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ON POLYNOMIAL COVERINGS AND THEIR CLASSIFICATION

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1. Introduction.

This paper consists of two independent notes on polynomial covering spaces in connection with the papers [3] and [4]. Roughly speaking, an n-fold polynomial covering space $\pi\colon E \to X$ over a space X is the zero set for a continuous family of simple, normed, complex polynomials of degree n parametrized by X. The first note (Section 2), which is the more substantial one and has given rise to the title, shows that polynomial covering spaces do not admit classifying spaces in the traditional sense, and the second note (Section 3) offers a criterion for certain orientation coverings to be polynomial.

Section 2 must be seen in connection with [4], which provides an algebraic classification of n-fold, polynomial coverings. As is stated in the very last sentence of [4], it would be nice also to have a geometric classification in terms of a classifying space for n-fold polynomial coverings. Theorem 2.4, however, shows that a classifying space of this kind does not exist in general.

Section 3 has its origin in [3, Theorem 6.1], which says that the orientation covering of a nonorientable closed surface is polynomial precisely when the surface has even genus. At the suggestion of V. L. Hansen, we have tried to find a proof of this theorem using characteristic classes with the purpose of finding a suitable criterion for more general orientation coverings to be polynomial. Theorem 3.1 provides such a criterion, and Example 3.2 shows that it covers [3, Theorem 6.1].

I am indebted to Vagn Lundsgaard Hansen for encouragement and guidance throughout this study.

2. On classifying spaces for polynomial coverings.

Let us first agree upon the notation. Throughout this paper B^n denotes the complement of the discriminant set Δ in complex n-space $\mathbf{C}^n$ (i.e. $B^n = \mathbf{C}^n \setminus \Delta$); $B(n)$ is the group of n-braids; and Σ_n is the full permutation group on n letters.

Σ_n acts freely on

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$$F_n(\mathbf{C}) = \{(\lambda_1,\ldots,\lambda_n) \in \mathbf{C}^n \mid \lambda_i \neq \lambda_j \text{ all } i \neq j\}$$

by permutation of coordinates. The orbit space $F_n(\mathbf{C})/\Sigma_n$ is canonically homeomorphic to B^n, and therefore we get an induced principal Σ_n-bundle $p_n\colon F_n(\mathbf{C}) \to B^n$.

Finally, let $\pi^n\colon E^n \to B^n$ be the canonical n-fold polynomial covering as defined in [3, Section 3].

The following theorem describes a relationship between π^n and p_n. In the theorem as well as in the rest of this section, X denotes a connected CW-complex with a non-degenerate base point $x_0 \in X$. The fundamental group of X with base point $x_0 \in X$ will be denoted by $\pi_1(X)$.

PROPOSITION 2.1. *Let α and β be two maps $X \to B^n$. Then the pull-back coverings $\alpha^*\pi^n$ and $\beta^*\pi^n$ are equivalent as coverings if and only if the pull-back bundles α^*p_n and β^*p_n are equivalent as principal Σ_n-bundles.*

PROOF. B^n is an Eilenberg–MacLane space of type $(B(n), 1)$. Hence the natural epimorphism $\tau_n\colon B(n) \to \Sigma_n$ induces a map $t_n\colon B^n \to B\Sigma_n$. As can be seen from ([1], Theorem 1), this map classifies p_n. Since $\tau_n \circ \alpha_*$ and $\tau_n \circ \beta_*\colon \pi_1(X) \to \Sigma_n$ are characteristic homomorphisms for $\alpha^*\pi^n$ and $\beta^*\pi^n$ ([4], Lemma 3.1), Proposition 2.1 follows.

REMARK 2.2. Letting β be a constant map, we see that $\alpha^*\pi^n$ is a trivial covering if and only if α^*p_n is a trivial principal Σ_n-bundle; i.e. if and only if $\alpha\colon X \to B^n$ admits a lifting $\alpha'\colon X \to F_n(\mathbf{C})$ over $p_n\colon F_n(\mathbf{C}) \to B^n$. Thus Proposition 2.1 generalizes [4, Theorem 4.1].

Now let $\mathrm{PC}_n(X)$ denote the set of equivalence classes of n-fold polynomial coverings over X, and let $[X, Y]$ be the set of free homotopy classes of maps of X into Y for an arbitrary space Y.

