallest possible h such that* A_3 *is an admissible basis.*—As usual, $[x]$ denotes the largest integer $\leqq x$.

We shall see later that it means a great simplification to restrict oneself to the case $h=h_0$. The importance of this was apparently first realized by Salié [10]. The extremal h-range of "h_0-bases" was determined by Hofmeister [3, Satz 3–4] for sufficiently large h_0.

Hofmeister [2] introduced the *regular* representation by a basis A_k as follows: First use a_k as often as possible, then a_{k-1} as often as possible, etc. In terms of (1.2), this means to impose the additional conditions

$$\sum_{i=1}^{j} x_i a_i < a_{j+1}, \quad j=1,2,\ldots,k-1 . \tag{2.6}$$

If only such representations are allowed, still restricted to at most h addends, we speak of the *regular h-range* $g_h(A_k)$. This was explicitly determined by Hofmeister [2, Satz 1]. He also [3] solved completely the problem of finding the *extremal* regular h-range $g_h(3)$ in the case $k=3$. (His Satz 5 is formulated with "for sufficiently large h", but does in fact hold for all h.)

Since clearly $n_h(A_2)=g_h(A_2)$ (use a_2 as often as possible), we note that the admissible bases A_3 are the *same* for regular as for ordinary representations. In particular, the minimal $h=h_0$ of (2.5) is the same in both cases.

It follows from Hofmeister [2] that

$$g_h(A_3) = (h+4-a_2-f)a_3-(r+2) , \tag{2.7}$$

provided $h \geqq h_0-1$. Choosing in particular $h=h_0$, this can be simplified by (2.5) to

$$g_{h_0}(A_3) = 2a_3-(r+2) . \tag{2.8}$$

For an arbitrary $h \geqq h_0$, it then follows from (2.7) that

$$g_h(A_3) = g_{h_0}(A_3)+(h-h_0)a_3 . \tag{2.9}$$

Defining h_0 suitably, a similar relation holds for *all* bases A_k (if a_3 is replaced by a_k).

A given integer may have several representations by a basis A_k. A *minimal* representation (not necessarily unique) is one with the smallest number of addends from the basis. Djawadi [1] called a basis A_k *pleasant* (German: "angenehm") if one minimal representation always coincides with the (unique) *regular* representation. This implies that $n_h(A_k)=g_h(A_k)$ for all h. Djawadi showed that A_3 *is pleasant if and only if*

$$s < q . \tag{2.10}$$

In particular, all bases A_3 with $a_2=2$ are therefore pleasant.

Combining (2.10) with (2.8). we get the following

THEOREM 2.1. *If $s<q$, that is, if $r=0$ or $r \geqq a_2-f$, then*

$$n_{h_0}(A_3) = 2a_3-(r+2) . \tag{2.11}$$

This result may, of course, be deduced independently of the formal apparatus for arbitrary k by Hofmeister and Djawadi. In fact, Salié [10] gave a simple proof (his Satz 2 and 3) of Theorem 2.1. He also showed (Satz 1) that (2.11) holds with $\geqq$ for an arbitrary basis A_3, with strict inequality when (2.10) fails. In other words, *we always have*

$$n_h(A_3) > g_h(A_3) \tag{2.12}$$

for a non-pleasant basis. (There are several ways to prove this result.)—Note that we may well have a regular representation (1.2) of the h-range $n_h(A_3)$, even if A_3 is a non-pleasant basis.

It follows from (2.2) that

$$n_{h+1}(A_2) = n_h(A_2)+a_2 .$$

For all k, we trivially have

$$n_{h+1}(A_k) \geqq n_h(A_k)+a_k \tag{2.13}$$

(assuming A_k admissible). For $k \geqq 4$. it is easy to give examples with strict inequality. A numerical search shows that the simplest possible case is given by

$$A_4 = \{1,2,5,6\},\ n_2(A_4) = 8,\ n_3(A_4) = 18 .$$

With $k=3$, numerical evidence indicates that we always have equality in (2.13) for $h \geqq h_0$. This was first shown by Windecker [15]. By a very complicated procedure, he could prove that

$$n_h(A_3) = ha_3+F(a_2,a_3) ,$$

where the function F depends only on the basis A_3.

Windecker's proof is extremely hard to read. At an early stage of the development, it was therefore decided to verify the result numerically, by means of Theorem 3.1 with $k=3$. Mossige [7] wrote a very efficient computer program to determine $n_h(A_k)$ directly from the definition, and used it to verify Theorem 2.2 for $h_0 \leqq 70$.

Recently, however, a much simpler proof of equality in (2.13) for $k=3$ has been given by Rödseth [9]. We formulate the result as

THEOREM 2.2. *For $h \geqq h_0$, we always have*

$$n_{h+1}(A_3) = n_h(A_3)+a_3 ,$$

and hence

$$n_h(A_3) = n_{h_0}(A_3)+(h-h_0)a_3 . \tag{2.14}$$

The important property (2.9) of regular representations thus holds also for ordinary representations in the case $k=3$.

We now introduce the *Frobenius number* $g(B)$ of a positive integral basis

$$B = \{b_1, b_2, \ldots, b_k\}, \quad (b_1, b_2, \ldots, b_k) = 1 .$$

This is the largest integer with *no* representation of the type

$$\sum_{i=1}^{k} y_i b_i; \quad y_i \geqq 0 .$$

This so called "coin exchange problem of Frobenius" has been extensively treated in the literature. For references, see Selmer [11] and Hofmeister [5].

The connection with the (local) postage stamp problem was discovered by Meures [6] (see also Rödseth [9]). To a given stamp basis A_k, we introduce the Frobenius basis

$$\bar{A}_k = \{a_k - a_{k-1}, \ldots, a_k - a_2, a_k - a_1, a_k\} , \tag{2.15}$$

where the basis elements are coprime since $a_1 = 1$. Meures then showed that

$$n_h(A_k) = ha_k - g(\bar{A}_k) - 1 , \tag{2.16}$$

if $h \geqq h_1$ (where the bound h_1 is very difficult to determine in the general case). We shall prove Meures' result in the next section.

For $k=3$, however, it follows from Theorem 2.2 that (2.16) holds for *all* $h \geqq h_0$. We thus get the important

THEOREM 2.3. *If* $h \geqq h_0$, *then*

$$n_h(1, a_2, a_3) = ha_3 - g(a_3 - a_2, a_3 - 1, a_3) - 1 . \tag{2.17}$$

To determine $n_h(A_3)$, we therefore need a formula for or a procedure to calculate $g(a,b,c)$ for a basis of three elements. The first such procedure was published by Beyer and the author [12]. Based on a certain continued fraction algorithm, we gave a formula for $g(a,b,c)$. The formula contained a function of an even number of arguments:

$$M\{x_1, x_2, \ldots, x_{2m}\} = x_{i_{m+1}} \text{ if } x_{i_1} \leqq x_{i_2} \leqq \cdots \leqq x_{i_{2m}} .$$

Our result was simplified by Rödseth [8]. By using negative division remainders in the continued fraction algorithm, he could replace our M-function by the minimum of two numbers only. For later use, we shall give a brief account of Rödseth's result (which, incidentally, had appeared earlier in an unpublished thesis by Siering [13]).

We assume $(a,b)=1$, and determine s_0 by

$$bs_0 \equiv c \pmod a,\ 0 \leqq s_0 < a. \tag{2.18}$$

We also introduce the integer

$$t = \frac{1}{a}(bs_0 - c), \tag{2.19}$$

which is positive if the basis $\{a,b,c\}$ is *independent*, that is, if none of the basis elements has a representation by the other ones.

The above-mentioned continued fraction algorithm is then applied to the ratio a/s_0:

$$\begin{cases} a = q_1 s_0 - s_1,\ 0 \leqq s_1 < s_0 \\ s_0 = q_2 s_1 - s_2,\ 0 \leqq s_2 < s_1 \\ s_1 = q_3 s_2 - s_3,\ 0 \leqq s_3 < s_2 \\ \dots\dots\dots\dots\dots\dots\dots \end{cases} \tag{2.20}$$

The continued fraction *convergents* P_i/Q_i are determined by

$$\begin{cases} P_0 = 1,\ P_1 = q_1,\ P_2 = q_1 q_2 - 1; & P_{i+1} = q_{i+1}P_i - P_{i-1} \\ Q_0 = 0,\ Q_1 = 1,\ Q_2 = q_2; & Q_{i+1} = q_{i+1}Q_i - Q_{i-1}. \end{cases} \tag{2.21}$$

In this notation, the result of Rödseth (and Siering) can be formulated as

THEOREM 2.4. *If*

$$\frac{P_{v+1}}{Q_{v+1}} \leqq \frac{b}{t} \leqq \frac{P_v}{Q_v}, \tag{2.22}$$

then

$$g(a,b,c) = -a + b(s_v - 1) + c(P_{v+1} - 1) - \min\{bs_{v+1}, cP_v\}. \tag{2.23}$$

Rödseth gave the condition (2.22) in the form

$$\frac{s_{v+1}}{P_{v+1}} \leqq \frac{c}{b} < \frac{s_v}{P_v}. \tag{2.24}$$

From his formulas (3.1–4), the equivalence of the left $\leqq$ in (2.22) and (2.24) follows easily. We also note that the right $<$ in (2.24) may be replaced by $\leqq$, since it is simple to show that in the case of equality, the formula (2.23) gives the same value of $g(a,b,c)$ if v is replaced by $v-1$. This observation will prove useful later.

We shall need one more result on the Frobenius number $g(a,b,c)$, due to Vitek [14, Th. 1]: For a coprime and *independent* basis with $a<b<c$, we have

$$g(a,b,c) \leqq (c-2)\left[\frac{a}{2}\right] - 1. \tag{2.25}$$

As mentioned in the Introduction, one main purpose of the present paper is to generalize the explicit formulas for $n_h(A_3)$ given by Salié [10]. He restricted himself to the case $h=h_0$, which of course suffices because of (2.14) (but this formula was not known when Salié wrote his paper).

Theorem 2.1, also proved by Salié, has already been stated. We conclude this section on known results with a summary of his remaining explicit formulas: If

$$h_0 \equiv \varrho \pmod{r+f-1},\ 0 \leqq \varrho < f, \tag{2.26}$$

then

$$n_{h_0}(A_3) = r(a_3-1)\left[\frac{h_0}{r+f-1}\right]-(r-2)a_3-2\ . \tag{2.27}$$

In particular,

$$r = 1 \Rightarrow n_{h_0}(A_3) = (a_3-1)\left[\frac{h_0}{f}\right]+a_3-2\ . \tag{2.28}$$

We further have

$$s = q \Rightarrow n_{h_0}(A_3) = (a_3-1)\left[\frac{h_0+2}{s}\right]+\left[\frac{h_0+1}{s}\right]+a_3-r-2\ . \tag{2.29}$$

$$f = 1 \Rightarrow n_h(1,h+1,h+r+1) \geqq (h+1)^2-r(r-1)-1\ , \tag{2.30}$$

with equality if $r \mid h$.

The formulas (2.27–30) correspond in this order to Satz 5–8 in Salié. Some of his formulations have been partly modified above, using (2.5).

3. Some general observations.

Before generalizing Salié's formulas, we present some apparently new results which apply to an arbitrary basis A_k.

The notion of an *admissible* basis extends immediately to the general case. An obvious generalization of (2.1) and (2.3) are the conditions

$$a_i < a_{i+1} < n_h(A_i)+2, \qquad i=1,2,\ldots,k-1\ . \tag{3.1}$$

These give the correct bounds (2.1) also for $i=1$, since $a_1=1$ and $n_h(A_1)=h$.

