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ALGEBRAS OF SPHERICAL FUNCTIONS ASSOCIATED WITH COVARIANT SYSTEMS OVER A COMPACT GROUP

MAGNUS B. LANDSTAD

Abstract.

If M is a C*-algebra, K a compact group and $\varrho\colon K \to \operatorname{Aut}(M)$ a homomorphism, one can form the covariance algebra $K \times_\varrho M$. In this note we show that the classification of the factor representations of $K \times_\varrho M$ (in particular the irreducible ones) can be reduced to the classification of all factor (or irreducible) representations of the algebras $M \otimes B(X(D))^{\varrho \otimes \operatorname{ad} D}$ $(D \in \hat{K})$ which can be considered as generalizations of the algebras of spherical functions defined by R. Godement for groups with a large compact subgroup. As a corollary we get that if K is abelian then $K \times_\varrho M$ is liminal or postliminal if and only if M^ϱ has the same property.

1. Introduction.

One of the main objects of any representation theory is to classify the most fundamental building blocks, for instance to find all irreducible or factor representations of a *-algebra or a group. In the theory of representations of groups this classification often can be done by a reduction process. For instance if a locally compact group G has a closed normal subgroup N the classification of the irreducible unitary representations of G (this set is called $\hat{G}$) can be reduced to the classification of $\hat{N}$ and certain projective representations of some subgroups of G/N, c.f. [11]. This theory has later been extended to more general systems, c.f. [5], [11], [13] and [16].

If the group G contains a "large" compact subgroup K, there is another method of classifying $\hat{G}$ first studied by R. Godement in [6]. Here the classification of $\hat{G}$ is reduced to that of determining $\hat{K}$ and to find all irreducible representations of certain algebras of "spherical functions". In this article we shall try to extend this method to covariant systems over a compact group.

Let M be a C*-algebra, K a compact group and ϱ a homomorphism of K into the group Aut (M) of *-automorphisms of M such that the map $k \to \varrho_k(a)$

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is norm-continuous for all $a \in M$. Let $L^1(K, M)$ be all integrable functions from K to M. With the following definitions $L^1(K,M)$ is a Banach*-algebra:

$$fg(x) = \int_K f(y)\varrho_y(g(y^{-1}x))\,dy$$
$$f^*(x) = \varrho_x(f(x^{-1}))^*$$
$$\|f\| = \int_K \|f(x)\|\,dx\,.$$

Let $\mathfrak{A}$ be the enveloping C*-algebra of $L^1(K,M)$ (c.f. [3, Chap. 2.7]). $\mathfrak{A}$ is called the *covariance algebra* of the *covariant system* (M,ϱ,K) and is also denoted $K \times_\varrho M$. These concepts were introduced by S. Doplicher, D. Kastler and D. Robinson in [4] and they showed that there is a one-to-one correspondance between non-degenerate *-representations T of $\mathfrak{A}$ and pairs (U,π) with U a continuous unitary representation of K, π a non-degenerate *-representation of M such that π and U acts on the same Hilbert space and

(1) $$U_k\pi(a)U_{k^{-1}} = \pi(\varrho_k(a)) \quad \text{for } k \in K,\ a \in M\,.$$

In fact if (U,π) is given, then T is defined by

$$T_f = \int \pi(f(k))U_k\,dk \quad \text{for } f \in L^1(K,M)\,.$$

The starting point in the Mackey–Takesaki theory is now to study the representation π of M. π is not necessarily irreducible even if T is and one looks at π's decomposition into irreducible representations. (This may not be possible so one has to make certain assumptions about M). The group K acts naturally on the set $\hat{M}$ of irreducible representations of M, and if this action is "nice" it is possible to describe $\hat{\mathfrak{A}}$ by $\hat{M}$ and certain projective representations of some subgroups of K.

We are instead going to look at the decomposition of U into irreducible representations of K, and we shall classify all pairs (π, U) as above having a given $D \in \hat{K}$ as a subrepresentation of U.

