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A NOTE ON THE CHARACTERIZATION OF STANDARD BOREL SPACES

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A measurable space (Borel space) is said to be a standard Borel space if its σ-algebra is σ-isomorphic to the usual σ-algebra in a Polish space [2]. We shall give below four equivalent characterizations of standard Borel spaces. The results are of some theoretical value, and seem not to be well known.

The following lemma is needed but will not be proved.

LEMMA. *Let $\mathscr{A}$ denote a countable algebra over a nonempty set X. Then the following are equivalent*:

a) *Every $\mathscr{A}$-measurable partition of X is finite.* ($\mathscr{D} \subseteq \mathscr{A}$ is a $\mathscr{A}$-measurable partition of X iff $X = \bigcup \mathscr{D}$, $\emptyset \notin \mathscr{D}$, and the sets in $\mathscr{D}$ are pairwise disjoint);

b) *Every covering $\mathscr{C} \subseteq \mathscr{A}$ of $A \in \mathscr{A}$ contains a finite subcovering*;

c) *$\mathscr{A}$ is semicompact.* ($\mathscr{A}$ is semicompact iff every sequence in $\mathscr{A}$ with the finite intersection property has a nonempty intersection);

d) *Every finite and finitely additive measure on $\mathscr{A}$ is σ-additive on $\mathscr{A}$.*

THEOREM. *Let $(X,\mathscr{F})$ denote a measurable space. Then $(X,\mathscr{F})$ is standard iff there exist a countable algebra $\mathscr{A}$ generating $\mathscr{F}$ and with one of the properties* a, b, c, *or* d *stated in the lemma.*

PROOF. "Only if". Note that if $\mathscr{F}$ contains only countably many atoms then $(X,\mathscr{F})$ is certainly standard.

Case 1. $\mathscr{F}$ contains only countably many atoms e_n, $n=1,2,3,\ldots$. Then the algebra generated by the sets $e_1 \cup \bigcup_{k \geqq n} e_k$, $n=1,2,3,\ldots$ serves our purpose.

Case 2. $\mathscr{F}$ contains uncountably many atoms. From the isomorphism theorem [2, p. 14, see also p. 147] it follows that the Borel-algebra over a complete separable metric space with uncountably many atoms is σ-isomorphic to the σ-algebra over M, where M is the space of binary sequences with the natural product topology. The algebra generated by the sets $\{(x_1,x_2,..) \in M : x_i=1\}$, $i \in \mathbf{N}$, satisfies the requirements.

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"If". Since $\mathscr{F}$ contains its atoms we shall work with the canonical representation of $(X, \mathscr{F})$ denoted by $(\tilde{X}, \tilde{\mathscr{F}})$ in the following [2, p. 133]. Suppose $\tilde{\mathscr{F}}$ is generated by a countable algebra $\tilde{\mathscr{A}}$ with the finite partition property. Let $\mathscr{T}$ denote the topology having $\tilde{\mathscr{A}}$ as a base. It is clear that $(\tilde{X}, \mathscr{T})$ is a (totally disconnected) compact Hausdorff space and $\tilde{\mathscr{F}} = \sigma(\mathscr{T})$. Hence $(\tilde{X}, \mathscr{T})$ is metrizàble as a complete metric space.

REFERENCES

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