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COHOMOLOGY OF SOME NON-SELFADJOINT OPERATOR ALGEBRAS

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In this paper we prove that a certain class of non-selfadjoint operator algebras have isomorphic continuous and normal cohomology for dual normal coefficient modules. We use this result to show that nest algebras have trivial cohomology groups for some dual normal coefficient modules.

The cohomological notation will be similar to that of [7]. The bar $-$ will always denote ultraweak closure. Throughout the paper H is a fixed Hilbert space, $B(H)$ the algebra of all bounded operators on H, and I will denote the identity operator on H. If $\mathcal{N} \subseteqq B(H)$ is an operator algebra $\mathcal{N}_{\mathcal{K}}$ will denote the algebra of compact operators in $\mathcal{N}$. If $\mathcal{N}$ contains I and $\mathcal{N}_{\mathcal{K}}^{-} = \mathcal{N}$ then we will call $\mathcal{N}$ a hypercompact algebra. (We see that $\mathcal{N}$ is hypercompact if and only if $\mathcal{N}$ is ultraweakly closed and $I \in \mathcal{N}_{\mathcal{K}}^{-}$.) In the following $\mathcal{N}$ will be a fixed hypercompact algebra of operators on H. If $\xi, \eta \in H$, $\xi \otimes \eta$ will denote the rank one operator in $B(H)$ defined by $\xi \otimes \eta(\gamma) = (\gamma \mid \xi)\eta$, $\gamma \in H$.

1.

In this section we prove that if ϱ is a n-cocycle from $\mathcal{N}$ into any dual normal $\mathcal{N}$-module then ϱ is cohomologous to a normal n-cocycle. This is then used to prove the existence of an isomorphism between the nth normal and continuous cohomology groups.

Let $\mathcal{M}$ be a dual normal $\mathcal{N}$-module with predual $\mathcal{M}_*$, $\mathcal{S}$ the set of singular states on $B(H)$ (see [8] for details), and, for each $f \in \mathcal{S}$, $\{\pi_f, H_f\}$ the GNS-construction with respect to f of $B(H)$. Let $\mathcal{H} = \sum_{f \in \mathcal{S}} \oplus H_f$ and Φ: $B(H) \to B(\mathcal{H})$ be the *-homomorphism defined by

$$\Phi(X)\left(\sum_{f \in \mathcal{S}} \oplus \xi_f\right) = \sum_{f \in \mathcal{S}} \oplus \pi_f(X)\xi_f .$$

Let K denote the Hilbert space $\mathcal{H} \oplus H$. Then we can regard the operators on K as 2×2 matrices in the usual way. We now define π: $B(H) \to B(K)$ by

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$$\pi(X) = \begin{pmatrix} \Phi(X) & 0 \\ 0 & X \end{pmatrix}, \quad X \in B(H) .$$

It is obvious that π is a faithful *-representation.

LEMMA 1. *Let φ be a bounded functional on $\pi(\mathcal{N})$. Then there exists an ultraweakly continuous functional ψ on $B(K)$ such that $\varphi = \psi|_{\mathcal{N}}$ and $\|\psi\| = \|\varphi\|$.*

PROOF. $\bar{\varphi} = \varphi \circ \pi$ is a bounded functional on $\mathcal{N}$ and $\|\bar{\varphi}\| = \|\varphi\|$ as π is isometric. Extend $\bar{\varphi}$ to a functional $\hat{\varphi}$ on $B(H)$ with $\|\hat{\varphi}\| = \|\varphi\|$. We define $\psi_1 \in \pi(B(H))^*$ by $\psi_1(\pi(X)) = \hat{\varphi}(X)$, $X \in B(H)$. Again, as π is an isometry $\|\psi_1\| = \|\varphi\|$. By [8], $\hat{\varphi} = f + g$ where f is singular (i.e. a linear combination of singular states) and $g \in B(H)_*$, so easy but tedious calculations show that ψ_1 is ultraweakly continuous on $\pi(B(H))$. Extending ψ_1 to an ultraweakly continuous functional ψ on $B(K)$ with $\|\psi\| = \|\varphi\|$ it is clear that ψ extends φ.