For $k=3$, the *last one* of the conditions (2.1) and (2.3) determines the smallest possible $h=h_0$ of (2.5), which also satisfies the *first* condition (2.1), since $f \geqq 1$. It was observed that this property holds in general. *The last condition* (3.1), *for* $i=k-1$, *determines the smallest* $h=h_0$ *such that all the conditions* (3.1) *are satisfied.*

The proof is trivial: If

$$a_k < n_{h_0}(A_{k-1})+2\ , \tag{3.2}$$

but $a_{i+1} \geqq n_{h_0}(A_i)+2$ for some $i<k-1$, then the h-range $n_{h_0}(A_i)$ will not be increased by addition of basis elements $a_{i+1},\ldots,a_k$. This would imply that

$$n_{h_0}(A_{k-1}) = n_{h_0}(A_i) \leqq a_{i+1}-2 < a_k-2 ,$$

contradicting (3.2).

Incidentally, the conditions (3.1) are equivalent to the *one* condition

$$n_h(A_k) \geqq a_k .$$

Another observation is perhaps more surprising: *When extending a basis* A_{k-1} *with a new element* a_k, *the conditions* (3.1) *do not ensure that the h-range increases.* In other words, we may have

$$n_h(A_{k-1} \cup \{a_k\}) = n_h(A_{k-1}) , \tag{3.3}$$

even if both bases are admissible.

We shall see soon that (3.3) is impossible for $k=3$. Already for $k=4$, however, we can give an example where *several* basis elements may be added without increasing the h-range. Let

$$a_2 = h+1,\ a_3 = h+2;\ n_h(A_3) = h(h+2) ,$$

and put

$$A_{h+2} = \{1, a_2, a_3, a_2+a_3, 2a_2+a_3, \ldots, (h-1)a_2+a_3\} .$$

Then

$$n_h(A_{h+2}) = n_h(A_3) ,$$

even if all extensions of A_3 are admissible.

To prove this, it suffices to show that

$$n_h(A_3)+1 = (h+1)^2 = (h+1)a_2$$

has no representation in at most h addends from A_{h+2}. Using a_2 alone, $h+1$ addends are necessary. All other basis elements are $\equiv 1 \pmod{a_2}$, and it is clearly impossible to form a multiple of a_2 by using at most $h=a_2-1$ of these.

The result bears a striking resemblance with an earlier observation by the author [11, § 4], concerning extension of *Frobenius* bases without altering (decreasing) the Frobenius number.

Considering only *regular* representations, the analogue of (3.3) does not hold. Indeed, we can show that

$$g_h(A_{k-1} \cup \{a_k\}) > g_h(A_{k-1}) , \tag{3.4}$$

assuming admissible bases.

Incidentally, this implies that

$$n_h(A_3) \geqq g_h(A_3) > g_h(A_2) = n_h(A_2),$$

showing that (3.3) is impossible for $k=3$.

To prove (3.4), we consider the (unique) regular representation of

$$g_h(A_{k-1} \cup \{a_k\}) = g_h(A_k) = x_1 \cdot 1 + x_2 a_2 + \ldots + x_k a_k.$$

The coefficients x_i were determined by Hofmeister [2, Satz 1]. It follows from his formulas that x_i depends only on $a_2, a_3, \ldots, a_{i+1}$ for $i<k$. It is an easy consequence that

$$g_h(A_k) - g_h(A_{k-1}) = x_k(a_k - a_{k-1}), \tag{3.5}$$

which implies (3.4) (since $x_k > 0$ for an admissible basis).

Our final observation concerns the possibility of equality in (2.13). We have already seen that this is always the case for $k=2$ and $k=3$, and shall now give a *sufficient* condition for arbitrary k:

THEOREM 3.1. *If* $h \geqq h_0 - 1$, *and*

$$n_h(A_k) \geqq (h+1)a_{k-1} - a_k, \tag{3.6}$$

then

$$n_{h+1}(A_k) = n_h(A_k) + a_k. \tag{3.7}$$

If h is increased by 1, the right hand side of (3.6) increases with a_{k-1}, while the left hand side increases with at least a_k. There is consequently a smallest h_S such that (3.6) and hence (3.7) are satisfied for *all* $h \geqq h_S$.—This has been used effectively by Mossige [7] in a computer program to determine $n_h(k)$ for arbitrary k.

Let us call the numbers (1.2) "h-representable". These integers form a certain *pattern* along the non-negative number axis, with the *first* "*gap*" at $n_h(A_k)+1$.

The $(h+1)$-representable numbers can be divided into two sets: 1) Those with $x_k > 0$, corresponding to a *translation* of the above pattern a_k units to the right. 2) The combinations with $x_k = 0$; these cover all non-negative integers $< a_k$ (since $h+1 \geqq h_0$), and do not exceed $(h+1)a_{k-1}$. The translated first gap is thus not affected by the second set if $n_h(A_k) + 1 + a_k > (h+1)a_{k-1}$.

This completes the proof of Theorem 3.1. Incidentally, the same argument shows that the inequality (2.13) is valid for $h \geqq h_0 - 1$.

We may say that $n_h(A_k)$ is "*stabilized*" for $h = h_1$ if (3.7) holds for all $h \geqq h_1$ but not for $h = h_1 - 1$. With $k=3$, it follows from Theorem 2.2 that $h_1 = h_0$ for a non-pleasant basis (since it is then easily seen that $n_{h_0}(A_3) > n_{h_0-1}(A_3) + a_3$). On the other hand, it is not difficult to show that $h_1 = h_0 - 1$ for all pleasant bases A_k (when $n_h(A_k) = g_h(A_k)$).

It is clear that h_1 also represents the bound for h in Meures' formula (2.16), for which we can now give a very simple proof.

If we choose $h \geqq h_1$, the difference $ha_k - n_h(A_k)$ is independent of h. We note that the largest h-representable number is ha_k, and that $ha_k - t$, $t \geqq 0$, is h-representable:

$$ha_k - t = \sum_{i=1}^{k} x_i a_i, \quad \sum_{i=1}^{k} x_i = h' \leqq h ,$$

if and only if

$$t = \sum_{i=1}^{k-1} x_i(a_k - a_i) + (h - h')a_k$$

has a representation by the basis $\bar{A}_k$ of (2.15)—with *no restriction on the number of addends*, since h can be chosen arbitrarily large. The largest non-representable t is thus the Frobenius number $g(\bar{A}_k)$, and all non-negative integers $< ha_k - g(\bar{A}_k)$ are then h-representable (by A_k). But this is just Meures' result (2.16).

It follows from (2.1–2) that (3.6) always holds—with strict inequality—for $k = 2$. For $k \geqq 3$, however, we *cannot reverse* Theorem 3.1:

For $k = 3$, we know that (3.7) always holds for $h \geqq h_0$. On the other hand, the discussion in connection with Theorem 10.1 below describes some general cases where (3.6) fails.

For $k = 4$, there are no counter-examples with $h_0 = 2$, but two with $h_0 = 3$:

$$A_4 = \{1, 3, 8, 9\}, \quad n_3(A_4) = 21, \; n_4(A_4) = 30$$

$$A_4 = \{1, 3, 8, 11\}, \quad n_3(A_4) = 17, \; n_4(A_4) = 28 .$$

4. Organizing the bases A_3.

We now turn to the case $k = 3$. For a given $h = H$, the conditions (2.1) and (2.3) give the total number

$$\frac{H(H^2 + 6H - 1)}{6} \tag{4.1}$$

of admissible bases. This number *includes* all the admissible bases with $h_0 \leqq H$.

To avoid this repetition, we shall focus attention on the "h_0-*bases*". For a given h, this means to consider only those bases for which this h is just the h_0 of (2.5). In other words, we choose the combinations

$$(4.2)\qquad \begin{cases} a_2 = 2, 3, \ldots, h+1 \\ a_3 = fa_2 + r = (h - a_2 + 2)a_2 + r,\ r = 0, 1, \ldots, a_2 - 1\ . \end{cases}$$

If we can determine the range $n_{h_0}(A_3)$ for such a basis, the general case is immediately covered by Theorem 2.2.

We may further restrict ourselves to *non-pleasant* bases, since the pleasant ones are covered by Theorem 2.1. In (4.2), it then suffices to choose

$$(4.3)\qquad r = 1, 2, \ldots, a_2 - f - 1 = 2a_2 - h - 3\ .$$

If this interval for r shall not be empty, we must have

$$2a_2 - h - 3 \geqq 1,\ \text{or}\ a_2 \geqq \left[\frac{h+5}{2}\right].$$

For non-pleasant h_0-bases, the conditions (4.2) are thus replaced by

$$(4.4)\qquad \begin{cases} a_2 = \left[\dfrac{h+5}{2}\right], \ldots, h+1 \\ a_3 = (h + 2 - a_2)a_2 + r,\ r = 1, 2, \ldots, 2a_2 - h - 3\ . \end{cases}$$

We note that $r = 2a_2 - h - 3$ corresponds to $s = q$, cf. (2.10).

A simple count shows that the number of non-pleasant bases A_3 with $h_0 \leqq H$ is given by

$$(4.5)\qquad \left[\frac{H(H+2)(2H-1)}{24}\right].$$

Comparing with (4.1), we see that *asymptotically* 50% *of all bases* A_3 *are pleasant.*

From a computational point of view, the conditions (4.4) are very simple to implement. So are of course also the conditions (2.1) and (2.3) for the set of *all* admissible bases for a given h. Some more care is required if we want to exclude the pleasant bases in the latter case.

This problem arises if we want to determine the *extremal h-range* $n_h(3)$ for a given h. We then need to scan all admissible bases, to find the largest $n_h(A_3)$. We may, however, leave out the pleasant ones, since it is easily seen that

$$(4.6)\qquad n_h(3) > g_h(3), \quad h \geqq 2\ .$$

One way to prove this inequality is to use two bounds of Hofmeister, one upper bound [2, (35a) p. 54] for $g_h(3)$, and one lower bound [3, Folgerung 1 p. 81] for $n_h(3)$. A comparison shows that (4.6) is always satisfied for $h \geqq 9$, and the cases $2 \leqq h < 9$ are easily dealt with numerically.

Incidentally, it seems a "safe bet" that we always have

$$n_h(k) > g_h(k), \qquad h \geqq 2,\ k \geqq 3\ ,$$

but apparently no proof of this inequality exists for arbitrary k.

To exclude the pleasant bases from (2.1) and (2.3), we note that (2.3) gives $f \leqq h+2-a_2$, while (4.3) shows that $f \leqq a_2-2$. All non-pleasant bases A_3 for a given h are thus determined by

$$(4.7) \qquad \begin{cases} a_2 = 3,4,\ldots,h+1;\ a_3 = fa_2+r \\ f = 1,2,\ldots,\min\{a_2-2, h+2-a_2\} \\ r = 1,2,\ldots,a_2-f-1\ . \end{cases}$$

For a chosen basis $A_3=\{1,a_2,a_3\}$, the simplest way to find $n_h(A_3)$ is to determine h_0 from (2.5), then calculate $n_{h_0}(A_3)$ from the formulas given later, and finally use (2.14) to determine $n_h(A_3)$.

We mention that this procedure was used successfully to decide a problem due to Hofmeister: He proved [3, Satz 2] a formula for the *extremal h-range* $n_h(3)$ and the corresponding extremal basis A_3^*, for h "sufficiently large". He also gave a table for $h \leqq 34$ (where, incidentally, the "anticipated" extremal basis $\{1,19,102\}$ for $h=22$ is missing). For $h>22$, all entries of the table confirm his general result on A_3^*.

Prompted by the possibility of further computer results, Hofmeister has recently taken a new look at his proof. He has informed me that it suffices to check separately the cases with $h \leqq 200$. This was done on our computer by Mossige [7], but no further exceptions were found. *For $h>22$, Hofmeister's result on the extremal h-range $n_h(3)$ thus covers all extremal bases.*

We note that Hofmeister's result *must* fail for some small h, since his bases A_3^* are *not admissible* for even $h \leqq 8$, and for odd $h \leqq 17$.

We conclude this section with Table 1, which gives all non-pleasant h_0-bases A_3 for $h_0 \leqq 10$, together with the corresponding values of $n_{h_0}(A_3)$.

