MATH. SCAND. 47 (1980), 91-115

3-DIMENSIONAL COHOMOLOGY OF THE mod p STEENROD ALGEBRA

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1. Introduction.

This paper is concerned with the differential algebra (E^1S, d^1S) related to the semisimplicial spectrum and with its application to the cohomology $H^{*,*}(A) = \mathrm{Ext}_A^{*,*}(\mathbf{Z}_p, \mathbf{Z}_p)$ of the mod p Steenrod algebra A ([9; Ch. 6, § 1]), where p is an odd prime, $\mathbf{Z}$ is the ring of integers and $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$. (Throughout this paper, except for this section, p always denotes an odd prime.)

To determine the stable homotopy groups of the sphere is one of the most important problems in algebraic topology. $H^{*,*}(A)$ is an important data related to this problem through the Adams spectral sequence ([1; Th. 2.1, 2.2]). Bousfield et al. [3; Prop. 2.4, 2.6, § 8, Prop. 2.4′] constructed the mod p restricted lower central series spectral sequence $\{E^rS, d^rS\}_{r \geqq 1}$. The cohomology of (E^1S, d^1S) as a cochain complex coincides with the cohomology $H^{*,*}(A)$ of the Steenrod algebra A as a Hopf algebra.

The known results on $H^{*,*}(A)$ are as follows. $H^{0,*}(A)$ is trivial by its definition. $H^{1,*}(A)$ is essentially determined by Steenrod ([9; Ch. 1, Lem. 4.2, Ch. 6, Th. 2.7]). Adams [2; Th. 2.5.1] determined $H^{2,*}(A)$, found a complete set of relations among decomposable elements in $H^{3,*}(A)$ and found a linearly independent subset of $H^{3,*}(A)$ for $p=2$; Liulevicius [5; Th. 3.0.1] did the same for an odd prime p; both using the Adams spectral sequence for a pair of Hopf algebras ([2; Th. 2.3.1]). Tangora [10; Appendix 2] determined $H^{s,t}(A)$ for $p=2$ and $t-s \leqq 70$; Nakamura [8; Th. 4.4] determined $E^0H^{*,*}(A)$ for an odd prime p and $t-s \leqq 2(p-1)(3p^2+3p+4)-2$ and found a linearly independent subset of $E^0H^{*,*}(A)$ (which did not contain $\tilde{\lambda}_i$, $h_{i;2,1}$ and $h_{i;1,2}$ for $i \geqq 4$ by the symbols in Theorem 7.2); both using May spectral sequence ([7]). Wang [11; Th. 2.11] determined $H^{3,*}(A)$ for $p=2$, using (E^1S, d^1S).

The main result of this paper is as follows.

THEOREM 8.1. *Let p be an odd prime. All generators listed in Table* 8.1 *form a basis of $H^{3,*}(A)$. All relations listed in Table* 8.2 *form a complete set of relations among decomposable elements in $H^{3,*}(A)$.*

Received January 8, 1978; in revised form August 30, 1979.

The method to determine $H^{s,*}(A)$ in this paper is as follows. We essentially use (E^1S, d^1S). But it is difficult to compute $H^{s,*}(E^1S, d^1S)$ directly. Accordingly, using the two facts that (E^1S, d^1S) has a totally ordered finite basis for each bidegree and that (E^1S, d^1S) essentially admits an endomorphism of a cochain complex preserving the order, we reduce the problem to find a basis (an infinite set) of $H^{s,*}(E^1S, d^1S)$ to the problem to find as small a subset B'_s of $H^{s,*}(E^1S,d^1S)$ as we can find a basis by an appropriate method. Theorem 2.6 gives such an appropriate method. But it is unsuitable to apply the method to (E^1S, d^1S), since the endomorphism θ' of (E^1S, d^1S) corresponding to θ defined before Proposition 5.1 cannot be defined such that $\theta'(E^1_{s,t}S) \subset E^1_{s,t'}S$ for some t' depending on (s, t). Therefore, instead of (E^1S, d^1S), we use a new differential algebra (E, d) characterized by Theorem 4.1 which is isomorphic to (E^1S, d^1S) not as a differential algebra but as a cochain complex. Note that (E, d) has no geometrical meaning.

2. A cochain complex with a totally ordered basis.

In this section we study a general method useful for determining the cohomology of a cochain complex over a field which has a totally ordered basis for each bidegree.

A pair (C, d) is called a cochain complex with a totally ordered basis if $C_{s,t}$ is a module over a field F with a totally ordered basis $K_{s,t}$ for each (s, t), if $C = \sum_{s,t} C_{s,t}$ (a direct sum), and if $d\colon C \to C$ is a homomorphism with $dd = 0$ and with $dC_{s,t} \subset C_{s+1,t}$. Then an element of $C_{s,t}$ is called to be of bidegree (s, t), of dimension s, and of degree t. Define the cohomology of (C, d) as follows:

$$H^{s,*}(C) = \sum_t H^{s,t}(C),\ H^{*,*}(C) = \sum_s H^{s,*}(C) \qquad \text{(direct sums)}\ .$$

We introduce a partial order in $C_{s,t}$ by defining that $y_1 > y_2$ if

$$y_k = \sum_{a \in K_{s,t}} c_{k,a}a,\ c_{k,a} \in F \quad (k = 1, 2), \quad \text{and} \tag{2.1}$$

$$c_{1,a} = c_{2,a}(a > a'), c_{1,a'} \neq 0, c_{2,a'} = 0\ . \tag{2.2}$$

The restriction of this order in $C_{s,t}$ to $K_{s,t}$ coincides with the total order in $K_{s,t}$. $y_1 \geqq y_2$ means $y_1 > y_2$ or $y_1 = y_2$.

Proposition 2.1. *Let (C, d) be a cochain complex with a totally ordered basis, y_1 and y_2 cocycles in $C_{s,t}$, $y_1 \sim y_2$ (cohomologous), $y_1 \neq y_2$, $y_1 \not> y_2$, and $y_1 \not< y_2$. Then there is a y in $C_{s,t}$ such that $y_1 > y$, $y_2 > y$ and $y \sim y_1$.*

Proof. Set y_1 and y_2 as in (2.1). Since $y_1 - y_2 = dx$ for some x in $C_{s-1,t}$, $c_{1,a} = c_{2,a}$ $(a > a')$, $c_{1,a'} \neq 0$, $c_{2a'} \neq 0$, and $c_{1,a'} \neq c_{2,a'}$, we can choose such a y.

For an $a \in K_{s,t}$, $(<a)$ stands for a linear combination of a' such that $a' < a$. If $y = ca + (<a)$ and $0 \neq c \in F$, then define $M(y) = a$. Let (C,d) be a cochain complex with a totally ordered basis.

(1) A cocycle y is called minimal if y is a non-bounding cocycle and if $y' \geqq y$ for each cocycle $y' \sim y$.

(2) A cocycle y is called semi-minimal if y is a non-bounding cocycle and if $M(y') \geqq M(y)$ for each cocycle $y' \sim y$.

(3) For a cochain $z_0 = ca + (<a)$, $0 \neq c \in F$, a cocycle y is called *z_0-minimal* if $y = ca + (<a)$, if y is a semi-minimal cocycle, and if $y' - z_0 \leqq y - z_0$ for each cocycle $y' \sim y$.

PROPOSITION 2.2. *Fix (s,t). Suppose (C,d) is a cochain complex with a totally ordered basis, $C_{s-1,t}$ or $C_{s,t}$ is finitely generated, and b is a non-zero cohomology class of bidegree (s,t).*

(1) *There is a unique minimal cocycle representing b.*

(2) *There is a unique pair (a,c) of $a \in K_{s,t}$ and of $0 \neq c \in F$ such that if y is a semi-minimal cocycle representing b, then $y = ca + (<a)$.*

(3) *If there is a semi-minimal cocycle of a form $ca + (<a)$ representing b, and $z_0 = ca + (<a)$ is a cochain, then there is a unique z_0-minimal cocycle representing b.*

PROOF. Use Proposition 2.1.

THEOREM 2.3. *Fix (s,t). Suppose that*

(i) *(C,d) is a cochain complex with a totally ordered basis;*

(ii) *$C_{s,t}$ or $C_{s-1,t}$ is finitely generated;*

(iii) *$B_{s,t} = \{b_i;\ i \in J_{s,t}\}$ is a subset of $H^{s,t}(C)$, $Y_{s,t} = \{y_i;\ i \in J_{s,t}\}$ is a set of semi-minimal cocycles of bidegree (s,t), and y_i represents b_i for each $i \in J_{s,t}$;*

(iv) *if $y_i \in Y_{s,t}$, $y_j \in Y_{s,t}$ and $M(y_i) = M(y_j)$, then $y_i = y_j$; and*

(v) *for each semi-minimal cocycle y of bidegree (s,t) there is an $i \in J_{s,t}$ such that $M(y) = M(y_i)$.*

Then $B_{s,t}$ is a basis of $H^{s,t}(C)$.