Let $\mathcal{C} = \pi(\mathcal{N})^- \cap \pi(\mathcal{N})^{-\prime}$. The projection

$$P = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \in B(K)$$

is in $\mathcal{C}$. It is obvious that P commutes with $\pi(\mathcal{N})^-$. As $\mathcal{N}$ is hypercompact there exists a net $(X_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{N}_{\mathcal{K}}$ such that $X_\lambda \to I$ ultraweakly. For $f \in \mathcal{S}$ we have $f(X_\lambda) = 0$, $\lambda \in \Lambda$. Hence (ultraweak convergence)

$$\pi(X_\lambda) = \begin{pmatrix} \Phi(X_\lambda) & 0 \\ 0 & X_\lambda \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & X_\lambda \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = P ,$$

so also $P \in \pi(\mathcal{N})^-$. $\pi(\mathcal{N})^- P$ consists of all operators of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}, \; X \in \mathcal{N} .$$

We define

$$\alpha \colon \pi(\mathcal{N})^- P \to \mathcal{N} \text{ by } \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix} \to X .$$

It is clear that α is an isometric algebraic isomorphism, is ultraweakly continuous, and $\alpha^{-1} \colon \mathcal{N} \to \pi(\mathcal{N})^- P$ satisfies the same conditions.

LEMMA 2. *Let $\mathcal{B} \subseteq \mathcal{N}$ be a norm closed subalgebra, $n \geqq 1$, and let $\varrho \in Z^n_C(\mathcal{N}, \mathcal{M})$ such that ϱ vanishes when any of its arguments lie in $\mathcal{B}$. Then there exists $\xi \in C_C^{n-1}(\mathcal{N}, \mathcal{M})$ such that $\varrho - \Delta\xi \in Z^n_W(\mathcal{N}, \mathcal{M})$ and $\varrho - \Delta\xi$ vanishes when any of its arguments lie in $\mathcal{B}$.*

PROOF. If $A \in \pi(\mathcal{N})^-$ then $\alpha(AP) \in \mathcal{N}$, so we can define module multiplications on $\pi(\mathcal{N})^- \times \mathcal{M}$ and $\mathcal{M} \times \pi(\mathcal{N})^-$ by $A.m = \alpha(AP)m$, $m.A = m\alpha(AP)$, $m \in \mathcal{M}$. In this way $\mathcal{M}$ becomes a dual $\pi(\mathcal{N})^-$-module. We define $\varrho_1 \in C_C^n(\pi(\mathcal{N}), \mathcal{M})$ by

$$\varrho_1(A_1,\ldots,A_n) = \varrho(\alpha(A_1P),\ldots,\alpha(A_nP)), \quad A_1,\ldots,A_n \in \pi(\mathcal{N}).$$

As α is a homomorphism and $P \in \mathscr{C}$, it follows that $\varrho_1 \in Z_C^n(\pi(\mathcal{N}), \mathcal{M})$. Let $\omega \in \mathcal{M}_*$, $A_j \in \pi(\mathcal{N})$, $1 \leqq j \leqq n$. For $1 \leqq j \leqq n$ define $\varphi_j \in \pi(\mathcal{N})^*$ by

$$\varphi_j(A) = \omega(\varrho_1(A_1,\ldots,A_{j-1},A,A_{j+1},\ldots,A_n)).$$

By Lemma 1 each φ_j is the restriction of an ultraweakly continuous functional on $B(K)$, so in particular is ultraweakly continuous. This shows that ϱ_1 is separately ultraweak-weak* continuous. As in [7, section 5], (but simpler in this special case) we see that ϱ_1 extends to a bounded n-linear mapping $\bar{\varrho}_1: \pi(\mathcal{N})^- \times \ldots \times \pi(\mathcal{N})^- \to \mathcal{M}$ which is also separately ultraweak-weak* continuous, that is, $\bar{\varrho}_1 \in C_W^n(\pi(\mathcal{N})^-, \mathcal{M})$. An easy continuity argument shows that $\bar{\varrho}_1 \in Z_W^n(\pi(\mathcal{N})^-, \mathcal{M})$ and that $\bar{\varrho}_1$ vanishes if any of its arguments lie in $\pi(\mathscr{B})$. By [7, Lemma 6.2] we see that there exists $\xi_1 \in C_W^{n-1}(\pi(\mathcal{N})^-, \mathcal{M})$ such that $\bar{\varrho}_1 - \Delta\xi_1$ vanishes when any of its arguments lie in either $\pi(\mathscr{B})$ or the subspace spanned by P and I. We now define $\xi \in C_C^{n-1}(\mathcal{N}, \mathcal{M})$ by