For $D \in \hat{K}$, let ψ_D be its character, i.e.

$$\psi_D(k) = \dim(D)\operatorname{tr}(D_{k^{-1}}) \quad \text{for } k \in K\,.$$

$L^1(K)$ can be embedded in $M(L^1(K,M))$ = the algebra of multipliers of $L^1(K,M)$ by defining

$$\varphi f(x) = \int_K \varphi(k)\varrho_k(f(k^{-1}x))\,dk$$
$$f\varphi(x) = \int_K \varphi(k)f(xk^{-1})\,dk, \quad \text{for } \varphi \in L^1(K),\ f \in L^1(K,M),\ x \in K\,.$$

K can also be embedded in $M(L^1(K,M))$ by

$$kf(x) = \varrho_k(f(k^{-1}x))$$
$$fk(x) = f(xk^{-1}) \quad \text{for } f \in L^1(K,M),\ k,\ x \in K\,.$$

Look at the two sided ideal

$$A(D) = \text{closed lin span}\{f\psi_D g \mid f,g \in L^1(K,M)\}$$

of $L^1(K,M)$. The importance of this ideal is seen from

LEMMA 1. *If T is a non-degenerate *-representation of $L^1(K,M)$ and U and π are the representations of K and M such that* (1) *holds, then a given $D \in K$ occurs in U if and only if the restriction of T to the two-sided ideal $A(D)$ is non-zero.*

PROOF. Let $P_D = \int_K \psi_D(k) U_k\, dk$, so P_D is a projection. Now D occurs in U if and only if $P_D \neq 0$. Since T is non-degenerate $P_D \neq 0$ if and only if there are f and g in $L^1(K,M)$ with $T_g P_D T_f \neq 0$. Using that $P_D T_f = T_{\psi_D f}$ it follows that $P_D \neq 0$ if and only if $T(A(D)) \neq 0$.

We can now describe the classification of all factor representations T of $L^1(K,M)$ more accurately. It is proved in [13] (Example 6.7 and 6.8) that $A(D)$ and $A_1(D) = \psi_D L^1(K,M)\psi_D$ are strongly Morita equivalent in the terminology of [14]. Since $B(X(D)) \cong \psi_D L^1(K)\psi_D$ is contained in the multiplier algebra of $A_1(D)$, it follows that $A_1(D) \cong B(X(D)) \otimes L_D^0(K,M)$. Here $L_D^0(K,M)$ is the commutant of $\psi_D L^1(K)\psi_D$ in $A_1(D)$, that is, $L_D^0(K,M)$ consists of all functions f in $L^1(K,M)$ satisfying (2) and (3):

$$\varrho_k(f(k^{-1}xk)) = f(x) \quad \text{for almost all } k, x \in K\,, \tag{2}$$

$$\psi_D f = f\,. \tag{3}$$

So $A(D)$ and $L_D^0(K,M)$ are strongly Morita equivalent. (We have here used the following: If A is an algebra and its multiplier algebra $M(A)$ contains a subalgebra B isomorphic to a full matrixalgebra, then $A \cong B \otimes C$ where $C = A \cap B'$.)

The algebra $L_D^0(K,M)$ is the analogue of the algebra $L^0(d)$ considered in § 10 of [6], justifying the title of this article.

We shall also prove that $L_D^0(K,M)$ is isomorphic to the algebra

$$B(D) = \{a \in M \otimes B(X(\bar{D})) \mid (\varrho_x \otimes \operatorname{ad} \bar{D}_x)(a) = a \text{ for all } x \in K\}\,.$$

($\bar{D}$ is the conjugate representation of D.)

So the classification of irreducible representations of the covariant system (M,ϱ,K) has been reduced to the classification of $\hat{K}$ and the irreducible representations of $B(D)$ for different $D \in \hat{K}$. If K in particular is abelian, this classification becomes very simple, then $B(D) \cong M^{\varrho}$. So the representation theory of (M,ϱ,K) is determined by the fixpoint algebra of M, and by $\hat{K}$.

This article is a revised and updated version of [10] which originated during the author's stay at the University of Pennsylvania, and I want to thank Professor J. M. G. Fell for introducing imprimitivity bimodules to me. I also want to acknowledge partial support from NAVF, Norway.

