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ACTIONS OF FINITE GROUPS ON C*-ALGEBRAS

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Let A be a C*-algebra, and let G be a locally compact group acting as automorphisms of A via the homomorphism α of G into Aut (A). Then there are two algebras commonly associated with this situation, namely the fixed point subalgebra A^G, and the crossed product algebra $A \times_\alpha G$. In recent years considerable progress has been made in understanding the relations between A and these algebras when G is Abelian, notably by D. Olesen and G. K. Pedersen (see [22, 23, 25] and references therein). In particular, they have investigated the question of when the crossed product will be simple or prime. However, in the case of non-Abelian groups much less in understood except in special cases [36, 5], though important progress has been made recently [11, 12]. The purpose of this paper is to show that at least in the case of finite groups, fairly detailed information can be obtained. This is done principally by showing, in Section 1, that a technique involving generalized notions of outer automorphisms, which has recently been introduced in the study of purely algebraic crossed products [9, 19, 20], works especially well for C*-algebras. In Sections 2 and 3 this technique is then applied to discuss when crossed products will be prime or simple. In Section 4 the technique is applied to show that if A, or A^G, or $A \times_\alpha G$ is type I, then the other two algebras are also. In Section 5 the same kind of results are obtained for the property of being liminal, or of having a T_1 primitive ideal space. These applications should be a good indication of how the technique can be used to answer other questions in the finite group case.

Undoubtedly many of the results of this paper can be generalized to the setting of crossed products twisted by cocycles [36], much as in [20], but I have not carried this out. In fact, the most general natural setting for many of these results may well be the C*-algebraic bundles of Fell [7, 8] over finite groups, just as in the purely algebraic case it may well be the algebraic bundles of Fell over finite groups [6] or the Clifford systems of Dade [2], though a few of the results may even generalize to the setting of normal subrings [32].

In the case of Abelian groups the principal tool has been the Connes spectrum [5, 21, 22, 25]. It is a very interesting open question as to how the

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Connes spectrum should best be defined in the non-Abelian case. But in whatever way it is defined and used, the results will have to be compatible with those obtained here, so that the present paper may provide some guidance in working on this question.

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1. Partly-inner automorphisms.

Let A be a C*-algebra. When A does not have an identity element, experience shows that in the place of inner automorphisms of A one should consider the *generalized inner* automorphisms, which are those obtained by conjugating with unitaries from the double centralizer algebra, $M(A)$, of A [1, 25]. (By "automorphism" we will always mean "*-automorphism".) In fact, it will be useful for us to consider automorphisms α having the property that the restriction of α to some non-zero α-invariant ideal of A is generalized inner. ("Ideal" will always mean "two-sided ideal".) We will call such automorphisms *partly inner*, and call automorphisms which are not partly inner, *purely outer*. It is a trivial matter to check that the set of partly inner automorphisms is closed under taking inverses and under conjugating by arbitrary automorphisms. But in general it is not closed under products. However, the subset consisting of those partly inner automorphisms α for which the ideal on which α is generalized inner can be chosen to be essential (i.e. has non-zero intersection with every proper ideal of A) will be closed under products and so is a normal subgroup of the automorphism group of A. (One can go further and take norm limits of these, which leads to considering the C*-algebra $M^{\infty}(A)$ of [4, 24], which is an analogue of the ring of left quotients of [9, 19, 20], but we will have no need for this.) In particular, if A is prime, then the set of all partly inner automorphisms is a normal subgroup.

Now let G be a (discrete) group and let α be a homomorphism of G into the group of automorphisms of A. The crossed product $A \times_{\alpha} G$ can be defined [34, 25] as the completion, for an appropriate C*-norm, of the purely algebraic crossed product, $C_c(G, A)$, consisting of the functions of finite support from G to A with product defined by

$$(f * g)(t) = \sum \{f(s)\alpha_s(g(s^{-1}t)) : s \in G\} .$$

In this section we would like to study (closed, two-sided) ideals in $A \times_{\alpha} G$. But such ideals can be elusive. In particular, there is no reason why such an ideal must contain any non-zero elements of $C_c(G, A)$. But the technique in [9, 19, 20] for algebraic crossed products, which we would like to adapt to C*-algebras, depends crucially on the fact that ideals there will contain functions of finite support. For this reason this technique seems to give information for C*-crossed products only in the case of finite groups, where the C*-crossed product coincides, as an algebra, with the algebraic crossed product. However, to make clear that the difficulty is with the existence of elements of finite support in ideals, and not with whether elements of G have finite order, we will describe the technique in terms of an arbitrary discrete group G, but for the algebra $C_c(G, A)$.