$$\xi(\alpha(A_1P),\ldots,\alpha(A_{n-1}P)) = \xi_1(A_1,\ldots,A_{n-1})$$

$A_1,\ldots,A_{n-1} \in \pi(\mathcal{N})$. Exactly as in [7, Lemma 6.3], $\varrho - \Delta\xi \in Z_W^n(\mathcal{N}, \mathcal{M})$ and $\varrho - \Delta\xi$ vanishes when any of its arguments lie in $\mathscr{B}$.

THEOREM 1. $H_W^n(\mathcal{N}, \mathcal{M}) \cong H_C^n(\mathcal{N}, \mathcal{M})$, $n \geqq 1$.

PROOF. As in [7, Lemma 6.5] we see that

$$B_C^n(\mathcal{N}, \mathcal{M}) \cap Z_W^n(\mathcal{N}, \mathcal{M}) = B_W^n(\mathcal{N}, \mathcal{M}). \tag{1}$$

The class $\varrho + B_W^n(\mathcal{N}, \mathcal{M}) \in H_W^n(\mathcal{N}, \mathcal{M})$ is contained in the class $\varrho + B_C^n(\mathcal{N}, \mathcal{M})$, so define a natural homomorphism

$$\Xi: H_W^n(\mathcal{N}, \mathcal{M}) \to H_C^n(\mathcal{N}, \mathcal{M})$$

by $\Xi: \varrho + B_W^n(\mathcal{N}, \mathcal{M}) \to \varrho + B_C^n(\mathcal{N}, \mathcal{M})$. From (1) we see that Ξ is injective, and by Lemma 2 (with $\mathscr{B} = \{0\}$) Ξ is surjective.

Now $\mathscr{D} = \mathcal{N} \cap \mathcal{N}^*$ is an ultraweakly closed selfadjoint operator algebra containing I, that is, $\mathscr{D}$ is a von Neumann algebra. Call $\mathscr{D}$ the diagonal of $\mathcal{N}$. Define

$$DZ^n_C(\mathcal{N},\mathcal{M}) = \{\varrho \in Z^n_C(\mathcal{N},\mathcal{M}) \mid \varrho \text{ vanishes when any of its arguments lie in } \mathcal{D}\},$$

and similarly for $DZ^n_W(\mathcal{N},\mathcal{M})$.

THEOREM 2. *Assume that the diagonal $\mathcal{D}$ of $\mathcal{N}$ is of class $\mathcal{F}$ (in the sense of* [7, p. 421]). *If $\varrho \in Z^n_C(\mathcal{N},\mathcal{M})$ then there exists $\xi \in C^{n-1}_C(\mathcal{N},\mathcal{M})$ such that $\varrho - \Delta\xi \in DZ^n_W(\mathcal{N},\mathcal{M})$.*

PROOF. There exists an amenable subgroup $\mathcal{V}$ of unitary operators in $\mathcal{D}$ such that the C*-algebra $\mathcal{B}$ generated by $\mathcal{V}$ is ultraweakly dense in $\mathcal{D}$. By [7], Theorem 4.3 there exists $\xi_1 \in C^{n-1}_C(\mathcal{N},\mathcal{M})$ such that $\varrho - \Delta\xi_1$ vanishes when any of its arguments lie in $\mathcal{B}$. Lemma 2 gives $\xi_2 \in C^{n-1}_C(\mathcal{N},\mathcal{M})$ such that $(\varrho - \Delta\xi_1) - \Delta\xi_2 \in Z^n_W(\mathcal{N},\mathcal{M})$ and $(\varrho - \Delta\xi_1) - \Delta\xi_2$ vanishes when any of its arguments lie in $\mathcal{B}$. By ultraweak continuity $(\varrho - \Delta\xi_1) - \Delta\xi_2$ will vanish when any of its arguments lie in $\mathcal{B}^- = \mathcal{D}$. Thus setting $\xi = \xi_1 + \xi_2$ we have $\varrho - \Delta\xi \in DZ^n_W(\mathcal{N},\mathcal{M})$.