2. The imprimitivity bimodule for a covariant system over a compact group.

We shall keep all definitions made in the introduction. In addition we make the following conventions: If S is a representation, we let $X(S)$ denote the corresponding Hilbert space. If X is a Hilbert space, $B(X)$ is the algebra of all bounded operators on X and $CC(X)$ is the subalgebra of all compact operators. For a covariant system (M,ϱ,K), M^{ϱ} denotes its fixpoint-algebra, i.e.

$$M^{\varrho} = \{a \in M \mid \varrho_x(a) = a \text{ for all } x \in K\} .$$

If D is a continuous representation of K, $\operatorname{ad} D$ is the map from K to $\operatorname{Aut} B(X(D))$ defined by

$$\operatorname{ad} D_x(R) = D_x R D_{x^{-1}} \qquad \text{for } x \in K,\ R \in B(X(D)) .$$

Lemma 1 showed the importance of the ideal $A(D)$. An application of the next lemma will give a more detailed description of the relation between the representations of $L^1(K,M)$ and $A(D)$. For C*-algebras this is essentially Lemma 2.10.3 and Proposition 2.10.4 in [3].

LEMMA 2. *Let A be a Banach *-algebra, J a closed self-adjoint two-sided ideal in A. Then there is a bijective correspondance between non-degenerate factor representations S of J and non-degenerate factor representations T of A with $T(J) \neq \{0\}$. If T is given, S is the restriction of T to J. If S is given, T is defined by*

$$T_a(S_b\xi) = S_{ab}\xi \qquad \text{for } a \in A,\ b \in J,\ \xi \in X(S) . \tag{4}$$

Furthermore, $T(A)'' = S(J)''$, so S and T are of the same type, and S is irreducible if and only if T is.

PROOF. If T is given, let S be the restriction of T to J. Then the closure $S(J)''$ is a two-sided ideal in $T(A)''$, so $S(J)'' = T(A)''$, S is non-degenerate, and S is also a factor representation.

Conversely, suppose S is a non-degenerate *-representation of J (not necessarily a factor representation), and without loss of generality we may assume that S has a cyclic vector ξ_0. On the dense subspace $X_0 = \{S_b\xi_0 \mid b \in J\}$ we define T by

$$T_a(S_b\xi_0) = S_{ab}\xi_0 \quad \text{for } a \in A,\ b \in J\ .$$

Now,

$$\begin{aligned}\|S_{ab}\xi_0\|^2 &= (S_{a^*ab}\xi_0, S_b\xi_0) \leqq \|S_b\xi_0\|\ \|S_{a^*ab}\xi_0\| \\ &\leqq \dots \leqq \|S_b\xi_0\|^{2-2^{-n}} \|S_{(a^*a)^{2^n}b}\xi_0\|^{2^{-n}} \\ &\leqq \|S_b\xi_0\|^{2-2^{-n}} \|a^*a\| (\|b\|\ \|\xi_0\|)^{2^{-n}} \quad \text{for } n=1,2,\dots,\end{aligned}$$

so

$$\|S_{ab}\xi_0\| \leqq \|a\|\ \|S_b\xi_0\| \quad \text{for all } a \in A,\ b \in J\ .$$

Hence T_a is well defined over X_0 and extends to a bounded operator over $X(S)$. T will obviously be a non-degenerate *-representation of A and (4) will hold for all $a \in A$, $b \in J$, $\xi \in X(S)$. Obviously $S(J)'' \subset T(A)''$. Conversely if $R \in S(J)'$, $a \in A$, $b \in J$ then

$$RT_aS_b\xi_0 = RS_{ab}\xi_0 = S_{ab}R\xi_0 = T_aS_bR\xi_0 = T_aRS_b\xi_0\ .$$

Thus $RT_a = T_aR$ for all $a \in A$, so $R \in T(A)'$. Hence $T(A)'' = S(J)''$, and in particular S is a factor representation if T is. To conclude, it is easy to check that the maps $S \to T$ and $T \to S$ are each others inverses.