Let P_α denote the set of $t \in G$ such that α_t is partly inner, so that P_α is a subset of G closed under taking inverses and conjugating by elements of G, but need not be closed under products. The main theorem of this section is an analogue of results in [9, 19, 20], such as Proposition 2 of [9].

1.1 THEOREM. *Let I be a non-zero ideal in $C_c(G, A)$. Then I contains a non-zero element supported on P_α.*

PROOF. We begin by imitating the proof of, for example, Proposition 2 of [9]. Let f be any non-zero element of I whose support has minimal cardinality among the supports of all non-zero elements of I. Since I is closed under translation, we may assume that the support of f contains the identity element, e, of G. We show that f is supported on P_α.

For $c \in A$ let $c\delta_e$ be the function with value c at e and zero elsewhere. It is easily seen that $(c\delta_e) * f$ and $f * (c\delta_e)$ have support contained in the support of f. It follows that if $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ are elements of A, and if

$$g = \sum (a_i\delta_e) * f * (b_i\delta_e) ,$$

then g has support contained in the support of f. Since $g \in I$ and f has minimal support, it follows that either $g \equiv 0$ or the support of g coincides with that of f. In other words, if t is in the support of f, then $g(t) = 0$ iff $g(e) = 0$. Computing these quantities, we find that

$$\sum a_i x b_i = 0 \quad \text{iff} \quad \sum a_i y \alpha_t(b_i) = 0 ,$$

where $x = f(e)$ and $y = f(t)$. Let us fix t, and so x and y, but let the a_i and b_i vary. If AxA and AyA denote the algebraic ideals generated by x and y, and if we try to define a linear map, T, from AxA to AyA by

$$T(\sum a_i x b_i) = \sum a_i y \alpha(b_i) ,$$

where $\alpha = \alpha_t$, then we see from the above computation that T is well-defined and injective. It is clear that T is also surjective, and that $T(ac)=aT(c)$, $T(ca)=T(c)\alpha(a)$ for all $a\in A$, $c\in AxA$.

It is at this point that the purely algebraic argument ends. To proceed further, we must use the special properties of C*-algebras. Our goal is to show that from T, which is probably highly discontinuous, we can obtain a unitary double centralizer on some ideal of A, conjugation by which will coincide with α on this ideal. Accordingly, let J and K denote the closures of AxA and AyA, so that J and K are themselves C*-algebras. Now G. K. Pedersen showed that any C*-algebra contains a minimal dense hereditary ideal, which was later shown by K. B. Laursen and A. M. Sinclair to be minimal among all dense ideals (for all of which see [25]). Let J_0 and K_0 denote the Pedersen ideals of J and K. By the Laursen–Sinclair result, $J_0\subseteqq AxA$ and $K_0\subseteqq AyA$. In particular, T is defined on J_0.

We would like to find a non-zero α-invariant ideal in J_0, since it will turn out that on such we can show that T is actually a double centralizer. To find such an ideal we will need to manipulate with J_0 and K_0 and their images under powers of α. For this purpose we need to use the fact [26] that Pedersen ideals, and also closed ideals, are algebraic ideals, as defined in [26], so that the product of any two such ideals is their intersection [26], and hence does not depend on the order in which the product is taken. We also use the fact that the product of the closures of two ideals is the closure of their product.

Let $L=T(J_0)^-$, which is clearly a non-zero closed ideal. Then actually

$$L = T(J_0)^- = T(J_0AxA)^- = (J_0T(AxA))^- = JK$$

(which can be seen to be essential in both J and K). Furthermore, for any integer n we have

$$\begin{aligned}\alpha^n(L)L &= (\alpha^n(L)T(J_0))^- = T(\alpha^n(L)J_0)^- \\ &= T(J_0\alpha^n(L))^- = (T(J_0)\alpha^{n+1}(L))^- \\ &= L\alpha^{n+1}(L) = \alpha^{n+1}(L)L\,.\end{aligned}$$

Setting $n=0,-1$, we find that L is α-invariant. Let L_0 be the Pedersen ideal of L, so that L_0 is also α-invariant. Since $L\subseteqq J\cap K$, it follows that $L_0\subseteqq J_0\cap K_0$, so that L_0 is in the domain of both T and T^{-1}. Furthermore $T(L_0)=T(L_0L_0)=L_0T(L_0)\subseteqq L_0$, and similarly $T^{-1}(L_0)\subseteqq L_0$. It follows that T is an invertible operator on L_0.