As shown in [3] and mentioned in [4, Section 2], pull-back of π^n along maps $\alpha\colon X \to B^n$ yields a surjective map

$$[X, B^n] \to \mathrm{PC}_n(X)\,.$$

This surjective map is not in general injective; see [4, Example 4.3] or Proposition 2.5 below. According to Proposition 2.1, there is an injective map

$$\mathrm{PC}_n(X) \to [X, B\Sigma_n]\,,$$

which to the equivalence class of $\alpha^*\pi^n$ associates the homotopy class of $t_n \circ \alpha$. These maps make the following diagram commutative:

$$(1)\qquad \begin{tikzcd} {[X,B^n]} \arrow[r] \arrow[d, "t_{n*}"'] & \mathrm{PC}_n(X) \arrow[dl] \\ {[X,B\Sigma_n]} & \end{tikzcd}$$

Here, t_{n*}: $[X,B^n] \to [X,B\Sigma_n]$ is induced by composition with t_n: $B^n \to B\Sigma_n$. Using this diagram we get:

LEMMA 2.3. i) *The map* $[X,B^n] \to \mathrm{PC}_n(X)$ *is bijective if and only if* t_{n*} *is injective.*

(ii) *Every n-fold covering over X is equivalent to a polynonial covering if and only if* t_{n*} *is surjective.*

The following theorem is the main result in this section. It shows that a classifying space in the traditional sense does not exist for n-fold polynomial coverings.

THEOREM 2.4. *There does not exist an n-fold polynomial covering* w^n: $E_n \to Z_n$, $n \geqq 2$, *such that the map*

$$[X,Z_n] \to \mathrm{PC}_n(X)$$

defined by pull-back of w^n *is bijective for all connected* CW*-complexes X.*

PROOF. Assume that w^n: $E_n \to Z_n$ is an n-fold polynomial covering with the property that the map $[X,Z_n] \to \mathrm{PC}_n(X)$ is bijective for all X.

We will bring this assumption to a contradiction in 3 steps. First of all, note that Z_n may be chosen to be a CW-complex.

STEP 1. Z_n is pathwise connected and $\pi_i(Z_n)=0$ for $i \geqq 2$.

PROOF OF STEP 1. Since any two maps from a one-point space into Z_n are homotopic, it easily follows that Z_n is pathwise connected.

Let $z_0 \in Z_n$ and $b_0 \in B^n$ be non-degenerate base points. Choose a map $(B^n,b_0) \to (Z_n,z_0)$ which induces π^n by pull-back of w^n, and a map $(Z_n,z_0) \to (B^n,b_0)$ which induces w^n by pull-back of π^n. Then the diagram

$$(2)\qquad \begin{tikzcd} & B^n \arrow[d] \\ Z_n \arrow[ur] \arrow[r, "1_{Z_n}"'] & Z_n \end{tikzcd}$$

is homotopy commutative. Here 1_{Z_n}: $Z_n \to Z_n$ is the identity on Z_n. Applying the functor $\pi_i(-,*)$ to (2) gives $\pi_i(Z_n,z_0)=0$ for $i \geqq 2$, since B^n is an Eilenberg–MacLane space of type $(B(n),1)$.

STEP 2. $H_1(Z_n; \mathbf{Z}) \cong \mathbf{Z}$.

PROOF OF STEP 2. Applying the functor $H_1(-; \mathbf{Z})$ to (2), we see that $H_1(Z_n; \mathbf{Z})$ is a subgroup of $H_1(B^n; \mathbf{Z})$, and since B^n is connected, we have

$$H_1(B^n; \mathbf{Z}) \cong B(n)/B(n)' \cong \mathbf{Z}\,.$$

For the last isomorphism, see e.g. [2].

Hence we only have to prove that $H_1(Z_n; \mathbf{Z}) \neq 0$. The maps from (2) build another homotopy commutative diagram

$$\begin{tikzcd} & Z_n \arrow[d] \\ B^n \arrow[ur] \arrow[r, "f"'] & B^n \end{tikzcd} \tag{3}$$

where f: $(B^n, b_0) \to (B^n, b_0)$ is a map such that $f^*\pi^n$ is equivalent to π^n. According to Proposition 2.1, $t_n \circ f$: $B^n \to B\Sigma_n$ is freely homotopic to t_n. Consider the following commutative diagram in which the vertical epimorphisms are Hurewicz homomorphisms

$$\begin{tikzcd} \pi_1(B^n, b_0) \arrow[r, "(t_n\circ f)_*"] \arrow[d] & \pi_1(B\Sigma_n, t_n(b_0)) \arrow[d] \\ H_1(B^n; \mathbf{Z}) \arrow[r, "H_1(t_n\circ f)"] & H_1(B\Sigma_n; \mathbf{Z}) \end{tikzcd}$$

Since $(t_n \circ f)_*$ is an epimorphism, $H_1(t_n \circ f) = H_1(t_n) \circ H_1(f)$ has to be surjective also. Since $H_1(B\Sigma_n; \mathbf{Z}) \neq 0$, this implies that $H_1(f) \neq 0$, so from (3) it follows that $H_1(Z_n; \mathbf{Z}) \neq 0$.