Remark. We have in fact proved that any non-degenerate *-representation S of J can uniquely be extended to a non-degenerate *-representation T of A with $T(A)'' = S(J)''$. The converse it not true in general, i.e. if T is not a factor representation of A, then we may have $T(J) \neq \{0\}$ but the restriction of T to J may be degenerate and $S(J)'' \neq T(A)''$.

In the introduction we indicated that $A(D)$ is strongly Morita equivalent to the subalgebra $L_D^0(K,M)$ of $L^1(K,M)$ defined by (2) and (3). Before showing this, we shall give an equivalent description of $L_D^0(K,M)$.

Suppose $D \in K$ and let $\bar{D}$ be the conjugate representation of D, i.e. $X(\bar{D}) = \overline{X(D)}$ = the conjugate Hilbertspace of $X(D)$, and if J is the natural conjugate linear map from $X(D)$ to $\overline{X(D)}$ then

$$\bar{D}_k = j^*D_kj \quad \text{for } k \in K\ .$$

Lemma 3. *Define a linear map* $\Phi\colon L^1(K,M) \to M \otimes B(X(\bar{D}))$ *by*

$$\Phi(f) = \int_K f(k^{-1}) \otimes \bar{D}_k\, dk$$

*then Φ is a *-isomorphism of $L_D^0(K,M)$ onto*

$$B(D) = M\otimes B(X(\bar{D}))^{\varrho\otimes \operatorname{ad}\bar{D}}$$

PROOF. Note that Φ is not a *-homomorphism of $L^1(K,M)$, but using that elements of $L_D^0(K,M)$ satisfy (2) and (3), straight forward calculations show that Φ is a *-homomorphism of $L_D^0(K,M)$ into $B(D)$.

Let $Q\colon M\otimes B(X(\bar{D})) \to M$ be defined by

$$Q(s\otimes a) = \dim(D)\operatorname{tr}(a)s \quad \text{for } s\in M,\ a\in B(X(\bar{D}))\ .$$

Then define $\Psi\colon M\otimes B(X(\bar{D})) \to L^1(K,M)$ by

$$\Psi(b)(x) = Q(b(I\otimes \bar{D}_x)) \quad \text{for } b\in M\otimes B(X(\bar{D})),\ x\in K\ .$$

It is now not difficult to see that Ψ maps $B(D)$ into $L_D^0(K,M)$ and that $\Psi\mid B(D)$ and $\Phi\mid L_D^0(K,M)$ are each others inverses.

In the introduction it was shown that $A(D)$ and $B(D)$ are Morita equivalent, both being equivalent to the algebra $A_1(D)$. How to find an imprimitivity bimodule connecting $A(D)$ and $A_1(D)$ is described in [5, Prop. 9.2] and [13, Example 6.7]. Using that $A_1(D)\cong B(X(D))\otimes L_D^0(K,M)$ and a procedure similar to the one given on [5, p. 90] one can get an imprimitivity bimodule connecting $A_1(D)$ and $L_D^0(K,M)$. However, we shall directly give an imprimitivity bimodule L between $A(D)$ and $B(D)$, i.e. $A(D)$ acts to the left on L and $B(D)$ acts to the right on L such that the axioms in [5, § 7] are satisfied.

Having this in mind, the following definitions should be rather natural. $L=M\otimes X(D)$ and we define actions of $L^1(K,M)$ and $M\otimes B(X(\bar{D}))$ on L by

$$(5)\qquad f(a\otimes\xi) = \int f(x)\varrho_x(a)\otimes D_x\xi\,dx \qquad \text{for } f\in L^1(K,M),\ a\in M,\ \xi\in X(D)\ ,$$

$$(6)\qquad (a\otimes\xi)(b\otimes s) = ab\otimes j^*s^*j\xi \qquad \text{for } a,b\in M,\ s\in B(X(\bar{D})),\ \xi\in X(D)\ .$$

Straight forward computations show that this really defines actions and that $f(rb)=(fr)b$ for $f\in L^1(K,M)$, $r\in L$ and $b\in B(D)$, but not for all $b\in M\otimes B(X(\bar{D}))$.