PROOF. First we will show that $B_{s,t}$ generates $H^{s,t}(C)$. Let b be a non-zero element in $H^{s,t}(C)$ represented by a semi-minimal cocycle y'_1. By (v), $M(y'_1) = M(y_{i(1)})$ for some $i(1) \in J_{s,t}$. Hence $M(y'_1 - c_1 y_{i(1)}) < M(y_{i(1)})$ for some c_1 with $0 \neq c_1 \in F$. If $y'_1 - c_1 y_{i(1)}$ is a coboundary, then $b = c_1 b_{i(1)}$. Suppose $y'_1 - c_1 y_{i(1)}$ is a non-bounding cocycle. Then $y'_2 = y'_1 - c_1 y_{i(1)} - dx_1$ is a semi-minimal cocycle for some $x_1 \in C_{s-1,t}$ by induction on the order of $M(y'_2)$. It follows from (v) that $M(y'_2) = M(y_{i(2)})$ for some $i(2) \in J_{s,t}$. By (ii), repeating this process, we see that there is a $j \in J_{s,t}$ such that $y'_j - c_j y_{i(j)}$ and $y'_k - c_k y_{i(k)} - y'_{k+1}$ are coboundaries for $1 \leqq k < j$. Hence $y'_1 \sim \sum_{k=1}^{j} c_k y_{i(k)}$ and $B_{s,t}$ generates $H^{s,t}(C)$. Secondly we will show that $B_{s,t}$ is linearly independent. Suppose $\sum_i c_i b_i = 0$ for $c_i \in F$. By the assumption, $\sum_i c_i y_i = dx$ for some $x \in C_{s-1,t}$. Suppose that there is an $i \in J_{s,t}$ such that $c_i \neq 0$. Set $y_k = \max\{y_i;\ i \in J_{s,t}, c_i \neq 0\}$. Then $M(y_k - c_k^{-1}dx) < M(y_k)$, this contradicts the assumption that y_k is semi-minimal.

PROPOSITION 2.4. *Fix* (s,t). *Suppose* (i)–(ii) *in Theorem* 2.3. *Then there is a pair* $Y_{s,t}$ *and* $B_{s,t}$ *satisfying* (iii)–(v) *in Theorem* 2.3.

PROOF. Choose a y_j of the maximal order in a basis of $H^{s,t}(C)$. Replace the others y_i for $M(y_i) = M(y_j)$ with $y_i - c_i y_j$ for some $c_i \in F$ such that $M(y_i - c_i y_j) < M(y_j)$. Repeat this process.

Let (C,d) and (C',d') be cochain complexes with totally ordered bases $K_{s,t}$ and $K'_{s,t}$, respectively, over a field F. Then $h\colon (C,d) \to (C',d')$ is called *a cochain map preserving orders* if

(i) $h\colon C \to C'$ is a homomorphism with $d'h = hd$;
(ii) for each (s,t) there is a $t' = t'(s,t)$ such that $h(C_{s,t}) \subset C_{s,t'}$;
(iii) if $t'(s,t_1) = t'(s,t_2)$, then $t_1 = t_2$ or $h(C_{s,t_1}) \cap h(C_{s,t_2}) = \{0\}$; and
(iv) if $y \in C_{s,t}$, $y' \in C_{s,t}$ and $y > y'$, then $h(y) > h(y')$.

Note that the condition (iv) may be replaced with the two conditions that

(v) if $a \in K_{s,t}$, $a' \in K_{s,t}$ and $a > a'$, then $M(h(a)) > M(h(a'))$; and
(vi) $h(a) \neq 0$ for each $a \in K_{s,t}$.

PROPOSITION 2.5. *Let* $h\colon (C,d) \to (C',d')$ *be a cochain map preserving orders.*

(i) *h is a monomorphism.*
(ii) $M(h(y)) = M(h(M(y)))$ *for each* $y \in C_{s,t}$.
(iii) *$h(y)$ is a cocycle if and only if y is a cocycle.*
(iv) *If $h(y)$ is a minimal cocycle, then so is y.*
(v) *If $h(y)$ is a semi-minimal cocycle, then so is y.*

PROOF. Standard.

Let $h\colon (C,d) \to (C,d)$ be a cochain map preserving orders. A cocycle y of C is called h-semi-minimal cocycle if $M(y - hy' - dx) \geqq M(y)$ for each cocycle y' of C and for each cochain x of C.

For the next theorem, fix the dimension s and suppose that

(i) (C,d) is a cochain complex with a totally ordered basis;
(ii) $C_{s,t}$ is finitely generated for each t;
(iii) $h\colon (C,d) \to (C,d)$ is a cochain map preserving orders;

(iv) $B'_s=\{b_i;\ i\in J'_s\}$ is a subset of $H^{s,*}(C)$, $Y'_s=\{y_i;\ i\in J'_s\}$ is a set of h-semi-minimal cocycles, and y_i represents b_i for each $i\in J'_s$;

(v) if $y_i\in Y'_s$, $y_j\in Y'_s$ and $M(y_i)=M(y_j)$, then $y_i=y_j$;

(vi) if y is a h-semi-minimal cocycle of dimension s, then $M(y)=M(y_i)$ for some $i\in J'_s$;

(vii) $k(i)$ is a natural number or $k(i)=\infty$ for each $i\in J'_s$;

(viii) if $i\in J'_s$ and $0\leqq k<k(i)$, then $h^k(y_i)$ is a semi-minimal cocycle;

(ix) if $i\in J'_s$ and $k(i)<\infty$, then $h^{k(i)}(y_i)$ is a coboundary; and

(x) $\bigcap_{k=0}^{\infty}h^k(C_s)=\{0\}$.

Set $B_s=\{(h^*)^k(b_i);\ i\in J'_s, 0\leqq k<k(i)\}$ and $B_{s,t}=B_s\cap H^{s,t}(C)$.

THEOREM 2.6. *Under the above hypotheses* (i)–(x), B_s *and* $B_{s,t}$ *are bases of* $H^{s,*}(C)$ *and* $H^{s,t}(C)$, *respectively.*

PROOF. It suffices to verify that $B_{s,t}$ and $Y_{s,t}=\{h^k(y_i);\ i\in J'_s, 0\leqq k<k(i)\}\cap C_{s,t}$ satisfy all assumptions (i)–(v) of Theorem 2.3. For the proof of (v) in Theorem 2.3, using (x), Proposition 2.5(v) and induction on k, we see that for each semi-minimal cocycle y there is an h-semi-minimal cocycle y' and an integer $k\geqq 0$ such that $y=h^k(y')+(<M(y))$.

3. Recollection on (E^1S, d^1S).

Let X be a semi-simplicial spectrum ([5; Def. 4.1]) whose homotopy groups $\pi_n(X)$ are finitely generated and vanish from some degree down. Bousfield et al. [3] constructed the mod p lower central series spectral sequence $\{E^rX, d^rX\}_{r\geqq 1}$. In this section we recall the result of [3] only for the case $X=S$, the sphere semi-simplicial spectrum.

CONVENTION. From now on a binomial coefficient $\binom{i}{j}$ is zero for $i<j$. $t(b)$ stands for the total degree (=deg. − dim.) of b in E^1S.

THEOREM 3.1 ([3; § 8, Prop. 2.4′, 2.5′, 2.6], [4; 14.1]). (1) (E^1S, d^1S) *is a graded differential algebra with unit* 1 *over* $\mathbf{Z}_p$ *such that*

(i) *as an algebra* E^1S *is generated by* λ_i *of bidegree* $(1, 2i(p-1))$ *for every integer* $i\geqq 1$ *and by* μ_i *of bidegree* $(1, 2i(p-1)+1)$ *for every integer* $i\geqq 0$;

(ii) *the product is given by*

$$\lambda_i\lambda_{pi+n} = \sum_{j\geqq 0}(-1)^{j+1}\binom{(n-j)(p-1)-1}{j}\lambda_{i+n-j}\lambda_{pi+j} \quad \text{for } i\geqq 1, n\geqq 0\,,$$

$$\lambda_i\mu_{pi+n} = \sum_{j\geqq 0}(-1)^{j+1}\binom{(n-j)(p-1)-1}{j}\lambda_{i+n-j}\mu_{pi+j}$$

$$+\sum_{j\geqq 0}(-1)^j\binom{(n-j)(p-1)}{j}\mu_{i+n-j}\lambda_{pi+j} \quad \textit{for } i\geqq 1,\ n\geqq 0\ ,$$

$$\mu_i\lambda_{pi+n+1} = \sum_{j\geqq 0}(-1)^{j+1}\binom{(n-j)(p-1)-1}{j}\mu_{i+n-j}\lambda_{pi+j+1} \quad \textit{for } i\geqq 0, n\geqq 0\ ,$$

$$\mu_i\mu_{pi+n+1} = \sum_{j\geqq 0}(-1)^{j+1}\binom{(n-j)(p-1)-1}{j}\mu_{i+n-j}\mu_{pi+j+1} \quad \textit{for } i\geqq 0, n\geqq 0;$$

(iii) *the differential* $d^1S=d^1$ *is given by*

$$d^1\lambda_n = \sum_{j\geqq 1}(-1)^j\binom{(n-j)(p-1)-1}{j}\lambda_{n-j}\lambda_j \quad \textit{for } n\geqq 1\ ,$$

$$d^1\mu_n = \sum_{j\geqq 0}(-1)^{j+1}\binom{(n-j)(p-1)-1}{j}\lambda_{n-j}\mu_j$$

$$+\sum_{j\geqq 1}(-1)^j\binom{(n-j)(p-1)}{j}\mu_{n-j}\lambda_j \quad \textit{for } n\geqq 0\ ,$$

$$d^1(1) = 0,\ d^1(bb') = (-1)^{t(b')}(d^1b)b'+b(d^1b') \quad \textit{for } b,b'\in E^1S\ .$$

(2) *There is a spectral sequence* $\{E^iS, d^iS\}_{i\geqq 1}$ *such that* (E^1S, d^1S) *is as in* (1) *and that* $\{E^iS, d^iS\}_{i\geqq 2}$ *is exactly the Adams spectral sequence for the sphere* ([1; Th. 2.1, 2.2]). *In particular,* $H^{s,t}(E^1S, d^1S)=E^2_{s,t}S=\mathrm{Ext}_A^{s,t}(\mathbf{Z}_p, \mathbf{Z}_p)=H^{s,t}(A)$.

REMARK. λ_i and μ_i in Theorem 3.1 correspond to λ_{i-1} and μ_{i-1} in [3; § 8, Prop. 2.4′]. Bousfield and Kan [4; p. 102, line 13] confessed that the right side of the formula for $d^1\mu_{n-1}$ in [3; p. 340] should have been multiplied by -1. We do not replace the signature in Theorem 3.1 differently from [4].