STEP 3. $\mathrm{PC}_n(S^1)$ is infinite. (The contradiction!)

PROOF OF STEP 3. Since Z_n is an Eilenberg–MacLane space, we may identify $[S^1, Z_n]$ with the set $\pi_1(Z_n, z_0)^{\mathrm{conj}}$ of conjugacy classes of elements in the fundamental group $\pi_1(Z_n, z_0)$. The Hurewicz epimorphism $\pi_1(Z_n, z_0) \to H_1(Z_n; \mathbf{Z})$ factors through $\pi_1(Z_n, z_0)^{\mathrm{conj}}$, so this set is infinite. By assumption, we may also identify $[S^1, Z_n]$ with $\mathrm{PC}_n(S^1)$; thus $\mathrm{PC}_n(S^1)$ is infinite.

Step 3 finishes the proof since the number of equivalence classes of n-fold coverings over S^1 is known to be finite.

Having recognized Theorem 2.4, the question arises: How far is B^n from being a classifying space for n-fold polynomial coverings? That is: For which spaces X is the map $[X, B^n] \to \mathrm{PC}_n(X)$ bijective?

Both $[X, B^n]$ and $\mathrm{PC}_n(X)$ have natural base points, so that $[X, B^n]$

$\to \mathrm{PC}_n(X)$ is a map between pointed sets. In Proposition 2.5 below we give a necessary and sufficient condition for the kernel to be trivial.

PROPOSITION 2.5. *The following two statements are equivalent:*

1) $H^1(X;\mathbf{Z})=0$.
2) *Any map* $\alpha\colon X \to B^n$, *which induces the trivial covering from* π^n, *is homotopic to a constant map.*

PROOF. For any two groups G and H, we let $\mathrm{Hom}\,(G,H)$ denote the set of homomorphisms of G into H, and $\mathrm{Hom}\,(G,H)^{\mathrm{conj}}$ the corresponding set of conjugacy classes. These sets are equiped with natural base points.

The epimorphism $\tau_n\colon B(n) \to \Sigma_n$ induces a map between pointed sets

$$\tau_{n*}\colon \mathrm{Hom}\,(\pi_1(X),B(n))^{\mathrm{conj}} \to \mathrm{Hom}\,(\pi_1(X),\Sigma_n)^{\mathrm{conj}}\,.$$

In more technical language statement 2) means that the kernel of τ_{n*} is trivial. This is the case if and only if $\mathrm{Hom}\,(\pi_1(X),\mathrm{kern}\,\tau_n)$ vanishes, and since kern τ_n has a normal series with free factors [2, p. 574], $\mathrm{Hom}\,(\pi_1(X),\mathrm{kern}\,\tau_n)=\{1\}$ if and only if $0=\mathrm{Hom}\,(\pi_1(X),\mathbf{Z})\cong H^1(X;\mathbf{Z})$.

Proposition 2.5 shows, that $H^1(X;\mathbf{Z})=0$ is a necessary condition for the n-fold polynomial coverings over X to be classified by B^n. On the other hand spaces with $H^1(X;\mathbf{Z})=0$ are apt to have few coverings—in particular few polynomial ones. This is expressed in

COROLLARY 2.6. *Assume that* $n\leqq 4$ *or that* $\pi_1(X)$ *is a finitely generated abelian group. Then the map* $[X,B^n] \to \mathrm{PC}_n(X)$ *is bijective if and only if all n-fold polynomial coverings over X are trivial.*

PROOF. If $H^1(X;\mathbf{Z})=0$ and $n\leqq 4$, then all n-fold polynomial coverings over X are trivial according to [2, Theorem 3.1]. If $H^1(X;\mathbf{Z})=0$ and $\pi_1(X)$ is a finitely generated abelian group, then $\pi_1(X)$ must be a torsion group. Since every element in $B(n)$ has infinite order, this implies that $\mathrm{Hom}\,(\pi_1(X),B(n))=\{1\}$.

REMARK 2.7. Questions similar to those discussed above arise when considering finite coverings over X that admit embeddings into the product bundle $\mathrm{pr}_1\colon X\times M \to X\colon (x,m) \to x$, where M is an arbitrary closed surface; cf. [5]. Without going into details, we mention that the analogues of Proposition 2.1, Remark 2.2, and Lemma 2.3 are valid. An analogue of Theorem 2.4 is valid at least when M is different from the 2-dimensional sphere S^2 and the real projective plane $\mathbf{R}\mathrm{P}^2$.