The pairs (a_2, a_3) are grouped in *intervals*, with a_2 and f fixed within each interval. The intervals are arranged according to *decreasing* f. In what follows, "interval" will always refer to this grouping. In particular, the "last interval" corresponds to

$$(4.8) \qquad f = 1,\ a_2 = h+1,\ a_3 = h+r+1,\ 1 \leqq r \leqq h-1\ .$$

Table 1. *Values of* $n_{h_0}(A_3)$, $h_0 \leqq 10$, *for non-pleasant bases.*

$h_0=2$

a_2	a_3	n_2
3	4	8

$h_0=3$

a_2	a_3	n_3
4	5	15
4	6	14

$h_0=4$

a_2	a_3	n_4
4	9	23
5	6	24
5	7	22
5	8	26

$h_0=5$

a_2	a_3	n_5
5	11	29
5	12	32
6	7	35
6	8	34
6	9	31
6	10	34

$h_0=6$

a_2	a_3	n_6
5	16	44
6	13	47
6	14	50
6	15	40
7	8	48
7	9	46
7	10	42
7	11	48
7	12	52

$h_0=7$

a_2	a_3	n_7
6	19	53
6	20	56
7	15	55
7	16	58
7	17	63
7	18	65
8	9	63
8	10	62
8	11	68
8	12	54
8	13	69
8	14	62

$h_0=8$

a_2	a_3	n_8
6	25	71
7	22	62
7	23	86
7	24	67
8	17	79
8	18	84
8	19	87
8	20	74
8	21	77
9	10	80
9	11	78
9	12	76
9	13	68
9	14	89
9	15	82
9	16	86

$h_0=9$

a_2	a_3	n_9
7	29	83
7	30	86
8	25	95
8	26	98
8	27	103
8	28	78
9	19	89
9	20	112
9	21	97
9	22	104
9	23	107
9	24	88
10	11	99
10	12	98
10	13	93
10	14	102
10	15	83
10	16	102
10	17	109
10	18	98

$h_0=10$

a_2	a_3	n_{10}
7	36	104
8	33	95
8	34	130
8	35	100
9	28	107
9	29	110
9	30	142
9	31	118
9	32	120
10	21	119
10	22	124
10	23	130
10	24	134
10	25	118
10	26	146
10	27	125
11	12	120
11	13	118
11	14	128
11	15	124
11	16	100
11	17	126
11	18	132
11	19	140
11	20	128

5. The Frobenius-dependent bases.

We shall now apply Theorem 2.3 to the determination of $n_h(A_3)$, and first consider the case where the Frobenius basis

(5.1) $$B = \{a_3 - a_2, a_3 - 1, a_3\}$$

is *dependent*. This means that one of the elements has a representation by the other ones, and it is easily seen that this occurs if and only if

(5.2) $$(a_3 - a_2) \mid (a_3 - 1) \quad \text{or} \quad (a_3 - a_2) \mid a_3 .$$

We then call the corresponding bases A_3 "Frobenius-dependent" or just "dependent".

It is clear that

$$(5.3) \qquad \begin{cases} (a_3-a_2)\,|\,(a_3-1) \Rightarrow g(B) = g(a_3-a_2, a_3) \\ (a_3-a_2)\,|\,a_3 \Rightarrow g(B) = g(a_3-a_2, a_3-1)\,. \end{cases}$$

The problem is thus reduced to a Frobenius basis of *two* elements. These are clearly coprime, and we may then use the well known formula

$$(5.4) \qquad g(b_1, b_2) = b_1 b_2 - b_1 - b_2\,.$$

The conditions (5.2) imply that $a_3 \leqq 2a_2$. We leave out the pleasant case $a_3 = 2a_2$, and can then confine ourselves to the *last interval* (4.8). Here $a_3 - a_2 = r$, and

$$(a_3-a_2)\,|\,(a_3-1) \Leftrightarrow r\,|\,h, \qquad (a_3-a_2)\,|\,a_3 \Leftrightarrow r\,|\,(h+1)\,.$$

Using (5.3), (5.4) and (2.17), a simple calculation shows that

$$(5.5) \qquad n_h(1, h+1, h+r+1) = \begin{cases} (h+1)^2 - r(r-1) - 1 & \text{if } r\,|\,h\,, \\ (h+1)^2 - r(r-2) - 2 & \text{if } r\,|\,(h+1)\,. \end{cases}$$

The case $r\,|\,h$ was already covered by Salié, cf. (2.30). His method is in a sense more "elementary", using only the definition of $n_h(A_3)$. We mention that a similar proof, independent of Theorem 2.3, is fairly simple to carry through also in the case $r\,|\,(h+1)$ (in the formulation (5.8) below).

When $r \nmid h$, we have strict inequality in (2.30). This was not stated explicitly by Salié, but is included in the following result:

The two formulas of (5.5) coincide for $r=1$. When $r>1$, the second expression is clearly larger than the first one, but *smaller than the range of any non-dependent basis*:

$$(5.6) \qquad n_h(1, h+1, h+r+1) > (h+1)^2 - r(r-2) - 2 \quad \text{if } r \nmid h, h+1\,.$$

To prove this, we use Vitek's bound (2.25), which shows that

$$g(r, h+r, h+r+1) \leqq (h+r-1)\left[\frac{r}{2}\right] - 1\,.$$

Substitution in (2.17) gives

$$(5.7) \qquad n_h(1, h+1, h+r+1) \geqq \left(h - \left[\frac{r}{2}\right]\right)(h+r-1) + 2h\,.$$

A simple calculation shows that this bound exceeds the right hand side of (5.6) when $r>2$. And the bases with $r=1$ or $r=2$ are always dependent.

We conclude this section with an alternative formulation, useful later, of (5.5): For any two positive integers p and r, we have

$$(5.8)\qquad \begin{cases} n_{pr}(1, pr+1, (p+1)r+1) = (p^2-1)r^2+(2p+1)r \\ n_{pr-1}(1, pr, (p+1)r) \quad = (p^2-1)r^2+2r-2\,, \end{cases}$$

where we must assume $p>1$ in the latter case.

6. General formulas for $n_h(A_3)$.

We shall now combine the Theorems 2.3 and 2.4 to obtain formulas for $n_h(A_3)$. It turns out that we get a substantial simplification of the general conditions in Theorem 2.4, implicitly due to the fact that *two of the Frobenius basis elements* (5.1) *differ by a unit.*

We note that there are a priori *six permutations* of these basis elements, to be used as the a, b. c of Theorem 2.4. It is possible (cf. [11, formula (3.1)]) to remove a common factor of two elements in a 3-element Frobenius basis. However, if we want to solve the congruence (2.18) for s_0 in the general case, we must be certain that a and b are *coprime.* In addition, this congruence should preferably have an *explicit* solution.

Fortunately, both these conditions are satisfied simultaneously by the two choices of a and b among the basis elements a_3-1 and a_3. At the same time, we get a simple expression for the t of (2.19):

$$(6.1)\qquad a = a_3,\ b = a_3-1,\ c = a_3-a_2;\ s_0 = a_2,\ t = a_2-1$$

$$(6.2)\qquad a = a_3-1,\ b = a_3,\ c = a_3-a_2;\ s_0 = t = a_3-a_2\,.$$

With the first combination (6.1), we shall perform the division algorithm (2.20) with $a=a_3$, $s_0=a_2$. For the numbers P_i and Q_i of (2.21), it is easily shown by induction that we then have

$$(6.3)\qquad a_2P_i-a_3Q_i = s_i, \quad i=0,1,2,\ldots\,.$$

This implies that

$$\frac{P_i}{Q_i}-\frac{b}{t} = \frac{(a_2-1)P_i-(a_3-1)Q_i}{tQ_i} = \frac{s_i-(P_i-Q_i)}{tQ_i}\,,$$

and the conditions (2.22) are thus equivalent to

$$s_v \geqq P_v-Q_v,\ s_{v+1} \leqq P_{v+1}-Q_{v+1}\,.$$

We can also give an explicit condition for the choice of argument in the minimum of (2.23):

$$\min\{bs_{v+1}, cP_v\} = \min\{(a_3-1)s_{v+1}, (a_3-a_2)P_v\}\,.$$

It follows from (6.3) that

$$(a_3-1)(P_v-Q_v)-(a_3-a_2)P_v = s_v-(P_v-Q_v) \geqq 0\,,$$

hence

$$\min\{\ \} = (a_3-a_2)P_v \text{ if } s_{v+1} \geqq P_v - Q_v .$$

On the other hand,

$$(a_3-a_2)P_v-(a_3-1)(P_v-Q_v-1) = a_3-s_v+P_v-Q_v-1 > 0 ,$$

since

$$s_v \leqq s_0 = a_2 < a_3,\ P_v-Q_v \geqq P_0-Q_0 = 1 .$$

This shows that

$$\min\{\ \} = (a_3-1)s_{v+1} \text{ if } s_{v+1} \leqq P_v-Q_v-1 .$$

Substituting a, b, c from (6.1) into (2.23), and using (2.17), we thus get

THEOREM 6.1. *In the notation of Theorem 2.4, put* $a=a_3$, $s_0=a_2$. *If*

$$s_v \geqq P_v-Q_v,\ s_{v+1} \leqq P_{v+1}-Q_{v+1} , \tag{6.4}$$

then

$$n_h(1,a_2,a_3) = (h+3)a_3-a_2-2-(a_3-1)s_v-(a_3-a_2)P_{v+1} + \begin{cases} (a_3-a_2)P_v & \text{if } s_{v+1} \geqq P_v-Q_v , \\ (a_3-1)s_{v+1} & \text{if } s_{v+1} < P_v-Q_v . \end{cases} \tag{6.5}$$

We next turn to the second combination (6.2), where now $a=a_3-1$, $s_0=a_3-a_2$, so that (6.3) is replaced by

$$(a_3-a_2)P_i-(a_3-1)Q_i = s_i, \quad i=0,1,2,\dots . \tag{6.6}$$

We then repeat the arguments that led to Theorem 6.1. Since now $s_0=c$, these arguments become even simpler, giving

THEOREM 6.2. *In the notation of Theorem 2.4, put* $a=a_3-1$, $s_0=a_3-a_2$. *If*

$$s_v \geqq Q_v,\ s_{v+1} \leqq Q_{v+1} , \tag{6.7}$$

then

$$n_h(1,a_2,a_3) = (h+3)a_3-a_2-2-a_3s_v-(a_3-a_2)P_{v+1} + \begin{cases} (a_3-a_2)P_v & \text{if } s_{v+1} > Q_v , \\ a_3s_{v+1} & \text{if } s_{v+1} \leqq Q_v . \end{cases} \tag{6.8}$$

As already mentioned in the Introduction, the Theorems 6.1–2 are not strictly "explicit", since the necessary number $v+1$ of division steps (2.20) cannot be determined a priori. Let us examine some cases where only one or two steps suffice:

In Theorem 6.1, we get $v=0$ if $s_1 \leqq P_1 - Q_1 = q_1 - 1$. In our standard notation (2.4), we now have $s_1 = s$, $q_1 = q$. A comparison with (2.10) shows that $v=0$ *in Theorem* 6.1 *corresponds exactly to the pleasant bases* A_3.

Let us also examine the case $v=1$, to illustrate a typical technique which will be used frequently below. For a non-pleasant basis, we now perform the next division step (2.20):

$$a_2 = q_2 s - s_2,\ 0 \leqq s_2 < s\ .$$

By (6.4), the condition for $v=1$ is

$$s_2 = q_2 s - a_2 \leqq P_2 - Q_2 = q q_2 - 1 - q_2\ ,$$

or

$$a_2 - 1 \geqq (s-q+1) q_2 = (s-q+1)\left\langle \frac{a_2}{s} \right\rangle . \tag{6.9}$$

As usual, $\langle x \rangle$ denotes the smallest integer $\geqq x$.

For real numbers x and y, it is easily seen that

$$y \geqq \langle x \rangle \Leftrightarrow x \leqq [y]\ . \tag{6.10}$$

The condition (6.9) for $v=1$ in Theorem 6.1 can thus be formulated alternatively as

$$a_2 \leqq s\left[\frac{a_2 - 1}{s-q+1}\right] \tag{6.11}$$

(assuming a non-pleasant basis, that is, $s \geqq q$).