All nest algebras (in the sense of [6]) are hypercompact, [3], and have diagonals of class $\mathcal{F}$. In fact the diagonal of a nest algebra is a type I von Neumann algebra.

2.

Let $\mathcal{N} \subseteq B(H)$ be a nest algebra and $\mathcal{M}$ an ultraweakly closed $\mathcal{N}$-module in $B(H)$ containing $\mathcal{N}$; that is, $\mathcal{M}$ is an ultraweakly closed subspace, $\mathcal{N} \subseteq \mathcal{M} \subseteq B(H)$ and $AM, MA \in \mathcal{M}$ when $A \in \mathcal{N}$, $M \in \mathcal{M}$. It is clear that $\mathcal{M}$ is a dual normal $\mathcal{N}$-module, the weak* topology on $\mathcal{M}$ coinciding with the ultraweak topology. The following theorem extends [2, Corollary 3.11] which proves the case $n = 1$.

THEOREM 3. $H^n_C(\mathcal{N},\mathcal{M}) = H^n_W(\mathcal{N},\mathcal{M}) = \{0\}, \quad n \in \mathbf{N}$.

PROOF. If $\sigma \in Z^n_C(\mathcal{N},\mathcal{M})$ then Theorem 2 gives $\varphi \in C^{n-1}_C(\mathcal{N},\mathcal{M})$ such that $\varrho = \sigma - \Delta\varphi \in DZ^n_W(\mathcal{N},\mathcal{M})$. For each $P \in \operatorname{Lat}(\mathcal{N})$ define

$$P_- = \sup\{Q : Q \in \operatorname{Lat}(\mathcal{N}), Q \subsetneqq P\}.$$

Let $P \in \operatorname{Lat}(\mathcal{N})$, $P_- \neq I$, and choose a fixed $\xi \in (I - P_-)H$, $\|\xi\| = 1$. For all $\eta \in H$, $\xi \otimes P\eta \in \mathcal{N}$; so define a $(n-1)$-linear mapping $\psi_P\colon \mathcal{N} \times \mathcal{N} \times \ldots \times \mathcal{N} \to B(H)$ by

$$\psi_P(A_1,\ldots,A_{n-1})\eta = \varrho(A_1,\ldots,A_{n-1}, \xi \otimes P\eta)\xi,$$

$A_1,\ldots,A_{n-1} \in \mathscr{N}$, $\eta \in H$ (compare with [4, Theorem 3.1]). In fact $\psi_P(A_1,\ldots,A_{n-1}) \in \mathscr{N}$, for if $Q \in \mathrm{Lat}(\mathscr{N})$ then by [7, Lemma 4.1],

$$\begin{aligned}\psi_P(A_1,\ldots,A_{n-1})Q\eta &= \varrho(A_1,\ldots,A_{n-1},\xi\otimes PQ\eta)\xi\\ &= \varrho(A_1,\ldots,A_{n-1},Q(\xi\otimes PQ\eta))\xi\\ &= \varrho(A_1,\ldots,A_{n-1}Q,\ \xi\otimes PQ\eta)\xi\\ &= \varrho(A_1,\ldots,QA_{n-1}Q,\ \xi\otimes PQ\eta) = \ldots\\ &= Q\varrho(QA_1Q_1,\ldots,QA_{n-1}Q,\ \xi\otimes PQ\eta)\xi = \ldots\\ &= Q\varrho(A_1,\ldots,A_{n-1},\ \xi\otimes PQ\eta)\xi\\ &= Q\psi_P(A_1,\ldots,A_{n-1})Q\eta\ .\end{aligned}$$

It is clear that

$$\|\psi_P(A_1,\ldots,A_{n-1})\eta\| \leqq \|\varrho\|\,\|A_1\|\ldots\|A_{n-1}\|\,\|P\|\,\|\eta\|\ ,$$

so

$$\psi_P \in C_C^{n-1}(\mathscr{N},\mathscr{M}), \qquad \|\psi_P\| \leqq \|\varrho\|\ . \tag{2}$$