If ξ and η are vectors in a Hilbert space H, let $E(\xi,\eta)$ be the rank one operator in $B(H)$ defined by

$$E(\xi,\eta)\zeta = \langle\zeta,\eta\rangle\xi\ .$$

The $A(D)$-rigging and $B(D)$-rigging are now defined by

$$[a\otimes\xi, b\otimes\eta]_A(x) = \langle\xi, D_x\eta\rangle a\varrho_x(b^*)$$

$$[a\otimes\xi, b\otimes\eta]_B = \int \varrho_x(a^*b)\otimes\bar{D}_x E(j\xi, j\eta)\bar{D}_{x^{-1}}\,dx$$

for $a, b \in M$ and $\xi, \eta \in X(D)$.

Obviously $[r,s]_B \in B(D)$ for $r,s \in L$. To see that $[r,s]_A \in A(D)$ note that if $\{\xi_i\}$ is an orthonormal basis in $X(D)$,

(7) $$[a_i\otimes\xi_i, a_j\otimes\xi_j]_A(x)$$
$$= \dim D \int \langle\xi_i, D_y\xi_1\rangle a_i\varrho_x(\langle\xi_j, D_{x^{-1}y}\xi_1\rangle a_j)^*\,dy = f_i f_j^*(x)$$

where

(8) $$f_i(x) = \dim D^{\frac{1}{2}}\langle\xi_i, D_y\xi_1\rangle a_i\,.$$

Since $f_i\psi_D = f_i$, it follows that $[r,s]_A \in A(D)$ for all $r,s \in L$.

It is more or less straightforward now to check the formulas (1)-(5) on page 72 of [5]. As an example we take (5):

$$[a\otimes\xi, b\otimes\eta]_A(c\otimes\zeta) = \int \langle\xi, D_x\eta\rangle a\varrho_x(b^*)\varrho_x(c)\otimes D_x\zeta\,dx$$
$$= \int a\varrho_x(b^*c)\otimes j^*\bar{D}_x E(j\zeta, j\eta)\bar{D}_{x^{-1}}j\xi\,dx$$
$$= (a\otimes\xi)(\varrho_x(b^*c)\otimes\bar{D}_x E(j\eta, j\zeta)\bar{D}_{x^{-1}})\,dx$$
$$= (a\otimes\xi)[b\otimes\eta, c\otimes\zeta]_B\,.$$

So $(L, [\cdot,\cdot]_A, [\cdot,\cdot]_B)$ is an imprimitivity bimodule. Since

$$\operatorname{lin\,span}\{a^*b\otimes E(j\xi, j\eta) \mid a,b \in M,\ \xi,\ \eta \in X(D)\}$$

is norm dense in $M\otimes B(X(\bar{D}))$ it should be obvious that $\operatorname{lin\,span}\{[r,s]_B \mid r,s \in L\}$ is norm dense in $B(D)$.

If $f \in L^1(K)$, $a \in M$ define $f\otimes a \in L^1(K, M)$ by

$$(f\otimes a)(x) = f(x)a\,.$$

Then with ξ_i as above

$$(f\otimes a)\Psi_D(g\otimes b)^* = \dim D \sum_i [a\otimes D_f\xi_i, b\otimes D_g\xi_i]_A\,,$$

where

$$D_f = \int f(x)D_x\,dx\,.$$

From this it follows that lin span $\{[r,s]_A \mid r,s \in L\}$ is norm dense in $A(D)$, so our imprimitivity bimodule is *topologically strict* as defined in [5, § 9].

LEMMA 4. *Every *-representation of $A(D)$ (respectively $B(D)$) is L-positive.* (Cf. [5, § 4]).