Let us also consider the case $v=0$ in Theorem 6.2. By (6.7), the condition for this is $s_1 \leqq Q_1 = 1$, where

$$a_3 - 1 = q_1(a_3 - a_2) - s_1,\ 0 \leqq s_1 < a_3 - a_2\ .$$

Now $s_1 = 0$ and $s_1 = 1$ are the only possibilities, and we see that

$$s_1 = 0 \Leftrightarrow (a_3 - a_2) \mid (a_3 - 1),\ s_1 = 1 \Leftrightarrow (a_3 - a_2) \mid a_3\ .$$

A comparison with (5.2) shows that $v=0$ *in Theorem* 6.2 *corresponds exactly to the Frobenius-dependent bases* A_3.

7. Two explicit formulas.

In this section, we shall prove the following generalizations of Salié's formulas (2.29) and (2.27):

THEOREM 7.1. *Let A_3 be a non-pleasant basis satisfying the condition*

$$a_2 \leqq s\left[\frac{a_2-1}{s-q+1}\right]. \tag{7.1}$$

Then

$$n_{h_0}(A_3) = (a_3-1)\sum_{i=0}^{s-q}\left[\frac{h_0+2+i}{s}\right] - (s-q+1)\left(a_3-\left[\frac{h_0+1}{s}\right]\right)+2a_3-r-2\,. \tag{7.2}$$

THEOREM 7.2. *Let A_3 be any basis satisfying the condition*

$$a_2-1 \leqq (r+f-1)\left[\frac{a_2}{r}\right]. \tag{7.3}$$

Then

$$n_{h_0}(A_3) = \left(\sum_{i=0}^{r-1}\left[\frac{h_0+i}{r+f-1}\right]-r+2\right)a_3-r\left[\frac{h_0}{r+f-1}\right]-2\,. \tag{7.4}$$

Inspired by Salié's formulas (2.27–29), and using numerical evidence provided by Mossige, I first *conjectured* the simplest formula (7.4), and later also (7.2). Both formulas were next verified numerically for a larger set of bases, before the proofs were found.

With $s=q$, the condition (7.1) is automatically satisfied, and (7.2) coincides with Salié's formula (2.29). Note that $s=q$ corresponds to the *last* (non-pleasant) basis in each interval. For a basis just above the last one, with $s=q+1$, the condition (7.1) fails only in the one case $A_3=\{1,4,5\}$, with $h_0=3$.

Salié's main formula (2.27) is a special case of Theorem 7.2. To see this, we note that the condition $\varrho<f$ of (2.26) implies

$$\left[\frac{h_0}{r+f-1}\right]=\left[\frac{h_0+1}{r+f-1}\right]=\ \dots\ =\left[\frac{h_0+r-1}{r+f-1}\right], \tag{7.5}$$

showing that the formulas (7.4) and (2.27) coincide.

We note that the condition (7.3) has the equivalent formulation

$$a_2 \geqq r\left\langle\frac{a_2-1}{r+f-1}\right\rangle, \tag{7.6}$$

cf. (6.10). On the other hand, (2.26) may by means of (2.5) be written as

$$h_0+r-1 \equiv a_2-2 \equiv \varrho_1 \pmod{r+f-1},\ r-1 \leqq \varrho_1 < r+f-1\,,$$

which is more in accordance with Salié's original formulation. It is now easily seen that his condition $\varrho_1 \geqq r-1$ implies—and is *stronger* than—the condition (7.6).

With $r=1$, the condition (7.3) is automatically satisfied, and we get Salié's formula (2.28). Note that $r=1$ corresponds to the *first* (non-pleasant) basis in each interval. The condition (7.3) is also always satisfied for $r=2$, and for $r=3$ if $f>1$.

To prove Theorems 7.1–2, we must first determine the sums of integer values. In Salié's case (7.5), these all coincide. In the general case, there may be a "jump" of one unit if the numerator "passes" a multiple of the denominator.

Consider first the sum in (7.2), where clearly the residue of $h_0+2 \pmod s$ is decisive. We prefer to use the negative residue

$$h_0+2 \equiv -\mu \pmod s, \quad 0 \leqq \mu < s .$$

If $m=\min\{\mu, s-q+1\}$, there is then a set of m equal addends:

$$\left[\frac{h_0+2}{s}\right] = \left[\frac{h_0+3}{s}\right] = \dots = \left[\frac{h_0+1+m}{s}\right], \text{ each } = \left[\frac{h_0+1}{s}\right].$$

If $\mu \leqq s-q$, there is also another set of $s-q+1-\mu$ equal addends:

$$\left[\frac{h_0+2+\mu}{s}\right] = \left[\frac{h_0+3+\mu}{s}\right] = \dots = \left[\frac{h_0+2+s-q}{s}\right], \text{ each } = \left[\frac{h_0+1}{s}\right]+1 .$$

Substituting in (7.2), an easy calculation shows that Theorem 7.1 is equivalent to the following

LEMMA 7.1. *Let A_3 be a non-pleasant basis satisfying the condition* (7.1), *and put*

$$h_0+2 \equiv -\mu \pmod s,\ 0 \leqq \mu < s;\ m = \min\{\mu, s-q+1\} .$$

Then

$$n_{h_0}(A_3) = a_3(s-q+1)\left[\frac{h_0+1}{s}\right] - m(a_3-1)+2a_3-a_2+q-3 . \tag{7.7}$$

To determine the sum in (7.4), we may similarly put $h_0 \equiv -\nu \pmod{r+f-1}$, and proceed as above. There is, however, some trouble with the case $\nu=0$. This trouble can be circumvented by using instead the smallest non-negative remainder $h_0 \equiv \varrho$. We then find that Theorem 7.2 is equivalent to the following

LEMMA 7.2. *Let A_3 be any basis satisfying the condition* (7.3), *and put*

$$h_0 \equiv \varrho \pmod{r+f-1},\ 0 \leqq \varrho < r+f-1;\ M = \max\{\varrho, f-1\} .$$

Then

$$(7.8) \qquad n_{h_0}(A_3) = r(a_3-1)\left[\frac{h_0}{r+f-1}\right]-a_3(r+f-M-3)-2 .$$

In particular, we get the correct formula (2.11) in all *pleasant* cases, since (7.3) is satisfied if $r=0$ or $r+f \geqq a_2$.

We note that the case $M=f-1$, that is, $\varrho<f$, corresponds exactly to Salié's main formula (2.27). As shown above, the condition (7.3) (in the form (7.6)) is then *automatically* satisfied.

There is in fact a similar case of Lemma 7.1, if $m=s-q+1$, that is, $\mu>s-q$. This is equivalent to

$$a_2 \equiv -\mu_1 \pmod s, \qquad 0 \leqq \mu_1 < q-1 .$$

The condition (7.1) (in the form (6.9)) is then automatically satisfied, and we get

$$n_{h_0}(A_3) = a_3(s-q+1)\left(\left[\frac{h_0+1}{s}\right]-1\right)+2a_3-(r+2) ,$$

in striking analogy with Salie's main formula (2.27).

Because of the congruences entering into Lemmas 7.1–2, they are "less explicit" than the original Theorems 7.1–2. On the other hand, the Lemmas are in most cases *simpler to apply* to a given basis A_3.

We next proceed to prove Lemma 7.1. Since the condition (7.1) coincides with (6.11), we must expect to need Theorem 6.1 for $v=1$. With

$$(7.9) \qquad a_2 = q_2 s - s_2, \; 0 \leqq s_2 < s ,$$

we then have

$$(7.10) \qquad n_{h_0}(A_3) = (h_0+3)a_3-a_2-2-(a_3-1)s-(a_3-a_2)(qq_2-1) + \begin{cases}(a_3-a_2)q & \text{if } s_2 \geqq q-1 ,\\ (a_3-1)s_2 & \text{if } s_2 < q-1 .\end{cases}$$

For a non-pleasant basis, $f=q-1$ and hence $h_0=f+a_2-2=q+a_2-3$ by (2.5). For the integer value in (7.7), we thus get

$$(7.11) \qquad \left[\frac{h_0+1}{s}\right] = \left[\frac{q+a_2-2}{s}\right] = \left[\frac{q+q_2 s-s_2-2}{s}\right] = q_2+\left[\frac{q-s_2-2}{s}\right],$$

using (7.9). On the other hand, $s \geqq q$ for a non-pleasant basis, and $q \geqq 2$, $0 \leqq s_2 < s$, so

$$-1 \leqq \frac{q-2}{s}-1 = \frac{q-s-2}{s} < \frac{q-s_2-2}{s} \leqq \frac{q-2}{s} < 1 .$$

The last integer value of (7.11) thus equals -1 or 0, and clearly

$$\left[\frac{h_0+1}{s}\right] = \begin{cases} q_2-1 & \text{if } s_2 \geqq q-1\,, \\ q_2 & \text{if } s_2 < q-1\,. \end{cases} \tag{7.12}$$

We must also determine the minimum m of Lemma 7.1. Now

$$h_0+2 = q+q_2s-s_2-1 \equiv q-s_2-1 \equiv -\mu \pmod{s}$$

shows that $\mu=s_2-q+1$ if $s_2\geqq q-1$, and $\mu=s+s_2-q+1$ if $s_2<q-1$. This gives

$$m = \min\{\mu, s-q+1\} = \begin{cases} s_2-q+1 & \text{if } s_2 \geqq q-1\,, \\ s-q+1 & \text{if } s_2 < q-1\,. \end{cases} \tag{7.13}$$

We thus get *the same two alternatives* in (7.12–13) as in (7.10), and it is now straightforward—but tedious—to verify that (7.7) and (7.10) coincide in both cases. For the verification, we must substitute

$$h_0 = q+a_2-3,\ a_3 = qa_2-s,\ s_2 = q_2s-a_2\,.$$

This completes the proof of Lemma 7.1 and thus of Theorem 7.1. We may safely say that this theorem would never have been formulated without first being conjectured from numerical evidence.

The same remark applies to Theorem 7.2, or the equivalent Lemma 7.2, which we shall now prove. This will give the first illustration of the important fact that *several division steps of Theorems* 6.1–2 *may sometimes be combined into a single formal operation.*

We now use Theorem 6.2, and shall apply the division algorithm (2.20) to the ratio

$$\frac{a}{s_0} = \frac{a_3-1}{a_3-a_2} = \frac{fa_2+r-1}{(f-1)a_2+r}\,.$$

The algorithm takes the form

$$\begin{aligned} fa_2+r-1 &= 2\{(f-1)a_2+r\} &&-\{(f-2)a_2+r+1\} \\ (f-1)a_2+r &= 2\{(f-2)a_2+r+1\} &&-\{(f-3)a_2+r+2\} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ 3a_2+r+f-4 &= 2\{2a_2+r+f-3\} &&-\{a_2+r+f-2\} \\ 2a_2+r+f-3 &= 2\{a_2+r+f-2\} &&-\underbrace{\{r+f-1\}}_{s_{f-1}} \end{aligned}$$

$$a_2+r+f-2 = q_f(r+f-1)-s_f,\ 0 \leqq s_f < r+f-1\,. \tag{7.14}$$

We see that $q_1 = q_2 = \ldots = q_{f-1} = 2$ (and it is easily shown that $q_f > 2$ for a non-pleasant basis). This gives

$$P_i = i+1,\ Q_i = i; \quad i=0,1,\ldots,f-1$$
$$P_f = fq_f-(f-1),\ Q_f = (f-1)q_f-(f-2)\,.$$

Since

$$s_{f-1} = r+f-1 \geqq Q_{f-1} = f-1 \tag{7.15}$$

(including the pleasant case $r=0$), it follows from (6.7) that we always have $v \geqq f-1$. It turns out that Theorem 7.2 corresponds to the choice $v=f-1$. The condition for this is

$$s_f \leqq Q_f = (f-1)q_f-(f-2)\,. \tag{7.16}$$

It follows from (7.14) that

$$q_f = \left\langle \frac{a_2-1}{r+f-1} \right\rangle + 1,\ s_f = q_f(r+f-1)-(a_2+r+f-2)\,.$$

Substituting this into (7.16), we get

$$a_2 \geqq r(q_f-1) = r\left\langle \frac{a_2-1}{r+f-1} \right\rangle,$$

which is just the condition (7.6) of Theorem 7.2.