A monomial $v_{n_1}v_{n_2}\dots v_{n_s}$ of generators (with each $v_{n_i}=\lambda_{n_i}$ or μ_{n_i}) is called admissible if $s=0$ or if

$$s>0,\ pn_k+\varepsilon_k > n_{k+1},\ \varepsilon_k = \begin{cases}0, & v_{n_k} = \lambda_{n_k}\\ 1, & v_{n_k} = \mu_{n_k}\end{cases} \quad \text{for } 1\geqq k<s\ .$$

All admissible monomials form a basis of E^1S ([3; § 8, Prop. 5.4′]).

4. A new differential algebra (E,d).

In this section we introduce a new differential algebra (E,d) such that $H^{*,*}(E,d)$ is isomorphic to $H^{*,*}(E^1S,d^1S)=H^{*,*}(A)$ as an ungraded module and that (E,d) is easy to apply the method in section 2. Let Z' denote the set of integers n such that $n=0,1 \pmod p$ and that $n\geqq 1$. Let $\sum'_{i,j}c_{i,j}T_iT_j$, $c_{i,j}\in\mathbf{Z}_p$, denote a sum running over all (i,j) such that $i,j\in Z'$ from now on.

THEOREM 4.1. *There is a graded differential algebra* (E,d) *with unit* 1 *over* $\mathbf{Z}_p$ *such that*

(i) *as an algebra* E *is generated by* T_n *of bidegree* $(1,n)$ *for* $n \in Z'$,

(ii) *the product and* d *are given by*

$$T_iT_{pi+n} = \sum_{j\geqq 0}{}' (-1)^{j+1}\binom{(n-j)(p-1)-1}{j} T_{i+n-j}T_{pi+j}$$

$$\text{for } i \in Z',\ n \equiv 0,1\ (p),\ n \geqq 0\ ,$$

$$dT_n = \sum_{j\geqq 1}{}' (-1)^{j}\binom{(n-j)(p-1)-1}{j} T_{n-j}T_j \quad \text{for } n \in Z'\ ,$$

$$d(1) = 0,\ d(bb') = b(db')+(-1)^{\dim(b')}(db)b' \text{ for } b,b' \in E, \text{ and}$$

(iii) (E,d) *is isomorphic to* (E^1S,d^1S) *as an ungraded differential algebra.*

PROOF. Let E denote the $\mathbf{Z}_p$-module generated by symbols $T(n_1,\ldots,n_s)$ such that

$$\text{(4.1)} \qquad s \geqq 0, \quad n_i \in Z'\ (1 \leqq i \leqq s), \quad pn_i > n_{i+1}\ (1 \leqq i < s)\ ,$$

where for the case $s=0$ there is only one $T(n_1,\ldots,n_s)$ and it is denoted by 1. Let f: $E^1S \to E$ be a homomorphism of $\mathbf{Z}_p$-modules defined by

$$f(T'_{n_1} \ldots T'_{n_s}) = (-1)^{\varepsilon(n_1,\ldots,n_s)}T(n_1,\ldots,n_s)\ ,$$

where $T'_{pi} = \lambda_i$ $(i \geqq 1)$, $T'_{pi+1} = \mu_i$ $(i \geqq 0)$ and

$$\varepsilon(n_1,\ldots,n_s) = \#\{(i,j)\ ;\ 1 \leqq i < j \leqq s, n_i \equiv 0(p),\ n_j \equiv 1(p)\}\ .$$

Then f is an isomorphism of ungraded $\mathbf{Z}_p$-modules. Define the product and the differential d in E such that f is an isomorphism of ungraded differential algebras. Let us denote $T(n) = T_n$ and define the bidegree of T_n is $(1,n)$. Then $T(n_1,\ldots,n_s) = T_{n_1}T_{n_2} \ldots T_{n_s}$ and (E,d) is a differential algebra as requested.

Let us denote $T(n_1,\ldots,n_s) = T_{n_1}T_{n_2} \ldots T_{n_s}$ for $n_i \in Z'$ $(1 \leqq i < s)$ if $pn_i \leqq n_{i+1}$ for some i. A monomial $T(n_1,\ldots,n_s)$ is called admissible if $s=0$ or if $s>0$ and $pn_i > n_{i+1}$ $(1 \leqq i < s)$.

COROLLARY 4.2. (1) *f induces an isomorphism* f^*: $H^{*,*}(E^1S,d^1S) \to H^{*,*}(E,d)$ *of ungraded modules.*

(2) *All admissible monomials* $T(n_1,\ldots,n_s)$ *form a basis of* E.

In general, $(\sum: T(n_1,\ldots,n_s), B)$ stands for a linear combination of admissible monomials $T(n_1,\ldots,n_s)$ satisfying the condition B. It follows from Corollary 4.2(2) that the set of admissible monomials in E of bidegree (s,t), that is, the set

$$K_{s,t} = \{T(n_1,\ldots,n_s)\ ;\ n_i \in Z'(1\leqq i\leqq s),\ \sum_{i=1}^{s} n_i=t,\ pn_i>n_{i+1}\ (1\leqq i<s)\}$$

is a basis of $C_{s,t}=E_{s,t}$. We introduce a total order into $K_{s,t}$ by defining $T(n_1,\ldots,n_s)>T(n'_1,\ldots,n'_s)$ if

$$1 \leqq j \leqq s, \quad n_i = n'_i\ (1\leqq i<j) \quad \text{and} \quad n_j>n'_j\,.$$

By the way in section 2, (E,d) is a cochain complex with a totally ordered basis and $E_{s,t}$ is finitely generated for each (s,t). Hence all results in section 2 hold for $(C,d)=(E,d)$.

If $1\leqq s\leqq r$, $T(n_1,\ldots,n_r)$ and $T(n'_1,\ldots,n'_s)$ are admissible, then in the notations

$$cT(n_1,\ldots,n_r)+(<(n'_1,\ldots,n'_s)), \qquad cT(n_1,\ldots,n_r)+(<\cdots)\,,$$
$$cT(n_1,\ldots,n_r)+(\leqq T(n'_1,\ldots,n'_s)), \qquad cT(n_1,\ldots,n_r)+(\ll\cdots)$$

for $0\neq c\in \mathbf{Z}_p$, the four sorts of the second summands stands for linear combinations of admissible monomials $T(n''_1,\ldots,n''_s)$ satisfying

$$T(n''_1,\ldots,n''_s) < T(n'_1,\ldots,n'_s), \qquad T(n''_1,\ldots,n''_r) < T(n_1,\ldots,n_r)\,,$$
$$T(n''_1,\ldots,n''_s) \leqq T(n'_1,\ldots,n'_s), \qquad T(n''_1) < T(n_1)\,,$$

respectively. If $y=\sum_{k=1}^{m} c_kT(n_{1,k},\ldots,n_{s,k})$, $0\neq c_k\in\mathbf{Z}_p$, is in admissible form, $m\geqq j\geqq 1$, $n_{1,1}=\ldots=n_{1,j}>n_{1,j+1}$ and

$$T(n_{1,k},\ldots,n_{s,k}) > T(n_{1,i},\ldots,n_{s,i}) \quad \text{for } k<i\,,$$

then define

$$M(y) = T(n_{1,1},\ldots,n_{s,1}),\ m_1(y) = n_{1,1}, \qquad M_2(y) = \sum_{k=1}^{j} c_kT(n_{2,k},\ldots,n_{s,k})\,.$$

FORMULAE 4.3. The following formulae hold under the convention that a summand in the right side exists only if it is admissible.

(1) If $n=n'p^2+kp$, $n'\geqq 0$, $0<k<p$, $i\equiv 0,1$ (p) and $i\geqq 1$, then

$$-T(i,pi+n) = \sum_{j=0}^{k}\binom{k}{j}T(i+n-pj,pi+pj)+(\leqq T(i+n-p^2))\,.$$

(2) If $n=n'p^2+kp+1$, $0<k<p$, $i\equiv 0(p)$ and $i\geqq p$, then

$$-T(i,pi+n) = \sum_{j=0}^{k-1}\binom{k-1}{j}T(i+n-pj,pi+pj)$$
$$+\sum_{j=0}^{k}\binom{k}{j}T(i+n-pj-1,pi+pj+1)+(\leqq T(i+n-p^2)).$$

(3) If $n=n'p^2+kp+1$, $0<k<p$, $i\equiv 1(p)$ and $i\geqq 1$, then

$$-T(i,pi+n) = \sum_{j=0}^{k}\binom{k}{j}T(i+n-pj-1,pi+pj+1)+(\leqq T(i+n-p^2)).$$

(4) If $n=n'p^2+kp$, $n'\geqq 0$ and $0<k<p$, then

$$dT_n = \sum_{j=1}^{k}\binom{k}{j}T(n-pj,pj)+(\leqq T(n-p^2)).$$

(5) If $n=n'p^2+kp+1$, $n'\geqq 0$ and $0<k<p$, then

$$dT_n = \sum_{j=1}^{k-1}\binom{k-1}{j}T(n-pj,pj)$$
$$+\sum_{j=0}^{k}\binom{k}{j}T(n-pj-1,pj+1)+(\leqq T(n-p^2)).$$

(6) If $n\equiv p(p^2)$, then

$$dT_n = T(n-p,p)+\sum_{j=1}^{p-1}\binom{(n-p^2-p)p^{-2}}{j}T(n-p^2j,p^2j)$$
$$+\sum_{j=1}^{k}\binom{(n-p)p^{-2}}{j}T(n-p^2j-p,p^2j+p)+(\leqq T(n-p^3)).$$

(7) If $n\equiv p+1(p^2)$, then

$$dT_n = T(n-1,1)+T(n-p-1,p+1)+(n-p^2-p-1)p^{-2}T(n-p^2,p^2)$$
$$+(n-p^2-p-1)p^{-2}T(n-p^2-1,p^2+1)$$
$$+(n-p-1)p^{-2}T(n-p^2-p-1,p^2+p+1)+(\leqq T(n-2p^2)).$$