PROOF. Let $\{\xi_i\}$ be an orthonormal basis of $X(D)$ as before, and let $r = \sum a_i \otimes \xi_i \in L$ with $a_i \in M$. Then

$$[r,r]_B = \int \sum_{i,k} \varrho_x(a_i{}^* a_k) \otimes \bar{D}_x E(j\xi_i, j\xi_k) \bar{D}_{x^{-1}}\, dx \ ,$$

so $[r,r]_B \geqq 0$ as an element of the C*-algebra $B(D)$. Take f_i as in (8) and $f = \sum f_i$. Then (7) gives that $[r,r]_A = ff^*$. Hence $S_{[r,r]_B} \geqq 0$ and $T_{[r,r]_A} \geqq 0$ for any *-representations S of $B(D)$ and T of $A(D)$. It should now be clear that $(L, [\,,]_A, [\,,]_B)$ satisfies the assumptions of Theorem 9.1 in [5], and combining this with Lemmas 2 and 4 we have

THEOREM 1. *For $D \in \hat{K}$, there is a bijective correspondance between (equivalence classes of) factor representations S of $B(D) = M \otimes B(X(\bar{D}))^{\varrho \otimes \mathrm{ad}\, \bar{D}}$ and factor representations T of $K \times_\varrho M$ such that D occurs in the restriction of T to K. If S and T correspond*

$$S(B(D))' \cong T(K \times_\varrho M)'$$

so S and T will be factor representations of the same type, and S is irreducible if and only if T is.

REMARK. The correspondance is given by first inducing S from $B(D)$ to a representation T of $A(D)$ as on page 51 of [5], then lifting T from $A(D)$ to $L^1(K,M)$ and $K \times_\varrho M$ by Lemma 2. This can also be considered as inducing S directly from $B(D)$ to $L^1(K,M)$, since L also is a left $L^1(K,M)$-module.

The next two corollaries show that taking the crossed product with a compact group will always give a "nicer" algebra, contrary to what happens with discrete groups.

COROLLARY 1. *$K \times_\varrho M$ is postliminal if and only if $M \otimes B(X(D))^{\varrho \otimes \mathrm{ad}\, D} = B(\bar{D})$ is postliminal for all $D \in \hat{K}$. In particular this is the case if M itself is postliminal.*

PROOF. The first part follows from the fact that a C*-algebra is postliminal if and only if all its factor representations are of type I. If M is postliminal, then so is $M \otimes B(X(D))$ and its C*-subalgebra $B(\bar{D})$.

COROLLARY 2. *$K \times_\varrho M$ is liminal if and only if $M \otimes B(X(D))^{\varrho \otimes \mathrm{ad}\, D} = B(\bar{D})$ is liminal for all $D \in \hat{K}$. In particular this is the case if M itself is liminal.*

PROOF. From Proposition 8.6 of [5] it follows that $A(D)$ is liminal if and only if $B(D)$ is liminal. If $K \times_\varrho M$ is liminal then so is $A(D)$. Conversely, suppose $A(D)$ is liminal for all $D \in K$ and that T is an irreducible *-representation of $K \times_\varrho M$ and thus also of $L^1(K, M)$. We want to prove that T_f is compact for all $f \in L^1(K, M)$. Let T^D be the restriction of T to $A(D)$, then T^D is irreducible for each D occurring in the restriction of T to K (and $T^D = 0$ otherwise). Now $f = \sum_{D \in \hat{K}} f\psi_D$ (convergence in norm) for each $f \in L^1(K, M)$, so to prove that T_f is compact it suffices to prove that $T_{f\psi_D}$ is compact for all $D \in \hat{K}$.. Now $f^*\psi_D f \in A(D)$ so $(T_{f\psi_D})^* T_{f\psi_D} = T^D_{f^*\psi_D f}$ is compact, thus $T_{f\psi_D}$ is compact for all $D \in \hat{K}$, and we have proved that $K \times_\varrho M$ is liminal.

If M itself is liminal, then obviously $M \otimes B(X(D))$ together with its C*-subalgebra $B(\bar{D})$ also will be liminal.

The next result is a slight generalization of a result in [15].

COROLLARY 3. *If $K \times_\varrho M$ is simple, then so are all $M \otimes B(X(D))^{\varrho \otimes \mathrm{ad}\, D}$. In particular M^ϱ is simple.*

PROOF. If $K \times_\varrho M$ is simple, then $B(D)$ also must be simple being Morita equivalent to the ideal $A(D)^-$ in $K \times_\varrho M$.