3. A criterion for certain orientation coverings to be polynomial.

In this section we use [3, Remark 7.5] to prove a necessary and sufficient condition for certain orientation coverings to be polynomial. Let M denote a connected nonorientable differentiable manifold with finitely generated homology groups $H_q(M; \mathbf{Z})$, $q=0,1,2$. The result is:

THEOREM 3.1. *Assume that* $H_1(M; \mathbf{Z})$ *contains no* 4*-torsion. Then the orientation covering* $\pi\colon \tilde{M} \to M$ *is polynomial if and only if the first Stiefel–Whitney class of* M *satisfies* $w_1(M)^2=0$.

PROOF. Let $\pi[\mathbf{R}]$ be the real line bundle associated with π when viewed as a principal $\mathbf{Z}/2$-bundle. As can be seen from the Gysin-sequence associated with π ([6, Corollary 12.3]), the Stiefel–Whitney classes $w_1(M)$ and $w_1(\pi[\mathbf{R}])$ are identical. The map between coefficient sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\cdot 2} & \mathbf{Z} & \longrightarrow & \mathbf{Z}/2 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbf{Z}/2 & \to & \mathbf{Z}/4 & \to & \mathbf{Z}/2 & \to & 0 \end{array}$$

induces a map between Bockstein sequences

$$\begin{array}{ccccccccccc} \longrightarrow H^1(M;\mathbf{Z}) & \longrightarrow & H^1(M;\mathbf{Z}/2) & \xrightarrow{\beta} & H^2(M;\mathbf{Z}) & \xrightarrow{\cdot 2} & H^2(M;\mathbf{Z}) & \xrightarrow{\lambda} & H^2(M;\mathbf{Z}/2) \to \\ \downarrow & & \downarrow & & {\scriptstyle\lambda}\downarrow & & \downarrow & & \downarrow \\ \to H^1(M;\mathbf{Z}/4) & \to & H^1(M;\mathbf{Z}/2) & \xrightarrow{\mathrm{Sq}^1} & H^2(M;\mathbf{Z}/2) & \to & H^2(M;\mathbf{Z}/4) & \to & H^2(M;\mathbf{Z}/2) \to \end{array}$$

where $\beta\colon H^1(M; \mathbf{Z}/2) \to H^2(M; \mathbf{Z})$ is a Bockstein homomorphism, and $\lambda\colon H^2(M; \mathbf{Z}) \to H^2(M; \mathbf{Z}/2)$ is reduction modulo 2 of the integral cohomology classes. The Bockstein homomorphism in the lower sequence is $\mathrm{Sq}^1\colon H^1(M; \mathbf{Z}/2) \to H^2(M; \mathbf{Z}/2)$: $x \to x^2$, and since $H^2(M; \mathbf{Z})$ contains no 4-torsion, we have $\operatorname{kern} \mathrm{Sq}^1 = \operatorname{kern} \beta$. Now [3, Remark 7.5], which shows that π is polynomial if and only if $\beta(w_1(\pi[\mathbf{R}]))=0$, finishes the proof.

In particular we get

EXAMPLE 3.2. (Compare with Section 6 in [3].) Let U_{g+1} denote the nonorientable closed surface of genus $g+1 \geqq 1$. The second Stiefel–Whitney class of U_{g+1}, $w_2(U_{g+1})=w_1(U_{g+1})^2$, is related to the fundamental class $[U_{g+1}] \in H_2(U_{g+1}; \mathbf{Z}/2)$ by the formula

$$\langle w_2(U_{g+1}), [U_{g+1}]\rangle = \chi(U_{g+1}) \bmod 2$$

where $\langle \cdot, \cdot \rangle$ is the Kronecker product, and $\chi(U_{g+1})$ is the Euler number of U_{g+1} ([6, Corollary 11.12]). Since $[U_{g+1}]$ generates $H_2(U_{g+1}; \mathbf{Z}/2)$ and $\chi(U_{g+1})=2-(g+1)$, we conclude that $w_2(U_{g+1})=0$ if and only if $2-(g+1)\equiv 0 \bmod 2$, i.e. g is odd.

The orientation covering of U_{g+1} is equivalent to the standard double covering $p_g\colon T_g \to U_{g+1}$ of U_{g+1} by the orientable closed surface T_g of genus $g \geqq 0$. Thus we have given an alternative proof of [3, Theorem 6.1].

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