Proof. As an example, we prove only (4) here. It follows from Theorem 4.1 that

$$dT_n = \sum_{j=1}^{p^2-1} Q_jT(n-j,j)+(\leqq T(n-p^2)),\ Q_j = (-1)^j\binom{(n-j)(p-j)-1}{j}.$$

The coefficient Q_j is converted to

$$Q_{pj} = (-1)^j\binom{(n'p^2+kp-jp)(p-1)-1}{jp}$$

$$= (-1)^j\binom{(n'p-n'+k-j-1)p^2+(p-k+j-1)p+(p-1)}{jp}$$

$$\equiv (-1)^j\binom{p-k+j-1}{j} = (-1)^j\frac{(p-k+j-1)(p-k+j-2)\ldots(p-k)}{j!}$$

$$\equiv \binom{k}{j} \pmod p$$

for $1 \leqq j \leqq k$, and to

$$Q_{pj} = (-1)^j\binom{n'p-n'+k-j)p^2+(p-k+j-1)p+(p-1)}{jp}$$

$$\equiv (-1)^j\binom{j-k-1}{j} \equiv 0 \pmod p$$

for $k<j<p$, where $n=n'p$. Here we use Lemma 1.2.6 of [9]. Using the lemma similarly, we can show that if $n=n'p+n''$, $0\leqq n''<p$, $n'\geqq 0$, $j=j'p+j''$, $j'\geqq 0$, $0 \leqq j''<p$ and $j''>n''$, then $Q_j\equiv 0(p)$.

5. Fundamental properties of the differential algebra (E, d).

In this section we study the fundamental properties of the product and the differential of (E, d).

For the next proposition, define a homomorphism θ: $E \to E$ of $\mathbf{Z}_p$-modules by

$$\theta(T(n_1,\ldots,n_s)) = T(pn_1,\ldots,pn_s),\ \theta(1) = 1$$

for each monomial $T(n_1,\ldots,n_s)$.

PROPOSITION 5.1. *θ is an endomorphism of a differential algebra. θ is a cochain map preserving orders.*

PROOF. It suffices to show that θ is compatible with the formulae of the product and the differential given in Theorem 4.1 (ii).

For the next proposition, define

$$e(T(n_1,\ldots,n_s)) = n_1+\sum_{i=1}^{s-1}(-pn_i+n_{i+1}) = n_s-\sum_{i=1}^{s-1}(p-1)n_i$$

for a monomial $T(n_1,\ldots,n_s)$.

PROPOSITION 5.2. *If* $T(n_1,\dots,n_{s-1})$ *is admissible*, $s\geqq 1$, $n\geqq n_1$ *and* $n\geqq e(T(n_1,\dots n_s))$, *then*

$$T(n_1,\dots n_s) = \left(\sum : T(n'_1,\dots,n'_s), n\geqq n'_1\geqq n_1\right).$$

PROOF. The proof is by induction on $s+n_s$. The case $s=1$ or $n_s=1$ is trivial. The case $s=2$ follows from Theorem 4.1 (ii). Assume that $m>4$ and that the proposition holds for all pairs (s,n_s) with $s+n_s<m$. Let $s+n_s=m$, $s>2$ and $n_s>1$. Rewrite $T(n_1,\dots,n_s)$ first by applying Theorem 4.1 (ii) to the $(s-1)$st $\sim$ sth factors, secondly by applying the induction hypothesis to the first $\sim(s-1)$st factors, and thirdly by applying the induction hypothesis to the first $\sim$ sth factors.

For the next theorem, define

$$m(T(n_1,\dots,n_s)) = \max\{e(T(n_1,\dots,n_i))\ ;\ 1\leqq i\leqq s\}$$

for a monomial $T(n_1,\dots,n_s)$.

THEOREM 5.3. *If* $T(n_1,\dots n_s)$ *is any monomial, then*

$$T(n_1,\dots,n_s) = \left(\sum : T(n'_1,\dots,n'_s), m(T(n_1,\dots,n_s))\geqq n'_1\geqq n_1\right).$$

PROOF. By induction on s. The theorem for the case $s=2$ is exactly Proposition 5.2 for the case $s=2$ and $n=m(T(n_1,n_2))$. For the proof of the case $s\geqq 3$, rewrite $T(n_1,\dots,n_s)$ first by applying the induction hypothesis to the first $\sim(s-1)$st factors, and secondly by applying Proposition 5.2 for the case $n=m(T(n_1,\dots,n_s))$ to the first $\sim$ sth factors.

PROPOSITION 5.4. *If* $T(n_1,\dots,n_s)$ *is admissible, then*

$$(dT(n_1))T(n_2,\dots,n_s) = \left(\sum : T(n'_1,\dots,n'_{s+1}), \quad n_1>n'_1>n_1/p\right).$$

PROOF. By Theorem 3.1 (ii),

$$dT(n_1) = \sum_k c_k T(n_{1,k},n_{2,k}),\ 0\neq c_k\in \mathbf{Z}_p, n_1 > n_{1,k} > n_1/p,$$

in admissible form. Here the last condition follows from the convention that $\binom{i}{j}=0$ for $i<j$. By Theorem 5.3,

$$T(n_{1,k},n_{2,k},n_2,\dots,n_s) = \sum_i c_{k,i}T(n_{1,k,i},\dots,n_{s+1,k,i}),$$

$$0\neq c_{k,i}\in \mathbf{Z}_p,$$

in admissible form and $n_1>n_{1,k,i}\geqq n_{1,k}>n_1/p$.

For $n \in Z'$, let $E(n)$ denote the $\mathbf{Z}_p$-submodule of E generated by all admissible monomials $T(n_1,\ldots,n_s)$ such that $s \geqq 0$ and $n_1 \leqq n$.

PROPOSITION 5.5. *$E(n)$ is a differential subalgebra of (E,d).*

PROOF. Use Theorem 5.3 to show that $E(n)$ is closed under the product. Use Proposition 5.4 and induction on dimension to show that $E(n)$ is closed under d.

PROPOSITION 5.6. *Let $y = T_n x + (<T_n)$ be in admissible form. If $dy = 0$, then $dx = 0$.*

PROOF. By Theorem 5.3 and Proposition 5.4,

$$dy \;=\; T_n(dx) \pm (dT_n)x + d(<T_n) \;=\; T_n(dx) + (<T_n)$$

in admissible form. If $dx \neq 0$, then $dy \neq 0$.

There is a non-bounding cocycle $y = T_n x + (<T_n)$ such that x is a coboundary. For example, $T(2p, p^2, p^2)$ is a minimal cocycle and $T(p^2, p^2) = 2^{-1} dT(2p^2)$.

PROPOSITION 5.7. *Let $y = T(n_1,\ldots,n_s)x + (<T(n_1,\ldots,n_s)) + \theta x'$ be in admissible form and either $T(n_1,\ldots,n_s) \notin \operatorname{Im}\theta$ or $\deg x \not\equiv 0\ (p)$. If $dy = 0$, then $dx = 0$.*

PROOF. Similar to the proof of Proposition 5.6.

6. Representing cocycles in E.

In this section we study in what range it suffices to find a set Y'_s of representatives in Theorem 2.6 in order to determine $H^{s,*}(E,d) \cong H^{s,*}(A)$.

THEOREM 6.1. *Suppose that*

(a) *$y = \sum_{k=1}^{m} c_k T(n_{1,k},\ldots,n_{s,k})$, $0 \neq c_k \in \mathbf{Z}_p$, is a minimal cocycle in admissible form of bidegree (s,t);*

(b) *$s < t$; and*

(c) *$T(n_{1,k},\ldots,n_{s,k}) > T(n_{1,i},\ldots,n_{s,i})$ for $k < i$.*

Set $$w \;=\; \max\{j\ ;\ n_{s,i} \equiv 0(p^2) \text{ for } 1 \leqq i < j\}\,,$$

$$v \;=\; \max\{j\ ;\ n_{s,i} \equiv 0(p^2),\ n_{s,i} > p^2 \text{ for } 1 \leqq i < j\}\,.$$

Then

(i) $n_{s,k} \equiv 0(p)$ *for all* k;
(ii) $m \geqq 1,\ 1 \leqq v \leqq w \leqq m+1$;
(iii) $n_{s,k} \equiv 0(p^2),\ n_{s-1,k} \equiv 0(p)$ *for* $k < w$,
(iv) *if* $v < w$, *then any one of* (1)–(3) *holds*:
  (1) $n_{s,v} = p^2,\ n_{s-1,v} \equiv 0(p^2)$,
  (2) $n_{s,v} = p^2,\ n_{s-1,v} \equiv p(p^2)$,
  (3) $n_{s,v} = p^2,\ n_{s-1,v} = 2p$;
(v) $n_{s,w} = p$;
(vi) *any one of* (1)–(4) *holds*:
  (1) $n_{s-2,w} \equiv 1(p),\ n_{s-1,w} = p+1$,
  (2) $n_{s-2,w} \equiv 1(p),\ n_{s-1,w} \equiv -p+1(p^2),\ c_{w+1} = -c_w$,
  (3) $n_{s-1,w} = p^2 - p,\ w+p-2 \leqq m,\ c_{w+i}T(n_{1,w+i},\ldots,n_{s,w+i})$
  $= c_w(-1)^i(i+1)^{-1}T(n_{1,w},\ldots,n_{s-2,w},\ p^2-(i+1)p,\ (i+1)p)$
  $(0 \leqq i \leqq p-2)$,
  (4) $n_{s-2,w} \equiv 1(p),\ pn_{s-2,w} - p = n_{s-1,w},\ s \geqq 3,\ n_{s-1,w} \equiv 0(p)$,
  $n_{s-1,w} \not\equiv -p(p^2)$.