Under this condition, (6.8) gives

$$n_{h_0}(A_3) = (h_0+3)a_3-a_2-2-a_3(r+f-1)-(a_3-a_2)(fq_f-f+1) + \begin{cases}(a_3-a_2)f & \text{if } s_f \geqq f, \\ a_3 s_f & \text{if } s_f < f.\end{cases}$$

For the integer value and the maximum of Lemma 7.2, we now find

$$\left[\frac{h_0}{r+f-1}\right] = \begin{cases} q_f-2 & \text{if } s_f \geqq f, \\ q_f-1 & \text{if } s_f < f.\end{cases}$$

$$M = \max\{\varrho, f-1\} = \begin{cases} r+2f-s_f-2 & \text{if } s_f \geqq f, \\ f-1 & \text{if } s_f < f,\end{cases}$$

and the proof of Lemma 7.2 is completed as for Lemma 7.1.

The total number of non-pleasant bases A_3 with $h_0 \leqq H$ is given by (4.5). A numerical investigation with $H=500$ indicates that each of the Theorems 7.1 and 7.2 covers approximately 60% of these bases (asymptotically, as $H \to \infty$).

This should be compared with a coverage of about 30% for Salié's main formula (2.27).

However, the figure 60% does not give a true picture of the "density" of bases covered, since there are many bases which *a priori* fall outside the scope of the conditions (7.1) or (7.3). In general, Theorem 7.1 fails completely in the beginning of most intervals, and Theorem 7.2 in the end. More precisely, the condition (7.1) of Theorem 7.1 is a priori possible only if $a_2 - 1 \geqq 2(s-q+1) = 2(a_2 - r - f)$, or

$$(7.17) \qquad r \geqq \left[\frac{a_2}{2}\right] - f + 1 .$$

Similarly, the condition (7.3) of Theorem 7.2 is a priori possible (for a non-pleasant basis) only if $a_2 \geqq 2r$, or

$$(7.18) \qquad r \leqq \left[\frac{a_2}{2}\right] .$$

An elementary but complicated count shows that the numbers of non-pleasant bases A_3 satisfying (7.17) or (7.18), accumulated for $h_0 \leqq H$, are given by

$$\left[\frac{H^2(2H+3)+9}{36}\right] \quad \text{and} \quad \left[\frac{H^2(H+3)}{18}\right],$$

respectively. Asymptotically, both these numbers represent only *two thirds* of the total number (4.5). The asymptotic coverage for each of the Theorems 7.1 or 7.2, in those regions where they are *a priori possible*, is therefore approximately 90%.

In every interval, there is always an *overlap* between the two regions determined by (7.17–18) (and the region of overlap increases with f for a given h_0). *In combination*, the Theorems 7.1–2 are therefore a priori possible for all non-pleasant bases A_3. A numerical investigation indicates an asymptotic combined coverage of approximately 86%. An improvement of this figure (to 98%) will be the aim of the next section.

It turns out that the asymptotic coverage is lower for small values of f, in particular for $f=1$. As a matter of fact, we can show that the coverage is 0% in the case of Theorem 7.2 for $f=1$.

This theorem was deduced from Theorem 6.2 with $v=f-1$, hence $v=0$ for $f=1$. By the concluding remark of Section 6, this corresponds to the *Frobenius-dependent* bases (5.5), where r is a divisor of h or $h+1$.

Let $\tau(n)$ denote the number of divisors of an integer n, including 1 and n. The number of different divisors $r<h$ of h and $h+1$ is then

$$\tau(h)+\tau(h+1)-3 .$$

We now accumulate over the last interval ($f=1$), of length h_0-1, for $h_0 \leqq H$. The total number of bases is then $\frac{1}{2}H(H-1)$. On the other hand, it is well known that

$$\sum_{h \leqq H} \tau(h) = H \log H + (2\gamma - 1)H + O(\sqrt{H}),$$

where $\gamma = 0.57721\ldots$ is the Euler constant. This shows that the relative frequency of non-pleasant bases with $f=1$ and $h_0 \leqq H$ covered by Theorem 7.2 is given by

$$\frac{4 \log H + 8\gamma - 10}{H} + O(H^{-3/2}). \tag{7.19}$$

Asymptotically, the coverage is thus 0%.

8. The main formulas.

We have now reached a stage where further progress was again based on numerical evidence. In the next section, we shall see that in the last interval, for $f=1$, $n_h(A_3)$ can be expressed as a *quadratic polynomial* in h. A study of the coefficients of these polynomials, as computed by Mossige, led me to conjecture the following result: In the last interval (4.8), we put

$$h \equiv \varrho \pmod r, \quad 0 \leqq \varrho < r. \tag{8.1}$$

If

$$r \leqq (\varrho+1)\left[\frac{r}{\varrho}\right], \tag{8.2}$$

then

$$n_h(1, h+1, h+r+1) = (h+r)(h+2-\varrho) - r - (h+r-\varrho)\left[\frac{r}{\varrho+1}\right]. \tag{8.3}$$

If $\varrho = 0$, we define (8.2) to be satisfied. This is the Frobenius-dependent case $r \mid h$, and (8.3) then coincides with the first formula (5.5). If similarly $r \mid (h+1)$, $\varrho = r-1$, we also have (8.2) satisfied, and get the second formula (5.5).

The proof of (8.2–3) turned out to be comparatively simple: By (8.1), we have

$$h = \tau r + \varrho, \quad 0 \leqq \varrho < r. \tag{8.4}$$

We now use Theorem 6.1, and shall apply the division algorithm (2.20) to the ratio

$$\frac{a_3}{a_2} = \frac{h+r+1}{h+1} = \frac{(\tau+1)r + \varrho + 1}{\tau r + \varrho + 1}.$$

The algorithm takes the form

$$\begin{aligned}
(\tau+1)r+\varrho+1 &= 2\{\tau r+\varrho+1\} \qquad -\{(\tau-1)r+\varrho+1\}\\
\tau r+\varrho+1 &= 2\{(\tau-1)r+\varrho+1\}-\{(\tau-2)r+\varrho+1\}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
3r+\varrho+1 &= 2\{2r+\varrho+1\} \qquad -\{r+\varrho+1\}\\
2r+\varrho+1 &= 2\{r+\varrho+1\} \qquad -\underbrace{\{\varrho+1\}}_{s_\tau}
\end{aligned}$$

(8.5) $$r+\varrho+1 = q_{\tau+1}(\varrho+1)-s_{\tau+1},\ 0 \leqq s_{\tau+1} < \varrho+1 .$$

We see that $q_1=q_2=\ldots=q_\tau=2$, and hence

$$P_i = i+1,\ Q_i = i; \quad i=0,1,\ldots,\tau$$

$$P_{\tau+1} = (\tau+1)q_{\tau+1}-\tau,\ Q_{\tau+1} = \tau q_{\tau+1}-(\tau-1) .$$

Since

$$s_\tau = \varrho+1 \geqq P_\tau-Q_\tau = 1 ,$$

it follows from (6.4) that we always have $v \geqq \tau$. It turns out that (8.2–3) correspond to the choice $v=\tau$. The condition for this is

(8.6) $$s_{\tau+1} \leqq P_{\tau+1}-Q_{\tau+1} = q_{\tau+1}-1 .$$

It follows from (8.5) that

$$q_{\tau+1} = \left\langle\frac{r}{\varrho+1}\right\rangle+1,\ s_{\tau+1} = q_{\tau+1}(\varrho+1)-(r+\varrho+1) .$$

Substituting this into (8.6), we get the condition

$$r \geqq \varrho(q_{\tau+1}-1) = \varrho\left\langle\frac{r}{\varrho+1}\right\rangle ,$$

which by (6.10) is equivalent to (8.2).

When this condition is satisfied, (6.5) gives

$$n_h(A_3) = (h+3)(h+r+1)-(h+1)-2-(h+r)(\varrho+1)-r\{(\tau+1)q_{\tau+1}-\tau\}$$
$$+\begin{cases} r(\tau+1) & \text{if } s_{\tau+1} \geqq P_\tau-Q_\tau = 1 ,\\ (h+r)s_{\tau+1} = 0 & \text{if } s_{\tau+1} = 0 .\end{cases}$$

Since clearly

$$q_{\tau+1} = \left[\frac{r}{\varrho+1}\right]+\begin{cases} 2 & \text{if } s_{\tau+1} \geqq 1 ,\\ 1 & \text{if } s_{\tau+1} = 0 ,\end{cases}$$

a simple calculation shows that we have obtained the formula (8.3).

The proof just completed is analogous to the earlier proof of Lemma 7.2. In both cases, several steps of Rödseth's division algorithm have been combined into one formal operation. In the case of Theorems 8.1–2 below, the same principle applies, and we shall then content ourselves with brief sketches of the proofs.

Having conjectured and proved the result (8.2–3), it is natural to try a generalization to $f>1$. The crucial point is then the extension of the division (8.4). After some trial and error, the answer turned out to be

$$a_2-1 = \tau(r+f-1)+\varrho, \quad 0 \leqq \varrho < r+f-1 . \tag{8.7}$$

We still use Theorem 6.1. Assuming so far $r>0$, we have $q_1=q=f+1$. If $\tau>1$, we further find

$$q_2 = q_3 = \ldots = q_\tau = 2 .$$

The last division (8.5) is replaced by

$$r = (q_{\tau+1}-1)s_\tau - s_{\tau+1}, \quad \text{where } s_\tau = \tau(f-1)+\varrho+1 = a_2-\tau r .$$

As above, we use $v=\tau$ in Theorem 6.1. A straightforward calculation then gives

THEOREM 8.1 (*First main theorem*). *Let*

$$\begin{aligned} a_2-1 &= \tau(r+f-1)+\varrho, \quad & 0 \leqq \varrho < r+f-1 \\ r &= \alpha(a_2-\tau r)-\beta, \quad & 0 \leqq \beta < a_2-\tau r . \end{aligned}$$

If

$$r+f-1 \geqq \alpha\varrho , \tag{8.8}$$

then

$$n_{h_0}(A_3) = (\tau r+2)(a_3-1)-r-\alpha(\tau f+1)(a_3-a_2) + \begin{cases} (\tau f+1)(a_3-a_2) & \text{if } \beta > \tau(f-1) , \\ \beta(a_3-1) & \text{if } \beta \leqq \tau(f-1) . \end{cases} \tag{8.9}$$

With $f=1$, we get the earlier case (8.2–3). However, two other specializations of Theorem 8.1 are more interesting:

First, Theorem 8.1 *contains Theorem* 7.1 *as a special case*, namely for $\tau=1$. This follows from the fact that the latter theorem corresponds to Theorem 6.1 for $v=1$. (There is one case with $\tau>1$ covered by Theorem 7.1, namely for $\tau=2$, $\varrho=0$ in (8.7). Then $s_2=P_2-Q_2$, and we may use either $v=1$ or $v=2$ in Theorem 6.1.)

Next, Theorem 8.1 *also contains Theorem* 7.2 (including all *pleasant* bases) *as a special case*. To prove this, we show that the condition (7.6) implies — and is *stronger* than — the condition (8.8):

For $\varrho=0$, (8.8) is always satisfied. For $\varrho>0$, it follows from (8.7) that

$$\left\langle \frac{a_2-1}{r+f-1} \right\rangle = \tau+1 ,$$

and (7.6) may be written as $r \leqq a_2-\tau r$. On the other hand, the condition (8.8) has the two equivalent forms

$$(8.10) \qquad r+f-1 \geqq \varrho\left\langle \frac{r}{a_2-\tau r} \right\rangle \Leftrightarrow r \leqq (a_2-\tau r)\left[\frac{r+f-1}{\varrho}\right].$$

Since $\varrho<r+f-1$, the last condition is certainly satisfied if $r \leqq a_2-\tau r$.