REMARK. It is not true that if $B(D)$ is simple for all $D \in \hat{K}$ then $K \times_\varrho M$ is simple. A counterexample is obtained by taking $\varrho = i$ and M simple.

If K is abelian $B(D) = M^\varrho$, so we have:

COROLLARY 4. *If K is a compact abelian group, $K \times_\varrho M$ is liminal (postliminal) if and only if M^ϱ is liminal (postliminal).*

Our algebras $B(D)$ should be considered as analogues of the algebras $L^0(d)$ of spherical functions defined in [6]. The following should therefore come as no surprise, c.f. Corollary on p. 522 of [6].

PROPOSITION 1. *If $\dim S \leqq n$ for every irreducible representation S of $B(D) = M \otimes B(X(\bar{D}))^{\varrho \otimes \mathrm{ad}\, \bar{D}}$ and T is an irreducible *-representation of $K \times_\varrho M$, then D occurs at most n times in the restriction of T to K.*

PROOF. Let T be an irreducible *-representation of $K \times_\varrho M$ and let X_0 be a non-zero subspace of $X(T)$ invariant under the restriction of T to K and such that

$$\int \psi_D(x) T_x \alpha \, dx = \alpha \quad \text{for all } \alpha \in X_0 .$$

Then the induced representation S of $B(D)$ corresponding to T in Theorem 1 is first defined on $\bar{L} \otimes X(T)$ ($\bar{L}$ = conjugate space of L, c.f. [5, page 73]) with inner product

$$\begin{aligned} (9) \qquad \langle a \otimes \xi \otimes \alpha, b \otimes \eta \otimes \beta \rangle &= \langle T_{[b \otimes \eta, a \otimes \xi]_A} \alpha, \beta \rangle \\ &= \int \langle \eta, D_x \xi \rangle b \varrho_x(a^*) \langle T_x \alpha, \beta \rangle \, dx \\ &\quad \text{for } a, b \in M, \xi, \eta \in X(D), \alpha, \beta \in X(T) . \end{aligned}$$

Take $X(S)$ to be the separated completion of $\bar{L} \otimes X$ with this inner product. S is then defined by

$$S_b(a \otimes \xi \otimes \alpha) = (a \otimes \xi) b^* \otimes \alpha \quad \text{for } b \in B(D) ,$$

for details see §§ 4 and 7 of [5].

Let Y_0 be the closed linear span in $X(S)$ of $\{r \otimes \alpha \mid r \in \bar{L}, \alpha \in X_0\}$. Then $Y_0 \neq \{0\}$, because if the expression (9) always is 0 we will have

$$\int \langle \eta, D_x \xi \rangle \langle T_x \alpha, \beta \rangle \, dx = 0 ,$$

so

$$\langle \alpha, \beta \rangle = \int \psi_D(x) \langle T_x \alpha, \beta \rangle \, dx = 0 \quad \text{for all } \alpha, \beta \in X_0 ,$$

a contradiction.

Furthermore, if $X_1 \perp X_0$ is another subspace of $X(T)$ with the same properties as X_0, define Y_1 to be the corresponding subspace of $X(S)$. (9) then shows that $Y_0 \perp Y_1$, so D can occur at most $\dim X(S)$ times in the restriction of T to K.

COROLLARY. *If $M \otimes B(X(\bar{D}))^{\varrho \otimes \operatorname{ad} \bar{D}}$ is abelian, then D occurs at most once in the restriction to K of any irreducible representation of $K \times_\varrho M$.*

We next give an application to W*-crossed products of compact groups:

PROPOSITION 2. *Suppose (M, ϱ, K) is a covariant system with M a von Neumann algebra and ϱ σ-weakly continuous. If each $D \in \hat{K}$ occurs only finitely* *many times in* ϱ, *then* $K \times_\varrho M$ (W*-*crossed product*) *is a type* I *von Neumann algebra.*