PROOF OF (i). Set

$$y = \sum_{i=0}^{q} y_i T_1^i,\quad y_i = (\textstyle\sum :\ T(n_1,\ldots,n_{s-i}),\ n_{s-i} \neq 1) \qquad \textit{for } 0 \leqq i \leqq q\ ,$$

and

$$y_q = \sum_{k=1}^{m'} c_k T(n_{1,k},\ldots,n_{s-q,k}), \qquad 0 \neq c_k \in \mathbf{Z}_p,\ m' \geqq 1\ ,$$

in admissible form, where $T(n_{1,k},\ldots,n_{s-q,k}) > T(n_{1,i},\ldots,n_{s-q,i})$ for $k < i$. It suffices to show the following two statements:
(A): $n_{s-q,k} \equiv 0(p)$ for all k;
(B): $q = 0$.
First we will show (A). Suppose $n_{s-q,k} \equiv 0(p)$ for $1 \leqq k < j$, $n_{s-q,j} \equiv 1(p)$ and $1 \leqq j \leqq m'$. It follows from Theorem 5.3 that

(6.1) $$dy = (\textstyle\sum :\ T(n_1,\ldots,n_{s-i})T_1^i,\ i \leqq q, n_{s-i} \geqq p) + c_j z + (<z) \neq 0\ ,$$

where $z = T(n_{1,j},\ldots,n_{s-q-1,j}, n_{s-q,j}-1)T_1^{q+1}$. This contradicts the assumption $dy = 0$. Secondly we will show (B). If $q \geqq 1$, then

$$y + d((-1)^q c_1 T(n_{1,1},\ldots,n_{s-q-1,1}, n_{s-q,1}+1)T_1^{q-1}) < y$$

which contradicts the assumption that y is minimal.

PROOF OF (iii) (*the second part*). The assumption that

$$n_{s-1,k} \equiv 0 \ (p) \text{ for } k<j, \quad n_{s,k} \equiv 0 \ (p^2) \text{ for } k \leqq j, \text{ and } n_{s-1,j} \equiv 1 \ (p)$$

incurs a contradiction similar to (6.1).

PROOF OF (iii) (*the first part*) *and* (v). The assumption that

$$n_{s,k} \equiv 0 \ (p^2) \text{ for } k < j, \quad n_{s,j} \not\equiv 0 \ (p^2) \text{ and } n_{s,j} \neq p$$

incurs a contradiction similar to (6.1).

PROOF OF (v). Suppose $n_{s-1,v} \not\equiv 0, p(p^2)$ and $n_{s-1,v} \neq 2p$. By Theorem 5.3, (iii) (the first part) and (v), set

$$\begin{aligned}(6.2) \qquad y &= (\textstyle\sum : T(n_1,\dots,n_s),\ n_{s-2}\equiv 0(p),\ n_{s-1}\equiv n_s \equiv 0(p^2),\ n_s > p^2) \\ &\quad + T(n_{1,v},\dots,n_{s-2,v})y_1 + (< T(n_{1,v},\dots,n_{s-2,v}))\,, \\ y_1 &= c_v T(n_{s-1,v}, n_{s,v}) + (\textstyle\sum : T(n_1,n_2),\ n_1 \equiv 0(p),\ n_2 \equiv 0(p^2)\,, \\ &\quad T(n_1,n_2) < T(n_{s-1,v}, n_{s,v}))\,.\end{aligned}$$

It follows from the reason of degree and $dy=0$ that $dy_1=0$. On the other hand,

$$dy_1 = -\binom{n_{s-1,v}p^{-1}}{2} c_v T(n_{s-1,v}-2p, 2p, p^2) + (< \dots) \neq 0\,.$$

PROOF OF (vi). First consider the case $n_{s-1,w} \equiv 1 \ (p)$. By the admissibility, $n_{s-1,w} \geqq p+1$. Suppose $n_{s-1,w} \geqq 2p+1$. Set

$$(6.3) \qquad c = \begin{cases} 1, & T(n_{1,w+1},\dots,n_{s,w+1}) = T(n_{1,w},\dots,n_{s-2,w},n_{s-1,w}-p,2p) \\ 0, & \text{otherwise}\,. \end{cases}$$

Since $dy=0$,

$$2c_{w+1}c - (n_{s-1w}-p-1)p^{-1}c_w = 0, \ c_{w+1}c - (n_{s-1,w}-1)p^{-1}c_w = 0\,.$$

Therefore $c=1$, $c_w = -c_{w+1}$, and $n_{s-1,w} \equiv -p+1 \ (p^2)$, In the case $n_{s-1,w} = p+1$ or $n_{s-1,w} \equiv -p+1 \ (p^2)$, by the method similar to (A) in (i), we see that $n_{s-2,w} \equiv 1 \ (p)$. Secondly consider the case $n_{s-1,w} \equiv 0 \ (p)$. Since $dy=0$,

$$(6.4) \qquad n_{s-1,w} = p \quad \text{or} \quad \text{“}2c_{w+1}c - c_w n_{s-1,w}p^{-1} = 0,\ n_{s-1,w} \geqq 2p\text{”}\,,$$

where c is defined in (6.3). Since y is minimal,

$$(6.5) \qquad n_{s-1,w} \equiv -p \ (p^2); \text{ or}$$

(6.6)

$$pn_{s-2,w}-p = n_{s-1,w}, \quad n_{s-2,w} \equiv 1 \ (p),\ s \geqq 3, \quad n_{s-1,w} \not\equiv -p \ (p^2) .$$

First consider the case (6.5). By (6.4), we have $c=1$ and $2c_{w+1}=-c_w$. By the method similar to (6.3), $y_1=c_wT(n_{s-1,w},p)+(<\dots)$ is a cocycle and $y_1=c_wL_0$ (see Remark after Theorem 7.2) and hence (3) in (vi) holds. In the case (6.6), according to (6.4), $c=0$. This completes the proof of Theorem 6.1.

COROLLARY 6.2. *Let y be a semi-minimal cocycle in admissible form in* (a) *of Theorem* 6.1 *satisfying* (b)–(c) *of Theorem* 6.1. *Then* (i)–(vi) *with* (3)–(4) *of* (vi) *omitted hold.*

PROPOSITION 6.3. *Let y be a coboundary, $s \geqq 1$ and*

$$y = (\textstyle\sum : T(n'_1,\dots,n'_{s+1}),\ n'_{s+1} \equiv 0 \ (p)) .$$

Then there is a cochain x such that $dx=y$ and that

$$x = (\textstyle\sum : T(n_1,\dots,n_s),\ n_s \equiv 0 \ (p)) .$$

PROOF. It suffices to show the two statements similar to (A) and (B) in the proof of Theorem 6.1 (i).

COROLLARY 6.4. *If $s \geqq 2$, y is a cocycle, and*

$$y = (\textstyle\sum ; T(n_1,\dots,n_s),\ n_i \equiv 1 \ (p) \quad \text{for } 1 \leqq i < s,\ n_s \equiv 0 \ (p)) ,$$

then y is a minimal cocycle.

PROOF. Let $y-dx<y$ for some cochain $x=cT(n'_1,\dots,n'_{s-1})+(<\dots)$, $0 \neq c \in \mathbf{Z}_p$. By the reason of degree, $n'_i \equiv 1$ (p) for $1 \leqq i \leqq s-1$ and $dx =cT(n'_1,\dots,n'_{s-2},n'_{s-1}-1,1)+(<\dots)$. This is a contradiction.

EXAMPLE 6.5. It follows from this proposition that $y=T(p+1,\dots,p+1,p)$ is a minimal cocycle of bidegree $(s,sp+s-1)$ for $s \geqq 2$. y corresponds to $y' =\mu_1\mu_1 \dots \mu_1\lambda_1$ in E^1S of bidegree $(s,2(p-1)s+s-1)$ through f. y' represents $-h_{-1;1,2}$ in Proposition 7.2 for $s=2$ and ϱ_3 in Theorem 8.1 for $s=3$.

7. Recollections on $H^{1,*}(A)$ and $H^{2,*}(A)$.

PROPOSITION 7.1 ([9; Ch. 6, Th. 2.7], [6; Th. 3.0.1]). *All generators listed in Table* 7.1 *form a basis of $H^{1,*}(A)$.*

Table 7.1. *A basis of* $H^{1,*}(A)$.

generator	corresponding cocycle in $E_{1,*}$	representative cocycle in $E^1_{1,t}S$	degree (t)	range of indices
h_i	$T(p^{i+1})$	$\lambda(p^i)$	$2(p-1)p^i$	$i \geqq 0$
		μ_0	1	$i=-1$

PROPOSITION 7.2 ([6; Th. 3.0.1]). (1) *All generators listed in Table 7.2 form a basis of* $H^{2,*}(A)$.

Table 7.2. *A basis of* $H^{2,*}(A)$.

generator	corresponding cocycle in $E_{2,*}$	representative cocycle in $E^1_{2,t}S$	degree (t)	range of indices
$h_i h_k$	$T(p^{k+1}, p^{i+1})$	$\lambda(p^k, p^i)$	$2(p-1)(p^k+p^i)$	$i-2 \geqq k \geqq 0$
		$\mu_0\lambda(p^i)$	$2(p-1)p^i+1$	$i \geqq 1$, $k=-1$
		$\mu_0\mu_0$	2	$i=k=-1$
$\tilde{\lambda}_k$	L_k	L'_k	$2(p-1)p^{k+1}$	$k \geqq 0$
$h_{k;2,1}$	$T(2p^{k+1}, p^{k+2})$	$\lambda(2p^k, p^{k+1})$	$2(p-1)(2p^k+p^{k+1})$	$k \geqq 0$
$h_{k;1,2}$	$T(p^{k+1}, 2p^{k+2})$	$\lambda(p^k, 2p^{k+1})$	$2(p-1)(p^k+2p^{k+1})$	$k \geqq 0$
		$\mu_0\lambda_2$	$4(p-1)+1$	$k=-1$

(2) *All relations listed in Table 7.3 form a complete set of relations among decomposable elements in* $H^{2,*}(A)$.