Now

$$\begin{aligned}\Delta\psi_P(A_1,\ldots,A_n)P\eta &= A_1\varrho(A_2,\ldots,A_n,\ \xi\otimes P\eta)\xi+\\ &\quad+\sum_{j=1}^{n-1}(-1)^j\varrho(A_1,\ldots,A_jA_{j+1},\ldots,A_n,\ \xi\otimes P\eta)\xi+\\ &\quad+(-1)^n\varrho(A_1,\ldots,A_{n-1},\ \xi\otimes PA_nP\eta)\xi\\ &= [A_1\varrho(A_2,\ldots,A_n,\ \xi\otimes P\eta)\xi+\\ &\quad+\sum_{j=1}^{n-1}(-1)^j\varrho(A_1,\ldots,A_jA_{j+1},\ldots,A_{n-1},\ \xi\otimes P\eta)\xi+\\ &\quad+(-1)^n\varrho(A_1,\ldots,A_{n-1},A_n(\xi\otimes P\eta))\xi+\\ &\quad+(-1)^{n+1}\varrho(A_1,\ldots,A_n)(\xi\otimes P\eta)\xi]+(-1)^n\varrho(A_1,\ldots,A_n)P\eta\\ &= \Delta\varrho(A_1,\ldots,A_n,\ \xi\otimes P\eta)\xi+(-1)^n\varrho(A_1,\ldots,A_n)P\eta\\ &= (-1)^n\varrho(A_1,\ldots,A_n)P\eta\ ,\end{aligned}$$

since ϱ is a cocycle. Thus we have

$$\Delta\psi_P(A_1,\ldots,A_n)P = (-1)^n\varrho(A_1,\ldots,A_n)P\ . \tag{3}$$

If $I_- \neq I$ then ψ_I is defined as above, and from (3), $\Delta\psi_I = (-1)^n\varrho$, and hence $\sigma = \Delta(\varphi + (-1)^n\psi_I)$.

Now assume $I_- = I$. Then $\mathscr{P} = \{P : P \in \mathrm{Lat}(\mathscr{N}), P \neq I\}$ with the natural direction is a net converging ultraweakly to I. We identify $C_C^{n-1}(\mathscr{N}, \mathscr{M})$ with the dual space of $\mathscr{N} \otimes \mathscr{N} \otimes \ldots \otimes \mathscr{N} \otimes \mathscr{M}_*$, the n-fold projective tensor product of $(n-1)$ copies of $\mathscr{N}$ and one copy of $\mathscr{M}_*$. As the net $(\psi_P)_{P \in \mathscr{P}}$ is bounded (by $\|\varrho\|$) there exists a weak* convergent subnet $(\psi_{P(\lambda)})_{\lambda \in \Lambda}$ with limit $\psi \in C_C^{n-1}(\mathscr{N}, \mathscr{M})$. In particular for $A_1, \ldots, A_n \in \mathscr{N}$ $\psi_{P(\lambda)}(A_1, \ldots, A_{n-1}) \to \psi(A_1, \ldots, A_{n-1})$ ultraweakly, and so

$$(4) \qquad \Delta\psi_{P(\lambda)}(A_1, \ldots, A_n) \to \Delta\psi(A_1, \ldots, A_n) \quad \text{ultraweakly}.$$

The set $H_0 = \bigcup_{P \in \mathscr{P}} P(H)$ is norm dense in H. It is easily seen from (4) that

$$(\Delta\psi_{P(\lambda)}(A_1, \ldots, A_n)P(\lambda)x_0 \mid y_0) \to (\Delta\psi(A_1, \ldots, A_n)x_0 \mid y_0)$$

whenever $x_0, y_0 \in H_0$. The argument of [5, Lemma 4.3] gives

$$(5) \qquad \Delta\psi_{P(\lambda)}(A_1, \ldots, A_n)P(\lambda) \to \Delta\psi(A_1, \ldots, A_n) \quad \text{ultraweakly}$$

$A_1, \ldots, A_n \in \mathscr{N}$. The net $(P(\lambda))_{\lambda \in \Lambda}$ converges ultraweakly to I so combining (3) and (5) gives $\varrho = (-1)^n \Delta\psi$, and hence $\sigma = \Delta(\varphi + (-1)^n \psi)$.

This shows $H_C^n(\mathscr{N}, \mathscr{M}) = \{0\}$. The rest of the theorem follows from Theorem 1.

Note that Theorem 3 shows that the hypothesis of [4 Theorem 2.1] are satisfied by nest algebras.

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