So far, the applications of Theorems 6.1 and 6.2 have worked in parallel. It is therefore natural to look for an analogue to (8.7), to be applied in connection with Theorem 6.2. It took some time to discover the very simple answer:

$$(8.11) \qquad a_2 = \lambda s+\sigma, \quad 0 \leqq \sigma < s .$$

We use Theorem 6.2, and assume so far $\lambda>1$. The incomplete quotients q_i, for a *non-pleasant* basis, now become

$$q_1 = q_2 = \dots = q_{f-1} = 2, \; q_f = 3 ,$$

and if $\lambda>2$ further

$$q_{f+1} = q_{f+2} = \dots = q_{f+\lambda-2} = 2 .$$

The calculations are straightforward but rather complicated. With $v=f+\lambda-2$ in Theorem 6.2, we find

THEOREM 8.2 (*Second main theorem*). *For a non-pleasant basis* A_3, *let*

$$a_2 = \lambda s+\sigma, \quad 0 \leqq \sigma < s$$

$$s-f = \gamma(\lambda f+\sigma-1)-\delta, \quad 0 \leqq \delta < \lambda f+\sigma-1 .$$

If

$$(8.12) \qquad s \geqq \gamma\sigma ,$$

then

$$(8.13) \qquad n_{h_0}(A_3) = \{\lambda(s-f)+2\}a_3-r-2-\gamma\{\lambda(f+1)-1\}(a_3-a_2) + \begin{cases} \{\lambda(f+1)-1\}(a_3-a_2) & \text{if } \delta \geqq \lambda f , \\ \delta a_3 & \text{if } \delta < \lambda f . \end{cases}$$

We have *not* excluded the case $\lambda=1$, even if the division algorithm then takes a different form. It is easily seen that $\lambda=1$ — and in addition $\lambda=2, \sigma=0$ — cover *all the non-pleasant cases of Theorem* 7.2 (and also a few pleasant ones). Quite unexpectedly, it turns out that these cases are all *contained in Theorem* 8.2, in spite of the formally different proofs.

In analogy with the arguments around (8.10), it is easily shown that Theorem 8.2 *also contains Theorem* 7.1 *as a special case*. Thus, each of the new main theorems comprises the non-pleasant cases of both of the earlier explicit theorems.

If both main theorems *fail*, we can improve the situation by strengthening Theorems 7.1–2 (which, of course, then also fail). The common failures have a tendency to occur when r is in the interval

$$\frac{a_2}{3} < r < \frac{2a_2}{3} .$$

This corresponds to $q_2=3$ in (7.9), or $q_f=4$ in (7.14). For these particular values, we therefore *extend* Theorem 7.1 from $v=1$ to $v=2$, and Theorem 7.2 from $v=f-1$ to $v=f$. With q_2 or q_f given, only *one division step* will then appear explicitly in the formulas. Standard arguments yield

THEOREM 8.3 (*Supplementary theorem*). *Let*

$$\frac{a_2}{2} < r \leqq \frac{2a_2}{3} - f ,$$

and put

$$a_2 - r = \eta(2a_2 - 3r) - \vartheta, \quad 0 \leqq \vartheta < 2a_2 - 3r .$$

If $\vartheta \leqq (3f-1)\eta - f$, *then*

$$n_{h_0}(A_3) = (3r+5f-a_2)a_3 + r - 2a_2 - 2$$
$$+ \varepsilon\vartheta(a_3-1) - (\varepsilon+\eta)(3f+2)(a_3-a_2) .$$

Let next

$$\frac{a_2}{3} < r \leqq \frac{a_2}{2} - f ,$$

and put

$$r+f-1 = \eta(3r+3f-a_2-2) - \vartheta, \quad 0 \leqq \vartheta < 3r+3f-a_2-2 .$$

If $\vartheta \leqq (3f-2)\eta - f + 1$, *then*

$$n_{h_0}(A_3) = (2a_2-3r+2f+\varepsilon\vartheta)a_3+2(2r-a_2-1)$$
$$-(\varepsilon+\eta)(3f+1)(a_3-a_2)\,.$$

In both cases, choose

$$\varepsilon = \begin{cases} 0 & \text{if } \vartheta \geqq 3f-1\,, \\ 1 & \text{if } \vartheta < 3f-1\,. \end{cases}$$

For small h_0, this theorem represents a great improvement. If we consider only $f>1$ (all intervals but the last one), the combined main theorems fail for 42 non-pleasant bases with $h_0 \leqq 50$. If we also use the supplementary theorem, there is only *one* common failure left:

$$h_0 = 47,\ a_2 = 47,\ a_3 = 106\,.$$

In the last interval ($f=1$), the theorems are much less effective. For $h_0 \leqq 50$, there are then 108 non-pleasant bases A_3 where Theorems 8.1–3 all fail. To improve the coverage, we argue as follows:

The supplementary theorem represents particular cases of $v=2$ and $v=f-1$ in the general Theorems 6.1 and 6.2. If we use these theorems *without restrictions* for $v \leqq 2$ and $v=1$ respectively (in addition to the main theorems), we cover all but 46 non-pleasant bases for $f=1$, $h_0 \leqq 50$. If we also extend Theorem 6.2 to $v \leqq 2$, there are only *four* common failures left:

$$(h_0, r) = (43,17), (43,19), (48,19), (50,22)\,.$$

Let us also indicate what happens for $50 < h_0 \leqq 100$. With $f>1$, Theorems 8.1–3 then fail for 217 non-pleasant bases A_3, of which 195 have $f=2$ and the remaining 22 have $f=3$. The first case with $f=3$ occurs for $h_0=76$. — With $f=1$, Theorems 8.1–2 in combination with $v \leqq 2$ in Theorems 6.1–2 fail in 194 cases.

As for Theorems 7.1–2, we shall also examine the *percentage* of bases A_3 covered by the different Theorems 8.1–3. The denominator of the percentage is then given by the total number (4.5) of non-pleasant bases with $h_0 \leqq H$.

Considering *all* such bases, the asymptotic behaviour of this percentage is apparently very regular, cf. Table 2. Even if the behaviour has not been studied theoretically, no great risk is involved by stating the following results:

Each of the two main theorems seem to cover asymptotically $\approx 91.3\,\%$ of all non-pleasant bases, and $\approx 96.7\,\%$ in combination. — The former number should be compared with the combined coverage $\approx 86\,\%$ of Theorems 7.1–2, which are both particular cases of each of the main theorems.

Table 2. *Percentage of non-pleasant bases* A_3 *with* $h_0 \leqq H$ *where the following theorems fail*:
I: Th. 8.1. II: Th. 8.2. III: Th. 8.1–2. IV: Th. 8.1–3.

$H=$	100	200	300	400	500	600	700	800	900	10000
I	7.80	8.27	8.42	8.49	8.53	8.559	8.578	8.592	8.604	8.612
II	8.26	8.47	8.54	8.57	8.59	8.608	8.618	8.625	8.631	8.636
III	2.78	3.05	3.14	3.18	3.21	3.224	3.235	3.243	3.250	3.254
IV	1.50	1.75	1.82	1.86	1.88	1.896	1.906	1.913	1.918	1.922

In combination, the main theorems and the supplementary theorem seem to fail for asymptotically 1.94% of all non-pleasant bases. Since we know (cf. the comment to (4.5)) that these represent 50% of all *admissible* bases, this means *an asymptotic failure of only* 0.97% when the pleasant bases—covered by Theorem 2.1—are included. The asymptotic behaviour in this case is illustrated in Fig. 1, where the percentage denominator is now given by the number (4.1).

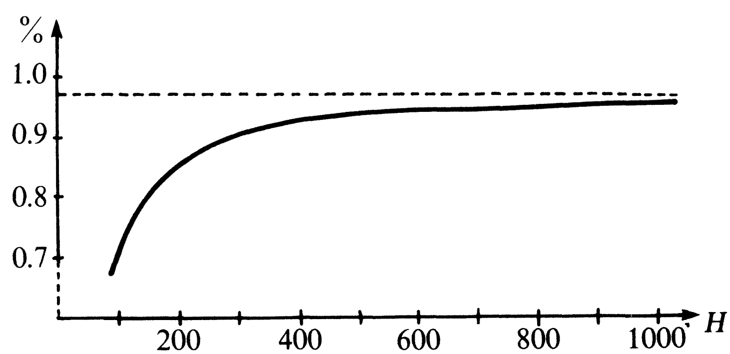

Fig. 1. Percentage of admissible bases A_3 with $h_0 \leqq H$ where the conditions of Theorems 2.1 and 8.1–3 all fail.

We have already noted that most failures occur for $f=1$. The length of the last interval is h_0-1, accumulating to a percentage denominator $\frac{1}{2}H(H-1)$ for $h_0 \leqq H$.

It turns out that for a chosen (small) f, the asymptotic behaviour of the covering percentage is much "slower" than in the case of all intervals combined. Even with $H \leqq 1000$, it is difficult to evaluate the asymptotic behaviour directly from a table.

Inspired by (7.19), we may try to approximate the percentage by an expression of the form

$$A+\frac{B\log H+C}{H},$$

where the coefficients A, B and C can be determined from three different "observations". This does, in fact, seem to give a fairly good estimate over a large range of values H. Based on such considerations, the following asymptotic results were obtained:

In combination, the main theorems and the supplementary theorem seem to cover approximately 15% of all non-pleasant bases with $f=1$, versus 27% for $f=2$. If we replace the supplementary theorem by Theorems 6.1–2 with $v\leqq 2$, the percentage for $f=1$ is approximately doubled.

As already mentioned in the Introduction, the application of Rödseth's division algorithm (2.20) in Theorems 6.1–2 does not lead to strictly "explicit" formulas for the h-range, since the number of necessary division steps cannot be estimated a priori. One purpose of Theorems 8.1–3 is just to reduce the number of (formal) division steps to at most *two*.

The same principle can of course be applied directly to the general Theorems 6.1–2. As above, we will *not* count the determination of f, r, s and q by (2.4) as a division step.

Considering first Theorem 6.1 (for non-pleasant bases), the two first division steps are then

$$a_2 = q_2 s - s_2, \quad s = q_3 s_2 - s_3 .$$

We have seen that the first step suffices, corresponding to $v=1$ (Theorem 7.1), if

$$a_2-1 \geqq (s-q+1)q_2 .$$

When this condition is not satisfied, a simple application of (6.4) shows that we have $v=2$ if

$$a_2-1 \geqq (s-q+1)\left(q_2-\frac{1}{q_3}\right).$$

Turning next to Theorem 6.2, we saw in the proof of Lemma 7.2 that an initial string of incomplete quotients $q_i=2$ can be formally combined with the first non-trivial division step (7.14):

$$a_2-1 = q'_f(r+f-1)-s_f \qquad (q'_f=q_f-1) .$$

This step suffices, corresponding to $v=f-1$ (Theorem 7.2), if

$$a_2 \geqq rq'_f .$$

When this condition is not satisfied, the next division step

$$r+f-1 = q_{f+1}s_f - s_{f+1}$$

will suffice, corresponding to $v=f$, if

$$a_2 \geqq r\left(q'_f - \frac{1}{q_{f+1}}\right).$$

Table 3. *Percentage of non-pleasant bases A_3 with $h_0 \leqq H$ where the following theorems fail:*

I: Th. 6.1 ($v=1,2$). II: Th. 6.2 ($v=f-1,f$). III: I and II.

$H=$	100	200	300	400	500	600	700	800	900	1000
I	19.87	20.17	20.29	20.35	20.38	20.409	20.427	20.441	20.453	20.462
II	19.31	19.99	20.19	20.29	20.35	20.388	20.414	20.433	20.447	20.458
III	1.96	2.23	2.32	2.36	2.38	2.397	2.408	2.415	2.421	2.426

Numerical results are given in Table 3. A comparison with Table 2 leads to some interesting observations:

Using only *one* of Theorems 6.1–2, with two division steps, we get a much *lower* asymptotic coverage ($\approx 79.5\%$) than with only one of the main theorems (which, incidentally, justifies the adjective "main"). In *combination*, however, Theorems 6.1–2 give a *larger* percentage ($\approx 97.5\%$) than Theorems 8.1–2, which need the auxiliary Theorem 8.3 to "take the lead" again. This indicates *less overlap* between Theorems 6.1–2.