PROOF. Suppose $M \subset B(H)$ and let

$$M_0 = \{a \in M \mid x \to \varrho_x(a) \text{ is norm continuous}\} .$$

This is a σ-weakly dense C*-subalgebra of M (c.f. [1, Theorem 9]), so (M_0, ϱ, K) is a C*-covariant system. For $D, E \in \hat{G}$ let

$$M(E) = \left\{a \in M \,\middle|\, \int \psi_E(x)\varrho_x(a)\,dx = a\right\}$$

and

$$B(X(D), E) = \left\{b \in B(X(D)) \,\middle|\, \int \psi_E(x) D_x b D_{x^{-1}}\,dx = b\right\} .$$

$M(E)$ is by assumption a finite dimensional subspace of M, and $M(E) \subset M_0$. For a given D, $B(X(D), E)$ is non-zero for only finitely many E in $\hat{K}$. It then follows from the theory of tensor-product representations of compact groups (c.f. [7, 27.34e]) that

$$B(\bar{D}) = M_0 \otimes B(X(D))^{\varrho \otimes \operatorname{ad} D} \subset \sum_{E \in \hat{G}} M(\bar{E}) \otimes B(X(D), E) .$$

So $B(\bar{D})$ is finite dimensional for all $D \in \hat{K}$, thus $K \times_\varrho M_0$ is a type I C*-algebra. Since $K \times_\varrho M$ is the σ-weak closure of $K \times_\varrho M_0$ over $L^2(K, H)$, $K \times_\varrho M$ is type I.

COROLLARY. *If* (M, ϱ, K) *is as in Proposition* 2 *and* ϱ *is ergodic* (*i.e.* $M^\varrho = \mathbf{C}I$), *then* $K \times_\varrho M$ *is a type* I *von Neumann algebra.*

PROOF. It is proved in [8] that ergodicity implies finite multiplicity, so the result follows from Proposition 2.

REMARK. I have been informed that this fact has also been observed independently by M. Takesaki.

EXAMPLE 1. M. Takesaki showed in [17] that if A is a uniformly hyperfinite C*-algebra there is a compact abelian group K and ϱ: $K \to \operatorname{Aut} A$ such that $K \times_\varrho A$ is liminal. It is not difficult to show that in this case A^ϱ is abelian, so Corollary 4 gives a different proof that $K \times_\varrho A$ is liminal.

EXAMPLE 2. Let H be Mautner's 5-dimensional non-type I solvable Lie group (for details, see [16, Appendix].) Then one can form a semi-direct product G of H and the circle group T exactly as in [16, § 9], except that we use

T instead of the real numbers. The action of T on H induces an action ϱ on $C^*(H)$ such that

$$C^*(G) \cong T \times_\varrho C^*(H) .$$

Also in this case one can show that $C^*(H)^\varrho$ is liminal so by Corollary 4 $C^*(G)$ (thus G) is liminal.

FINAL REMARKS. If T is a factor representation of $K \times_\varrho M$, usually more than one D in $\hat{K}$ will occur in the restriction of T to K. Therefore the description of all factor representations T of $K \times_\varrho M$ is repetitious, and in general we have no way of telling when a representation induced from $B(D)$ is equivalent to one induced from $B(E)$ for two inequivalent representations D and E of K. So if M is liminal, the Mackey–Takesaki method in [16] is superior to the one given here, since they describe the factor representations of $K \times_\varrho M$ by a process producing each representation only once. But if M is not liminal, our method can give much information of the representation theory of $K \times_\varrho M$ even if the representationtheory of M is very inaccessible, c.f. the two examples above.

Corollary 4 tells us that there is a close correspondance between the representation theories of $K \times_\varrho M$ and M^ϱ when K is abelian. They are however not Morita equivalent (e.g. take $M = \mathbf{C}$), but A. Kishimoto and H. Takai has shown in [9] that under certain assumptions $K \times_\varrho M \cong M^\varrho \otimes CC(L^2(K))$ so $K \times_\varrho M$ and M^ϱ are Morita equivalent.

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MATEMATISK INSTITUTT
NORGES LÆRERHØGSKOLE
UNIVERSITETET I TRONDHEIM
7055 DRAGVOLL
NORWAY