Table 7.3. *A complete set of relations among decomposable elements in* $H^{2,*}(A)$.

$h_k h_k = 0$ $(k \geqq 0)$ $h_{k+1} h_k = 0$ $(k \geqq -1)$,

the relations by the fact that $H^{*,*}(A)$ is commutative.

REMARK. h_k $(k \geqq 0)$, $h_{-1}, \tilde{\lambda}_k, h_{k;2,1}$, $h_{k;1,2}$ $(k \geqq 0)$ and $h_{-1;1,2}$ in Propositions 7.1–7.2 corresponds to h_k $(k \geqq 0)$, α_0, λ_k, μ_k, ν_k $(k \geqq 0)$ and ϱ in [6; Th. 3.0.1], respectively. Define

$$L_k = \sum_{i=1}^{p-1} (-1)^{i+1} i^{-1} T(p^{k+1}(p-i), p^{k+1} i) ,$$

$$L'_k = \sum_{k=1}^{p-1} (-1)^{i+1} i^{-1} \lambda(p^k(p-i), p^k i) .$$

For a ring R, let $R\{\eta_1, \eta_2, \dots\}$ denote the free module generated by $\eta_1, \eta_2, \dots$. $H^{2,*}(E)$ is expressed as a module over the polynomial ring $\mathbf{Z}_p[\theta^*]$ as follows:

$$H^{2,*}(E) = \mathbf{Z}_p[\theta^*]\{h_{-1}h_{-1}, h_i h_{-1}, \tilde{\lambda}_0, h_{0;2,1}, h_{-1;1,2}\ ; i \geqq 1\}/(\theta^*(h_{-1}h_{-1})) ,$$

where for a base $\eta \in H^{2,*}(A)$ the same symbol η stands for $f^*(\eta)$.

8. Determination of $H^{3,*}(A)$.

In this section we determine $H^{3,*}(A)$, using Theorem 2.6.

THEOREM 8.1. *Let p be an odd prime. All generators listed in Table* 8.1 *form a basis of* $H^{3,*}(A)$. *All relations listed in Table* 8.2 *form a complete set of relations among decomposable elements in* $H^{3,*}(A)$.

REMARK. As seen in Table 8.1, we can choose a basis of $H^{3,*}(A)$ such that each generator in the basis, except for $h_k\lambda_i$'s, is represented by a monomial in $E^1 S$. $h_{k;j_1,j_2,j_3}$ stands for the indecomposable generator in $H^{3,*}(A)$ represented by a monomial in $E^1 S$ corresponding to a monomial $T(j_1 p^{k+1}, j_2 p^{k+2}, j_3 p^{k+3})$ for $1 \leqq j_i < p$ $(i = 1, 2, 3)$.

PROOF. Use Theorem 2.6 for the case $C_{s,t} = E_{s,t}$ and $h = \theta$. Trivially (i)–(iii) of Theorem 2.6 hold. We will find B'_3 and Y'_3 satisfying (iv)–(vi) of Theorem 2.6. It follows from Corollary 6.2 that each representative y_i in Y'_3 is written as

$$y_i = cT(n_1, n_2, n_3) + (< T(n_1, n_2, n_3)) + \theta x' ,$$

$$0 \neq c \in \mathbf{Z}_p,\ T(n_1, n_2, n_3) \notin \operatorname{Im} \theta ,$$

in admissible form satisfying any one of the following conditions:

(8.0) $$n_1 = n_2 = n_3 = 1 ,$$

(8.1) $$n_3 \equiv 0 \ (p^2),\ n_3 > p^2 ,$$

(8.2) $$n_3 = p^2 ,$$

(8.3) $$n_3 = p .$$

First, in section 9, we will find a set $Y'_3(1)$ of θ-semi-minimal cocycle y_i under the condition (8.1) such that $Y'_3(1)$ satisfies the assumptions (iv)–(vi) of Theorem 2.6 with Y'_3 replaced with $Y'_3(1)$. Secondly we find similar sets $Y'_3(2)$ and $Y'_3(3)$ under (8.2) and (8.3), respectively. Put Y'_3 to be the union of $Y'_3(1)$ – $Y'_3(3)$ and of $T(1,1,1)$; that is, as Y'_3 we may choose the set listed in Table 8.3. Verify that Y'_3 and their cohomology classes B'_3 satisfy (iv)–(vi) of Theorem 2.6. It follows from Proposition 8.2 that for each y_i in Y'_3 there is a $k(i)$ satisfying (vii)–(ix) of Theorem 2.6, that $k(i)=1$ for any one of

(8.4) $T_1^3, T(1,1,p^k)$ for $k \geqq 2$, $\quad T(1,1,3p)$ for $p \neq 3$,

$T(1,1,6p)$ for $p = 3$,

and that $k(i)=\infty$ for the others in Y'_3. According to Theorem 2.6 and the isomorphism f, we can choose representatives Y_3 of the basis B_3, the basis B''_3 of $H^{3,*}(A)$ and their representatives Y''_3 corresponding to B_3 and Y_3 as the columns "corresponding cocycle in $E_{3,*}$", "generator" and "representative in $E^1_{3,t}S$" in Table 8.1, respectively. For the relations, use the differential algebra structure of (E^1S, d^1S).

Proposition 8.2. (i) *If y is any one of cocycles listed in* (8.4), *then y is a minimal cocycle and $\theta(y)$ is a coboundary.*

(ii) *If $k \geqq 0$ and y is any one of cochains in Table* 8.3 *except for* (8.4) *and for* $L_0T(p^i)$, $i \geqq 4$, *then $\theta^k(y)$ is a minimal cocycle.*

(iii) *If $k \geqq 0$ and $y = L_0T(p^i)$, $i \geqq 4$, then $\theta^k(y)$ is a semi-minimal cocycle.*

Proof. As an example, we prove that $y = T(p^k, p^j, p^i)$ is a minimal cocycle for $0 \leqq k \leqq j-2 \leqq i-4$. By the direct calculation, y is a cocycle. Suppose that $y - dx < y$ for some cochain x. Calculating $dT(p^i+p^j+p^k-n, n)$ for all $n \in Z'$, we see that

$$x = \begin{cases} cT(p^i+p^j+p^k-p^h, p^h) + (<\ldots), & 0 \leqq h < k \\ cT(p^i+p^j, p^k) + (<\ldots) \\ cT(p^i+p^j-p^{k+1}+p^k, p^{k+1}) + (<\ldots), \end{cases}$$

for some $0 \neq c \in \mathbf{Z}_p$. In the first and third cases, $M(dx) > M(y)$ and hence y is not a minimal cocycle. In the second case, $x = cdT(p^i+p^j+p^k) + (<T(p^i+p^j, p^k))$, which is reduced to the other cases. Hence the three cases incurs contradictions.

For the next proposition, suppose that

(8.5) $y = z_0 + \sum_{k<n_1} T_k x_k$ is a cocycle, where $\sum_k$ is in admissible form,

and

(8.6) $z_0 = cT(n_1,\dots,n_s)+(<\dots)$, $0 \neq c \in \mathbf{Z}_p$, is a cochain in admissible form .

If there is a natural number

$$k_0 = \max\{k_0 \in Z' \ ; \ n_1 > k_0 > m_1(d(T_{k_0}x_{k_0})) \geqq m_1(dz_0),\ m_1(d(T_{k_0}x_{k_0})) \geqq m_1(d(T_k x_k)) \text{ for all } k \in Z'\} ,$$

then $T_{k_0}x_{k_0}$ is called the maximal summand for (y, z_0) and is denoted by $S(y, z_0)$. By the definition, x_{k_0} is a cocycle.

PROPOSITION 8.3. *Under the assumption* (8.5)–(8.6), *let* $T_{k_0}x_{k_0} = S(y, z_0)$ *and set* $k'' = m_1(d(T_{k_0}x_{k_0}))$.

(a) x_k *is a cocycle for* $k > k''$.

(b) $M_2(d(\sum_k T_k x_k)) + dx_{k''} = 0$ *if* $k'' > m_1(dz_0)$.

(c) $M_2(d(\sum_k T_k x_k)) + M_2(dz_0) + dx_{k''} = 0$ *if* $k'' = m_1(dz_0)$.

Here the two summations run over all $k \in Z'$ *such that*

$$k_0 \geqq k > k'',\ m_1(d(T_k x_k)) = k'' .$$

PROOF. (a) is proved by induction on k. (b) is proved similarly to (c). (c) is proved by the equation

$$0 = dy = T_{k''}\left(M_2\left(d\left(\sum_k T_k x_k + z_0\right)\right) + dx_{k''}\right) + (<\dots) .$$

PROPOSITION 8.4. *Under the assumption* (8.5)–(8.6), *if there is not the maximal summand for* (y, z_0), *then* x_k *is a cocycle for* $k > m_1(dz_0)$.

PROOF. Easy.

9. Preparations for the proof of Theorem 8.1.

PROPOSITION 9.1. *Suppose* (a)–(c) *of Theorem* 6.1. *Set*

$$v(r) = \max\{j \ ; \ n_{s,i} \equiv 0 \ (p^{r+1}) \ \text{for} \ 1 \leqq i < j\} .$$

Then

(i) $n_{s-2,k} \equiv 0 \ (p)$, $n_{s-1,k} \equiv 0 \ (p^2)$, $n_{s,k} \equiv 0 \ (p^3)$ *for* $v(3) \leqq k < v(2)$;

(ii) $v(2)$ *and* $v(1)$ *are exactly* v *and* w *in Theorem* 6.1.