9. Polynomial formulas.

The aim of this section is to express $n_h(A_3)$ as a *quadratic polynomial* in h, at least under certain conditions.

We first use Lemma 7.1. Substituting

$$a_2 = h_0 - q + 3,\ a_3 = qa_2 - s,\ \left[\frac{h_0+1}{s}\right] = \frac{h_0+\mu+2}{s} - 1$$

in (7.7), we get

$$n_{h_0}(A_3) = \frac{(s-q+1)q}{s}h_0^2 + F_1(s,q,\mu)h_0 + G_1(s,q,\mu)\,, \tag{9.1}$$

assuming (7.1) to be satisfied. The functions F_1 and G_1 are rather complicated. The simplest case is for $s=q$, corresponding to the *last* (non-pleasant) basis in each interval (when (7.1) is automatically satisfied). In particular, we then get *integer* coefficients in (9.1).

Similarly, Lemma 7.2 gives

$$n_{h_0}(A_3) = \frac{rf}{r+f-1}h_0^2 + F_2(r,f,\varrho)h_0 + G_2(r,f,\varrho)\,, \tag{9.2}$$

assuming (7.3) to be satisfied. The simplest case is now for $r=1$, corresponding to the *first* basis in each interval.

The above results are, of course, only alternative ways of writing Lemmas 7.1–2. More interesting is a completely different approach in the *last interval* ($f=1$). We shall prove the following

THEOREM 9.1. *In the last interval* (4.8), *put*

$$h \equiv \varrho \pmod r, \quad 0 \leqq \varrho < r\,. \tag{9.3}$$

Then

$$n_h(1, h+1, h+r+1) = h^2 + A(r,\varrho)h - B(r,\varrho)\,, \tag{9.4}$$

where the coefficients A and B are integer functions of r and ϱ.

As before, we apply Theorem 2.3, but now *not* with any of the permutations (6.1–2) of the Frobenius basis. We use instead one of the cases with $a = a_3 - a_2 = r$, for instance

$$a = a_3 - a_2 = r,\ b = a_3 = h+r+1,\ c = a_3 - 1 = h+r\,. \tag{9.5}$$

The congruence (2.18) then becomes

$$(\varrho+1)s_0 \equiv \varrho \pmod r, \quad 0 \leqq s_0 < r\,. \tag{9.6}$$

So far, we will assume $(\varrho+1, r) = 1$. We can exclude $\varrho = 0$ and $\varrho = r-1$, since the Frobenius-dependent cases (5.5) already give polynomials of the form (9.4).

When s_0 is determined by (9.6), we apply the division algorithm (2.20), with $a=r$. The numbers s_i and P_i are then functions of r and ϱ only.

When v is determined by (2.22), we see from (9.5) and (2.23) that $g(a,b,c)$ is a linear polynomial in h, and substitution in (2.17) yields (9.4). It only remains to show that for given v, the minimum of (2.23) is *independent of h*:

$$\min\{(h+r+1)s_{v+1}, (h+r)P_v\} = \begin{cases}(h+r)P_v & \text{if } P_v \leqq s_{v+1}\,,\\ (h+r+1)s_{v+1} & \text{if } P_v > s_{v+1}\,.\end{cases}$$

The first case is trivial, and the second case follows from

$$s_i \leqq s_0 < r \Rightarrow s_{v+1} < h+r \Leftrightarrow (h+r+1)s_{v+1} < (h+r)(s_{v+1}+1)\,.$$

If $(\varrho+1, r)=d>1$, then also $(a,b)=d$. We remove the common factor d of the Frobenius basis elements a and b by means of formula (3.1) in [11], and then proceed as above. This completes the proof of Theorem 9.1.

Both for small ϱ and for large ϱ, it is possible to give *explicit expressions* for the coefficients of (9.4). For $\varrho \leqq 3$ or $\varrho \geqq r-4$, the results are shown in Table 4.

Table 4. *Some general coefficients in Theorem* 9.1.

ϱ	$A(r,\varrho)$	$B(r,\varrho)$	Remark
0	2	$r(r-1)$	
1	$\left[\frac{r+3}{2}\right]$	$(r-1)\left[\frac{r}{2}\right]$	
2	$\left[\frac{2r+2}{3}\right]$	$(r-2)\left[\frac{r}{3}\right]+r$	
3	$\left[\frac{3r-1}{4}\right]$	$(r-3)\left[\frac{r}{4}\right]+2r$	$r \neq 5$
....	....	....	
$r-4$	$\left[\frac{3r-1}{4}\right]$	$(r+4)\left[\frac{r}{4}\right]+\begin{cases}4, & r \not\equiv 3 \pmod 4\\ 7, & r \equiv 3 \pmod 4\end{cases}$	$r \neq 5$
$r-3$	$\left[\frac{2r+2}{3}\right]$	$r\left[\frac{r}{3}\right]+\begin{cases}3, & r \not\equiv 1 \pmod 3\\ 2, & r \equiv 1 \pmod 3\end{cases}$	
$r-2$	$\left[\frac{r+3}{2}\right]$	$r\left[\frac{r}{2}\right]-r+2$	
$r-1$	2	$(r-1)^2$	

The upper part of the table follows easily from (8.1–3). Using Theorem 6.2 with $v=1$ on the last interval (4.8), standard methods give the following similar result: Let

$$h \equiv -s_1 \pmod r, \quad 1 < s_1 < r$$

$$r = q_2 s_1 - s_2, \quad 0 \leqq s_2 < s_1 .$$

If

$$r \leqq s_1\left[\frac{r}{s_1-1}\right],$$

then

$$n_h(1,h+1,h+r+1) = \begin{cases} (h+r)(h+4-s_1-q_2)-s_2 & \text{if } s_2>1\ , \\ (h+r)(h+3-s_1-q_2+s_2)-s_1 & \text{if } s_2\leqq 1\ . \end{cases}$$

By means of this, it is easy to prove the apparent *symmetry* of Table 4, both for $A(r,\varrho)$ and the excluded values of r. The verification of the expressions for $B(r,\varrho)$ in the lower part is also straightforward but slightly tedious.

Extending the table towards the middle, stronger conditions on r will result. As it is, the two conditions $r \neq 5$ can be circumvented, since the combination $\varrho=3$, $r=5$ is covered by $\varrho=r-2$, and $\varrho=r-4$, $r=5$ by $\varrho=1$.

The cases with $r\leqq 20$ not covered by Table 4 are given in Table 5.

Table 5. *Pairs* (A,B) *of coefficients of* $n_h(1,h+1,h+r+1)=h^2+Ah-B, \quad h\equiv\varrho \pmod r$.

r \ ϱ	4	5	6	7	8	9	10	11	12	13	14	15
9	6 32											
10	6 42	6 45										
11	8 36	7 50	8 37									
12	8 52	7 62	7 66	8 51								
13	9 57	9 55	8 72	9 56	9 54							
14	10 62	10 59	8 86	8 91	10 60	10 61						
15	10 78	10 80	10 78	9 98	10 80	10 78	10 80					
16	11 84	11 86	12 68	9 114	9 120	12 69	11 82	11 84				
17	12 90	12 92	12 89	12 88	10 128	12 91	12 89	12 91	12 88			
18	13 96	12 111	12 114	13 94	10 146	10 153	13 94	12 111	12 114	13 92		
19	14 102	14 99	13 121	14 98	13 117	11 162	13 120	14 100	13 116	14 100	14 100	
20	14 124	14 125	14 128	14 125	14 124	11 182	11 190	14 125	14 124	14 127	14 124	14 125

10. Some inequalities for $n_h(A_3)$.

With $k=3$, Theorem 3.1 states that

$$n_h(A_3) \geqq (h+1)a_2-a_3 \Rightarrow n_{h+1}(A_3) = n_h(A_3)+a_3\ .$$

It follows from Theorem 2.2 that the latter statement is always valid (for $h \geqq h_0$), but we *cannot reverse* the implication. On the other hand, overwhelming numerical evidence indicates that we always have

$$n_h(A_3) \geqq (h+1)a_2-a_3, \quad h\geqq h_0\ , \tag{10.1}$$

for a *non-pleasant* and *non-dependent* basis A_3. The main purpose of the present section is to prove this result. The proof caused considerable difficulties.

We noted in connection with Theorem 3.1 that if (10.1) holds for $h=h_S$, it will hold for all $h\geqq h_S$. It therefore suffices to prove (10.1) for $h=h_0=a_2+f-2$.

Let us first discuss the *exceptions* to (10.1). For a *pleasant* basis A_3, it follows from (2.11) that (10.1) fails for $h=h_0$ if

$$2a_3-(r+2) < (h_0+1)a_2-a_3 ,$$

which with $fa_2=a_3-r$ may be written as

$$2a_3 < a_2^2-a_2+2 .$$

The first pleasant bases A_3 satisfying this condition are given by

a_2	5	5	6	6	7	7	7	7	8 ...
a_3	9	10	11	12	13	14	19	20	15 ...

The number of possible a_3 is a non-decreasing function of a_2.

Next, we consider the *Frobenius-dependent* bases A_3, as given by the two forms (5.8). It is easily verified that in both cases, (10.1) fails if and only if

$$r \geqq p+3 .$$

Disregarding these exceptions, we shall in fact prove a *stronger* inequality than (10.1):

THEOREM 10.1. *For a non-pleasant and non-dependent basis A_3, we always have (for $h \geqq h_0$)*

$$n_h(A_3) > ha_2 \tag{10.2}$$

$$n_h(A_3) \geqq (h+4)a_2-2a_3 , \tag{10.3}$$

with equality only in the cases

$$a_2 = h+1, \; a_3 = h+5 .$$

In the last interval ($f=1$), $a_3<2a_2$, and (10.2) follows from the stronger (10.3). In the remaining intervals, $a_3>2a_2$, and (10.3) then follows from (10.2). In both cases, we always have

$$n_h(A_3) > (h+2)a_2-a_3 , \tag{10.4}$$

which is stronger than (10.1).

It is in fact possible to strengthen the result even more, but the proof then becomes very complicated. On the other hand, it is not much simpler to prove only (10.1)—our main object—than to prove the full Theorem 10.1.

We need two lemmas:

LEMMA 10.1. *For a non-pleasant basis A_3, we have*

$$n_{h_0}(A_3) \geqq \frac{s-q+1}{s} r a_3 + 2a_3 - r - 2 . \tag{10.5}$$

LEMMA 10.2. *For any basis A_3, we have*

$$n_{h_0}(A_3) \geqq \frac{r}{r+f-1}(a_2-1)(a_3-1)-(r-2)a_3-2 . \tag{10.6}$$

To prove Lemma 10.1, we use Lemma 7.1, and assume so far that the condition (7.1) is satisfied. Substituting

$$\left[\frac{h_0+1}{s}\right] = \left[\frac{h_0+2+\mu}{s} - \frac{\mu+1}{s}\right] = \frac{h_0+2+\mu}{s} - 1$$

in (7.7), the right hand side will represent *linear functions* $d\mu + e$ of μ in each of the two cases

$$\mu \geqq s-q+1 = m, \quad d = a_3 \frac{s-q+1}{s} > 0$$

$$s-q+1 \geqq \mu = m, \quad d = a_3 \frac{s-q+1}{s} - (a_3-1) < 0 .$$

In both cases, the linear functions thus attain their (common) minimum for $\mu = s-q+1$. This minimum is just the right hand side of the inequality (10.5) (which is consequently *sharp*).

Note that the condition (7.1) of Theorem 7.1 does *not* enter into Lemma 10.1 after all, for the following reason: Since A_3 is non-pleasant, we only know that $v \geqq 1$ in Theorem 6.1, while the formula (7.7) corresponds to the possibly too small value $v=1$. However, it follows from the method of Rödseth [9] that if we use a too small $v_1 < v$ in (2.23), the resulting value of g becomes too large. By (2.17), the calculated value of n_h is then consequently too small, and Lemma 10.1 holds *a fortiori* for $v > 1$.