Table 8.1. *A basis of* $H^{3,t}(A)=\mathrm{Ext}_A^{3,t}(\mathbf{Z}_p,\mathbf{Z}_p)$.

generator	corresponding cocycle in $E_{3,*}$	representative cocycle in $E^1_{3,t}S$	degree (t)	range of indices
$h_ih_jh_k$	$T(p^{k+1},p^{j+1},p^{i+1})$	$\lambda(p^k,p^j,p^i)$	$2(p-1)(p^k+p^j+p^i)$	$0\leqq k\leqq j-2\leqq i-4$
		$\mu_0\lambda(p^j,p^i)$	$2(p-1)(p^j+p^i)+1$	$k=-1,\ 1\leqq j\leqq i-2$
		$\mu_0\mu_0\lambda(p^i)$	$2(p-1)p^i+2$	$j=k=-1,\ i\geqq 1$
		$\mu_0\mu_0\mu_0$	3	$i=j=k=-1$
$\tilde{\lambda}_ih_j$	$T(p^{j+1})L_i$	$\lambda(p^j)L'_i$	$2(p-1)(p^{i+1}+p^j)$	$i\geqq 0,\ j\geqq 0,\ j\neq i+2$
		$\mu_0L'_i$	$2(p-1)p^{i+1}+1$	$i\geqq 0,\ j=-1$
$h_{i;2,1}h_j$	$T(p^{j+1},2p^{i+1},p^{i+2})$	$\lambda(p^j,2p^i,p^{i+1})$	$2(p-1)(p^{i+1}+2p^i+p^j)$	$i\geqq 0,\ j\geqq 0,\ j\neq i+2,\ i,\ i-1$
		$\mu_0\lambda(2p^i,p^{i+1})$	$2(p-1)(p^{i+1}+2p^i)+1$	$j=-1,\ i\geqq 1$
$h_{i;1,2}h_j$	$T(^{j+1},p^{i+1},2p^{i+2})$	$\lambda(p^j,p^i,2p^{i+1})$	$2(p-1)(2p^{i+1}+p^i+p^j)$	$i\geqq 0,\ j\geqq 0,\ j\neq i+2,\ i\pm 1,\ i$
		$\mu_0\lambda(p^i,2p^{i+1})$	$2(p-1)(2p^{i+1}+p^i)+1$	$i\geqq 1,\ j=-1$
		$\mu_0\lambda(2,p^j)$	$2(p-1)(p^j+2)+1$	$i=-1,\ j\geqq 2$
$h_{i;1,2,3}$	$T(p^{i+1},2p^{i+2},3p^{i+3})$	$\lambda(p^i,2p^{i+1},3p^{i+2})$	$2(p-1)(3p^{i+2}+2p^{i+1}+p^i)$	$p\neq 3,\ i\geqq 0$
		$\mu_0\lambda(2,3p)$	$2(p-1)(3p+2)+1$	$p\neq 3,\ i=-1$
$h_{i;1,2,0,2}$	$T(p^{i+1},2p^{i+2},2p^{i+4})$	$\lambda(p^i,2p^{i+1},2p^{i+3})$	$2(p-1)(2p^{i+3}+2p^{i+1}+p^i)$	$p=3,\ i\geqq 0$
		$\mu_0\lambda(2,2p^2)$	$2(p-1)(2p^2+2)+1$	$p=3,\ i=-1$
$h_{i;1,3,1}$	$T(p^{i+1},3p^{i+2},p^{i+3})$	$\lambda(p^i,3p^{i+1},p^{i+2})$	$2(p-1)(p^{i+2}+3p^{i+1}+p^i)$	$p\neq 3,\ i\geqq 0$
		$\mu_0\lambda(3,p)$	$2(p-1)(p+3)+1$	$p\neq 3,\ i=-1$
$h_{i;2,1,2}$	$T(2p^{i+1},p^{i+2},2p^{i+3})$	$\lambda(2p^i,p^{i+1},2p^{i+2})$	$2(p-1)(2p^{i+2}+p^{i+1}+2p^i)$	$i\geqq 0$
$h_{i;3,2,1}$	$T(3p^{i+1},2p^{i+2},p^{i+3})$	$\lambda(3p^i,2p^{i+1},p^{i+2})$	$2(p-1)(p^{i+2}+2p^{i+1}+3p^i)$	$i\geqq 0,\ p\neq 3$
ϱ_3	$T(1,1,3p)$	$\mu_0\mu_0\lambda_3$	$6(p-1)+2$	$p\neq 3$
ϱ'_3	$T(1,1,6p)$	$\mu_0\mu_0\lambda_6$	26	$p=3$
f_i	M_i	M'_i	$2(p-1)(p^{i+1}+2p^i)$	$i\geqq 1$
g_i	N_i	N'_i	$2(p-1)(2p^{i+1}+p^i)$	$i\geqq 1$

REMARK TO TABLE 8.1. M_i, N_i, M'_i and N'_i are defined as follows.

$$M_i = \sum_{k=1}^{p-1} \frac{(-1)^{i+1}}{i}\{T(2p^{i+1}, p^{i+2}-kp^{i+1}, kp^{i+1}) - 2T(2p^{i+1}-kp^i, kp^i, p^{i+2}) - 2T(p^{i+1}-kp^i, p^{i+1}+kp^i, p^{i+2})\},$$

$$N_i = \sum_{k=1}^{p-1} \frac{(-1)^{i+1}}{i}\{2T(p^{i+1}, 2p^{i+2}-kp^{i+1}, kp^{i+1}) + 2T(p^{i+1}, p^{i+2}-kp^{i+1}, p^{i+2}+kp^{i+1}) - T(p^{i+1}-kp^i, kp^i, 2p^{i+2})\},$$

$$M'_i = \sum_{k=1}^{p-1} \frac{(-1)^{i+1}}{i}\{\lambda(2p^i, p^{i+1}-kp^i, kp^i) - 2\lambda(2p^i-kp^{i-1}, kp^{i-1}, p^{i+1}) - 2\lambda(p^i-kp^{i-1}, p^i+kp^{i-1}, p^{i+1})\},$$

$$N'_i = \sum_{k=1}^{p-1} \frac{(-1)^{i+1}}{i}\{2\lambda(p^i, 2p^{i+1}-kp^i, kp^i) + 2\lambda(p^i, p^{i+1}-kp^i, p^{i+1}+kp^i) - \lambda(p^i-kp^{i-1}, kp^{i-1}, 2p^{i+1})\}.$$

Table 8.2. *Relations among decomposable elements in* $H^{3,*}(A)$.
Relations following from the fact that $H^{*,*}(A)$ is commutative.

$h_{i+1}h_ih_j = 0,$
$h_k^2h_j = 0,$
$h_{k+2}\tilde{\lambda}_k = \tilde{\lambda}_{k+1}h_{k+1},$
$h_{k+1}\tilde{\lambda}_k = h_{k+1;2,1}h_k,\ p = 3,$
$\tilde{\lambda}_{i+1}h_i = h_{i+1}h_{i;1,2},\ p = 3,$

$h_{k;2,1}h_{k+2} = 0,$
where $i \geqq -1$, $j \geqq -1$ and $k \geqq 0$.

$h_{k;2,1}h_{k-1} = 0,$
$h_{k;2,1}h_k = 0,\ p \neq 3,$
$h_{k+1}h_{k;2,1} = h_{k;1,2}h_k,$
$h_{i;1,2}h_{i+2} = 0,$
$h_{i;1,2}h_{i+1} = 0,\ p \neq 3,$
$h_{i;1,2}h_{i-1} = 0,$
$h_{-1;1,2}h_{-1} = 0,$

PROOF OF PROPOSITION 9.1. Let $n_{s-2,k} \equiv 0\ (p)$, $n_{s-1,k} \equiv 0\ (p^2)$ and $n_{s,k} \equiv 0\ (p^3)$ for $k<j$, and $n_{s-1,j} \equiv 0\ (p)$, $n_{s,j} \equiv 0\ (p^3)$. It suffices to show that $n_{s-2,j} \equiv 0\ (p)$ and $n_{s-1,j} \equiv 0\ (p^2)$. Since y is in admissible form, $n_{s-1,j} > p$. First suppose $n_{s-1,j} \not\equiv 0\ (p^2)$. By Theorem 5.3, (iii) (the first part) and (vi) both in Theorem 6.1, set y as (6.2) with v replaced with j. It follows from the reason of degree and $dy=0$ that $dy_1=0$. But a contradiction

$$dy_1 = c_j n_{s-1,j} p^{-1} T(n_{s-1,j}-p, n_{s,j}-p^2+p, p^2) + (<\dots) \neq 0$$

occurs. Hence $n_{s-1,j} \equiv 0\ (p^2)$. Secondly suppose $n_{s-2,j} \not\equiv 0\ (p)$. Set

Table 8.3. *θ-semi-minimal cocycles.*

θ-semi-minimal cocycle y	its maximal monomial $M(y)$	range of indices
(1) $T(1,p^j,p^i)$	$T(p^i-p^{j+1}+p^j-p+1,p^{j+1}-p^2+p,p^2)$	$2\leqq j\leqq i-2$
$T(1,1,p^i)$	$T(p^i-2p+1,p+1,p)$	$i\geqq 2$
$T(1,1,1)$	$T(1,1,1)$	
(2) T_1L_i	$T(p^{i+2}-p^{i+1}-p+1,p^{i+1}-p^2+p,p^2)$	$i\geqq 1$
$2T_1L_0+dT(p^2-2p+1,2p)$	$T(p^2-2p,p+1,p)$	
$L_0T(p^i)$	$T(p^i-p^3+p^2,p^3-p^2,p^2)$	$i\geqq 4$
	$T(p^2-2p,2p,p^2)$	$i=2$
T_pL_0	$T(p,p^2-p,p)$	
(3) $T(2p,p^2,p^i)$	$T(p^i-p^3-p^2+2p,2p^2,p^3)$	$i\geqq 4$
	$T(2p,p^2,p^2)$	$i=2$
$T(1,2p^i,p^{i+1})$	$T(2p^i-p+1,p^{i+1}-p^2+p,p^2)$	$i\geqq 2$
(4) $T(1,2p,p^i)$	$T(p^i-2p^2+p+1,p^2+p,p^2)$	$i\geqq 3$
$T(1,p^i,2p^{i+1})$	$T(p^{i+1}+p^i-p+1,p^{i+1}-p^2+p,p^2)$	$i\geqq 2$
(5) $T(1,2p,3p^2)$	$T(p^2+p+1,p^2+p,p^2)$	$p\neq 3$
(6) $T(1,2p,2p^3)$	$T(p^3+p^2+p+1,p^2+p,p^2)$	$p=3$
(7) $T(1,3p,p^2)$	$T(p+1,2p,p^2)$	$p\neq 3$
(8) $T(2p,p^2,2p^3)$	$T(p^3-p^2+2p,2p^2,p^3)$	
(9) $T(3p,2p^2,p^3)$	$T(3p,2p^2,p^3)$	$p\neq 3$
(10) $T(1,1,3p)$	$T(p+1,p+1,p)$	$p\neq 3$
(11) $T(1,1,6p)$	$T(4p+1,p+1,p)$	$p=3$
(12) M_1	$T(2p^2,p^3-p^2,p^2)$	
(13) N_1	$T(p^3+p^2,p^3-p^2,p^2)$	

$$
\begin{aligned}
y &= (\textstyle\sum : \ T(n_1,\dots,n_s),\ n_{s-2}\equiv 0\ (p),\ n_{s-1}\equiv 0\ (p^2),\ n_s\equiv 0\ (p^3)) \\
&\quad + T(n_{1,j},\dots,n_{s-3,j})y_1 + (< T(n_{1,j},\dots,n_{s-3,j}))\,, \\
y_1 &= T(n_{s-2,j})y_2 + T(n_{s-2,j}-1)y_3 + (< T(n_{s-2,j}-1))\,, \\
y_2 &= c_jT(n_{s-1,j},n_{s,j}) + (< \dots)\,, \\
y_3 &= (\textstyle\sum : \ T(n_1,n_2),\ n_1\equiv 1\ (p))\,.
\end{aligned}
$$

It follows from the reason of degree and $dy=0$ that $dy_1=0$. By Proposition 5.6, $dy_2=0$. By $dy_1=0$ and by the reason of degree, $dy_3=-T_1y_2$. By Theorem 7.2,

$$(9.1)\quad y_2 = -c_jT(p^r,p^i),\ 2 \leqq r \leqq i-2;\ c_jL_i,\ i \geqq 2;\ c_jT(2p^i,p^{i+1}),\ i \geqq 2;$$
$$-c_jT(p^i,2p^{i+1}),\ i \geqq 2;\ c_jm^{-1}dT(mp^i)+(<T((m-1)p^i,p^i)),$$
$$m \not\equiv 0(p),\ m \geqq 2\ .$$

If y_2 is non-bounding, then so is T_1y_2 by Proposition 8.2, which is a contradiction. Hence

$$y_2 = c_jm^{-1}dT(mp^i),\ m \not\equiv 0\ (p),\ m \geqq 2\ ,$$
$$y_3 = -c_jm^{-1}T(1,mp^i)+c'dT(mp^i+1),\ c' \in \mathbf{Z}_p\ .$$

$c' \neq 0$ contradicts (i) of Theorem 6.1. Since y is minimal, $mp^i \geqq pn_1$. Since $T(n_1-1)y_3$ is in admissible form, $p(n_1-1) > mp^i-p+1$. These are contradictions. Hence $n_{s-2,j} \equiv 0\ (p)$. The proof of (ii) is easy.

PROPOSITION 9.2. *Suppose* (a)–(c) *of Theorem* 6.1. *Set* $v(r)$ *as in Proposition* 9.1. *If* $s=3$, *then*

$$n_{1,k} \equiv 0\ (p^2),\ n_{2,k} \equiv 0\ (p^3),\ n_{3,k} \equiv 0\ (p^4) \quad \textit{for}\ k<v(3)\ .$$

PROOF. Similar to the proof of Proposition 9.1. Raise the power of p by one.

PROPOSITION 9.3. *If*

$$y = cT(n_1,n_2,p^3)+(<T(n_1,n_2))+\theta x, \qquad 0 \neq c \in \mathbf{Z}_p\ ,$$
$$= T(n_1)x'+(<T(n_1))+\theta x$$

is a semi-minimal cocycle in admissible form, then

$$x' = cT(2p^2,p^3), \quad n_1 \equiv 2p(p^2) \quad \textit{or}$$
$$x' = cT(2p^2,p^3), \quad n_1 = 3p,\ p \neq 3\ .$$

PROOF. Since y is admissible, $n_1 \geqq 3p$, $n_2 \geqq 2p^2$ and $n_1 \geqq 4p(n_2 > 2p^2)$. It follows from Theorem 6.1 and Proposition 9.1 that $n_1 \equiv 0\ (p)$ and $n_2 \equiv 0\ (p^2)$. The case $n_1 \equiv 0, p\ (p^2)$ is reduced to (8.2)–(8.3) and is omitted. Set $y = \sum_{k \leqq n_1} T_kx_k$ in admissible form. By Proposition 5.6, x_{n_1} is a cocycle. By Theorem 7.2,

$$x_{n_1} = cT(2p^2,p^3);\ -cT(p^2,p^3);\ -cT(p^2,p^i),\ i \geqq 4;\ cL_2;$$
$$cm^{-1}dT(mp^3)+(<T((m-1)p^3,p^3)),\ m \not\equiv 0\ (p),\ m \geqq 2\ .$$

Suppose that $n_1 \not\equiv 2p\ (p^2)$ and $n_1 \geqq 4p$. Then

$$d(T_{n_1}x_{n_1}) = c'T(n_1-3p,3p,2p^2,p^3)+(\ll\dots);$$
$$c'T(n_1-2p,2p,p^2 2p^3)+(\ll\dots);$$
$$c'T(n_1-2p,2p,p^2,p^i)+(\ll\dots);\ c'T(n_1-p,p)L_2+(\ll\dots);$$
$$c'T(n_1-p,p)(dT(mp^3))+(<T((m-1)p^3,p^3))+(\ll\dots),$$

for $0\neq c'\in \mathbf{Z}_p$, respectively. By Theorems 7.2, 6.1, $x_k=0$ for $n_1>k>n'' = m_1(d(T_{n_1}x_{n_1}))$. Thus $dx_{n''}=-M_2(d(T_{n_1}x_{n_1}))$. It follows from Proposition 8.2 and $dy=0$ that x_{n_1} is a coboundary. Since y is semi-minimal and in admissible form, $mp^3\geqq pn_1>(m-1)p^3$. Therefore $x_{n_1-p}=-cm^{-1}T(p,mp^3)+$ (a cocycle). By Theorems 7.2, 6.1, (a cocycle)$=0$. Hence $x_{n_1-p}=cm^{-1}T(mp^3-p^2+p,p^2)+(<\dots)$. This contradicts the assumption that y is in admissible form.

PROPOSITION 94. *If*

$$y = cT(n_1,2p^2,p^3)+(<\dots)+\theta x,\quad 0\neq c\in \mathbf{Z}_p,\ n_1\equiv 2p\ (p^2),$$

is a semi-minimal cocycle in admissible form, then

$$y = cT(2p,p^2,2p^3)+(<M(T(2p,p^2,2p^3)))\quad \textit{or}$$
$$y = cT(2p,p^2,p^q)+(<M(T(2p,p^2,p^q))),\quad q\geqq 4.$$

PROOF. Set y as in (8.5)–(8.6),

$$z_0 = cT(2p,\ p^2,np^q),\ n'' = m_1(dz_0),\quad \text{and}$$
$$np^q = n_1+p^3+p^2-2p,\ n\not\equiv 0\ (p),\ n\geqq 1,\ q\geqq 2,\ np^q\geqq p^3+2p^2.$$

Let y be z_0-minimal for a change. By Theorem 5.3 and Proposition 5.5,

$$(9.1)\quad dz_0 = \begin{cases} cnT((n-1)p^q-p^3-p^2+2p,2p^2,p^3,p^q)+(\ll\dots), \\ \qquad q=2,\ n\geqq p+3;\ q=3,\ n\geqq 3;\ q\geqq 5,\ n\geqq 2 \\ cnT((n-1)p^4-2p^3-2p^2+2p,3p^2,2p^3,p^4)+(\ll\dots), & q=4,\ n\geqq 2 \\ -cT(2p,p^2,2p^2,p^3), & q=2,\ n=p+2 \\ 0, & (q,n)=(3,2);\ n=1,\ q\geqq 4. \end{cases}$$

First consider the first three cases of (9.1). Using Propositions 8.3–8.4, there is no $S(y,z_0)$ and $dx_{n''}=M(dz_0)$. Therefore, either of the followings follows from (9.1):

(1) $q=2,\ n\geqq p+3,\ n''=(n-p-2)p^2+2p,\ p\neq 3,$

$$x_{n''} = -cT(2p^2,p^3+p^2)-2c3^{-1}T(3p^2,p^3);$$

(2) $q = 2,\ n = p+2,\ n'' = 2p,\ p \neq 3,\ x_{n''} = -c3^{-1}T(3p^2, p^3)$.

The second case contradicts the assumption that y is in admissible form. Secondly consider the first case. Set $z_1 = z_0 + T_{n''}x_{n''}$ and let y be z_1-minimal for a change. By theorem 5.3,

$$dz_1 = -c3^{-1}T(n_1 - p^2 - p, p, 3p^2, p^3) + (\ll \ldots) .$$

Using Proposition 8.3–8.4, we see that there is no $S(y, z_1)$ and

$$dx_{n''-1} = -M_2(dz_1) + cT(1, 3p, p^2) .$$

Since $p \neq 3$, this contradicts Proposition 8.2. In the last case of (9.1), z_0 is a minimal cocycle by Proposition 8.2 and hence $y = z_0 + (< M(z_0))$.

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