In exactly the same manner, Lemma 10.2 follows from Lemma 7.2.

To prove Theorem 10.1, we first assume $f > 1$, when it suffices to establish the inequality (10.2). We may confine ourselves to the interval $r \in [1, a_2-f-1]$ for non-pleasant bases, cf. (4.3). The *middle* of this interval (possibly fractional) is

$$r_m = \tfrac{1}{2}(a_2-f) .$$

Let the right hand sides of (10.5) and (10.6) be denoted by $\bar{R}_1$ and $\bar{R}_2$, respectively. We shall show that

(10.7) $$R_1 > h_0 a_2 \quad \text{for } r_m \leqq r \leqq a_2-f-1$$

(10.8) $$R_2 > h_0 a_2 \quad \text{for } 1 \leqq r \leqq r_m .$$

Substituting

$$s = a_2-r,\ q = f+1,\ h_0 = a_2+f-2,\ a_3 = fa_2+r ,$$

both sides of (10.7–8) can be expressed as functions of r, with coefficients depending on a_2 and f. Considering first (10.8), we form the difference

$$R_2-h_0a_2 = F(r) = -\frac{f(f-1)(a_2-1)^2}{r+f-1}$$
$$-r^2-\{(f-1)a_2-1\}r+(f-1)a_2^2-(f-2)(a_2-1) .$$

Since

$$F''(r) = -\frac{2f(f-1)(a_2-1)^2}{(r+f-1)^3}-2 < 0 ,$$

$F(r)$ is a *concave* function, and in particular

$$F(r) \geqq \min\{F(1), F(r_m)\} \quad \text{for } 1 \leqq r \leqq r_m .$$

Then (10.8) will follow if we can show that both arguments of the minimum are positive.

The first argument is simple, since $F(1)=a_2-1>0$. The second argument is much more complicated:

$$F(r_m) = \frac{(a_2-f-2)^2\{(2f-3)a_2-f\}+4(a_2-2)^2+4fa_2}{4(a_2+f-2)} .$$

For $f=1$, the factor in curly brackets is negative, and the proof fails. For $f>1$, however, we clearly get a positive expression, and (10.8) holds.

Turning to (10.7), we similarly form

$$R_1-h_0a_2 = G(r) = -\frac{f(f+1)a_2^2}{a_2-r}$$
$$+r^2+(fa_2+f+1)r-a_2^2+(f^2+2f+2)a_2-2 .$$

The function $G(r)$ is not necessarily concave, but it is simple to show (using $G'(r_m)>0$) that

$$G(r) \geqq \min\{G(r_m), G(a_2-f-1)\} \quad \text{for } r_m \leqq r \leqq a_2-f-1 .$$

Here $G(a_2-f-1)=a_2-2>0$ (since $a_2>2$ for a non-pleasant basis), and

$$G(r_m) = \frac{(a_2-f-2)^2\{(2f-3)a_2-f-2\}+4(f-1)(a_2+f+2)+16}{4(a_2+f)}$$

is positive for $f > 1$. This completes the proof of (10.7) and thus of (10.2) in all intervals but the last one.

When $f = 1$, we must prove the inequality (10.3). The above method then fails, and we resort to Vitek's bound (5.7).—Incidentally, his general bound (2.25) is *not* strong enough to prove (10.1) in most cases with $f > 1$.

By (5.7) (which assumes *non-dependent* bases), it suffices to show that

$$\left(h - \left[\frac{r}{2}\right]\right)(h+r-1) + 2h \geqq (h+4)a_2 - 2a_3 = (h+4)(h+1) - 2(h+r+1) .$$

For *r odd*, this may be written as

$$(r-3)(h-r+3)+4 \geqq 0 ,$$

which is always satisfied with strict inequality (since $r = 1$ gives a Frobenius-dependent basis).

For *r even*, we similarly get

$$(r-4)(h-r+1) \geqq 0 ,$$

which holds with strict inequality if $r > 4$. Since $r = 2$ gives a dependent basis, it only remains to examine the case $r = 4$. It turns out that we then get equality in (10.3), since

$$n_h(1, h+1, h+5) = h^2 + 3h - 6$$

follows from Table 4 with $r = 4$, $\varrho = 1, 2$ (and the cases $\varrho = 0, 3$ give dependent bases). This completes the proof of (10.3) and thus of Theorem 10.1.

We mention that Vitek's bound (5.7) may also be used to extend Theorem 10.1 with

$$f = 1 \Rightarrow n_h(A_3) \geqq (h-1)a_2 + a_3 ,$$

with equality only in the cases

$$h \text{ odd}, \; a_2 = h+1, \; a_3 = 2h .$$

This result is *stronger* than (10.3) if $3a_3 > 5a_2$, that is, in the last third of the last interval.

We conclude this section with some inequalities of a completely different kind. If we perform the division

(10.9) $$n_{h_0}(A_3) = \eta a_3 - \vartheta, \quad 0 \leqq \vartheta < a_3 ,$$

it is easily seen that we can have $\vartheta = 0$ only in the case

(10.10) $$n_h(1, h+1, h+2) = h(h+2) = ha_3 .$$

We have $\eta=\vartheta$ in the pleasant case $r=0$, where by (2.11)

$$n_{h_0}(1, a_2, fa_2) = 2a_3-2 . \tag{10.11}$$

In all other cases, numerical evidence indicates that always $\vartheta>\eta$, and we shall prove this below. — My interest for this problem was promted by Salié's paper [10], where he discusses at length *some* cases satisfying the condition $\vartheta>\eta$.

We shall prove the following

THEOREM 10.2. *In the notation* (10.9), *we have*

$$\vartheta > \eta , \tag{10.12}$$

except in the cases (10.10–11). *We also have*

$$\vartheta \geqq r+2 , \tag{10.13}$$

except in the case (10.10).

We note that (10.13) holds also if h_0 is replaced by h in (10.9), because of (2.14). There is equality in (10.13) in all pleasant cases (by (2.11)), but also in many non-pleasant cases.

To prove (10.12), we use Theorem 6.2. By (6.6), we may express P_v and P_{v+1} by Q_v and Q_{v+1}. We also use $h=h_0=a_2+f-2$. Substitution in (6.8) then gives

1) $s_{v+1}>Q_v$: $n_{h_0}(A_3)=\eta_1 a_3-\vartheta_1$, where

$$\eta_1 = a_2+f+1-s_v+Q_v-Q_{v+1},\ \vartheta_1 = a_2+2-s_v+s_{v+1}+Q_v-Q_{v+1} .$$

2) $s_{v+1} \leqq Q_v$: $n_{h_0}(A_3)=\eta_2 a_3-\vartheta_2$, where

$$\eta_2 = a_2+f+1-s_v+s_{v+1}-Q_{v+1},\ \vartheta_2 = a_2+2+s_{v+1}-Q_{v+1} .$$

We must show that η_i and ϑ_i, $i=1,2$, may be used as η and ϑ of (10.9), or in other words, that $0\leqq\vartheta_i<a_3$. The first inequality follows from $\vartheta_i>\eta_i$, to be established later. To show that $\vartheta_i<a_3$, we use the conditions (6.7), but now with

$$s_v > Q_v,\ s_{v+1} \leqq Q_{v+1} . \tag{10.14}$$

We have excluded the possibility $s_v=Q_v$, which by (7.15) corresponds to the exception (10.11), for $r=0$.

In case 1 above, it follows from (10.14) that $\vartheta_1\leqq a_2+1<a_3$, since $a_3=a_2+1$ gives the exception (10.10).

In case 2, we only get $\vartheta_2\leqq a_2+2$, and the case $a_3=a_2+2$ must be considered separately. With $f=1$, $r=2$, we have a dependent basis, and it follows easily from (5.5) that $\vartheta>\eta$ both for h even and h odd.

In the remaining cases, we have now shown that $\vartheta_i < a_3$, $i=1,2$, and must finally prove that $\vartheta_i > \eta_i$. From the comments to (7.15), it follows that $v \geqq f-1$, hence $Q_v \geqq Q_{f-1} = f-1$. We conclude that

$$1)\quad s_{v+1} > Q_v \Rightarrow s_{v+1} > f-1 \Leftrightarrow \vartheta_1 > \eta_1 .$$

$$2)\quad s_v > Q_v \Rightarrow s_v > f-1 \Leftrightarrow \vartheta_2 > \eta_2 .$$

This completes the proof of (10.12). It is rather unsatisfactory to involve a "deep" result like Theorem 6.2 in the proof of such a simple inequality.

We finally turn to (10.13), which now follows very simply from (10.12). We form

$$(10.15)\qquad n_{h_0}(A_3)+1 = \eta a_3 - \vartheta + 1 = (r-\vartheta+1)\cdot 1 + f a_2 + (\eta-1)a_3 .$$

Considered as a representation by the basis A_3, the sum of the coefficients does not exceed $h_0 = a_2 + f - 2$ if and only if

$$r \leqq a_2 + \vartheta - \eta - 2 .$$

Since $r < a_2$, this condition is satisfied because of (10.12). But $n_{h_0}(A_3)+1$ has by definition no representation in at most h_0 addends, and thus (10.15) must be an "illegal" representation. The only possibility for this is a negative constant term, $r - \vartheta + 1 < 0$, which is just (10.13).

It is also possible to give a nontrivial *upper bound* for ϑ:

THEOREM 10.3. *In the notation* (10.9), *we always have*

$$(10.16)\qquad \vartheta \leqq \eta + r < a_3 .$$

The proof runs as for (10.12), by means of the formulas for η_i and ϑ_i in the cases 1 and 2. The first inequality follows from

$$s_{v+1} < s_v \leqq s_{f-1} = r + f - 1 ,$$

cf. (7.15). The second inequality of (10.16) follows from

$$s_v > Q_v,\ s_v > s_{v+1};\ Q_{v+1} \geqq Q_f \geqq f ,$$

where the final inequality is strict for a non-pleasant basis A_3. The pleasant case $a_3 = 2a_2 - 1$ must be treated separately.

REFERENCES

1. M. Djawadi, *Kennzeichnung von Mengen mit einer additiven Minimaleigenschaft*, J. Reine Angew. Math. 311/312 (1979), 307–314.
2. G. Hofmeister, *Über eine Menge von Abschnittsbasen*, J. Reine Angew. Math. 213 (1963), 43–57.

3. G. Hofmeister, *Asymptotische Abschätzungen für dreielementige Extremalbasen in natürlichen Zahlen*, J. Reine Angew. Math. 232 (1968), 77–101.
4. G. Hofmeister, *Vorlesungen über endliche Zahlentheorie* (duplicated), Joh. Gutenberg-Universität, Mainz, 1976.
5. G. Hofmeister, *Lineare diophantische Probleme* (duplicated), Joh. Gutenberg-Universität, Mainz, 1978.
6. G. Meures, *Zusammenhang zwischen Reichweite und Frobeniuszahl*, Staatsexamensarbeit, Mainz, 1977.
7. S. Mossige, *Algorithms for computing the h-range of the postage stamp problem*, Math. Comp. (to appear).
8. Ö. Rödseth, *On a linear diophantine problem of Frobenius*, J. Reine Angew. Math. 301 (1978), 171–178.
9. Ö. Rödseth, *On h-bases for n*, Math. Scand. (to appear).
10. H. Salié, *Reichweite von Mengen aus drei natürlichen Zahlen*, Math. Ann. 165 (1966), 196–203.
11. E. S. Selmer, *On the linear diophantine problem of Frobenius*, J. Reine Angew. Math. 293/294 (1977), 1–17.
12. E. S. Selmer and Ö. Beyer, *On the linear diophantine problem of Frobenius in three variables*, J. Reine Angew. Math. 301 (1978), 161–170.
13. E. Siering, *Über lineare Formen und ein Problem von Frobenius*, Dissertation, Mainz, 1974.
14. Y. Vitek, *Bounds for a linear diophantine problem of Frobenius*, J. London Math. Soc. (2) 10 (1975), 79–85.
15. R. Windecker, *Zum Reichweitenproblem*, Dissertation, Mainz, 1978.

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