Define an operator, S, on L_0 by $S(c)=T(\alpha^{-1}(c))$ for any $c\in L_0$. Note the need for L_0 to be α-invariant. Then

$$S(ca) = T(\alpha^{-1}(ca)) = S(c)a\,,$$

$$cS(d) = cT(\alpha^{-1}(d)) = T(c\alpha^{-1}(d)) = T(c)d\,,$$

for $a \in A$, $c, d \in L_0$. This says, in particular, that the pair (S, T) is a double centralizer on L_0. Furthermore, from the invertibility of T it is easily seen that the double centralizer is invertible.

Now double centralizers on Pedersen ideals have been extensively studied by A. J. Lazar and D. C. Taylor [17]. As they indicate, if $\Gamma(L_0)$ denotes the set of double centralizers on L_0, then it is easily seen that under the evident operations $\Gamma(L_0)$ is a *-algebra in which L_0 sits as an essential ideal. We will also need the fairly deep theorem 5.42 of [17], which states that the center of $\Gamma(L_0)$ can be identified in a natural way with the algebra of all continuous complex-valued functions (bounded or not) on $\check{L}$, the primitive ideal space of L.

Viewing L_0 as an ideal in $\Gamma(L_0)$, we will find it convenient to denote the double centralizer (S, T) simply by w. Thus w is an invertible element of $\Gamma(L_0)$, and $T(c) = cw$ for $c \in L_0$. Then the relation $T(cd) = T(c)\alpha(d)$ beomes $cdw = cw\alpha(d)$. Since this is true for all $c \in L_0$, it follows that $dw = w\alpha(d)$ for all $d \in L_0$. Since w is invertible, this becomes $\alpha(d) = w^{-1}dw$, so that α is generalized inner in a sense appropriate for Pedersen ideals. Now because α is a *-homomorphism,

$$w^{-1}d^*w = (w^{-1}dw)^* = w^*d^*w^{*-1}$$

for all $d \in L_0$, so that $d^*ww^* = ww^*d^*$. It follows easily that ww^* is in the center of $\Gamma(L_0)$, and so corresponds to an invertible function on $L^\vee$ which is easily seen to be positive. Thus $v = (ww^*)^{-\frac{1}{2}}$ exists in the center of $\Gamma(L_0)$. Let $u^* = vw$. Then $u^* = u^{-1}$ so that u is unitary, and also $\alpha(d) = udu^*$ for all $d \in L_0$. From the fact that u is unitary it is easily seen that, as a double centralizer, u is norm continuous on L_0, and so extends to a unitary double centralizer on L. By continuity we then have $\alpha(a) = uau^*$ for all $a \in L$, so that α is partly inner. Since α was α_t for any t in the support of f, this concludes the proof of Theorem 1.1.

We remark that from Proposition 4.5 of [23] it immediately follows that if A is a type I C*-algebra, then an automorphism α of A is partly inner iff the fixed-point set for the action of α on $A^\vee$ has non-empty interior. This fact can be combined with Theorem 1.1 to give interesting applications. As an elementary application, for which 4.5 of [23] is not really needed, let A be the C*-algebra of continuous functions on the circle, and let the group of integers, $\mathbf{Z}$, act on A by powers of some fixed irrational rotation. Then $C_c(\mathbf{Z}, A)$ will be simple, in analogy with the corresponding fact for the C*-crossed product [36].

2. Prime crossed products.

In this section we use the results of the previous section to give a fairly complete analysis of how the crossed product of a C*-algebra with a finite group can be prime. In this analysis we will use without reference the standard facts contained in [3] concerning primitive ideal spaces of C*-algebras. Throughout this section G will denote a finite group which acts on a C*-algebra A, the action being denoted by α.

To begin with, in order for $A \times_\alpha G$ to be prime it is necessary for A to be G-prime, in the sense that any two non-zero G-invariant ideals must have non-zero intersection. This is because G-invariant ideals of A give ideals in the crossed product in an evident way. So we suppose from now on that A is G-prime.

Now the action of G on A permutes the ideals of A, and, in particular, gives an action of G as homeomorphisms of the primitive ideal space, $A^\vee$, of A with the Jacobson topology. The assumption that A is G-prime then becomes the assumption that every G-invariant open subset of $A^\vee$ is dense. In working with this we will, for ease of notation, write tJ instead of $\alpha_t(J)$ for $t \in G$ and $J \in A^\vee$.

It is a well-known open problem as to whether prime ideals of non-separable C*-algebras must be primitive. To avoid the difficulties which this would cause us here, we assume (only through the next proposition) that the Jacobson topology on $A^\vee$ is second countable (as it will be if A is separable or simple). Accordingly, let $\{U_n\}$ be a countable base for the topology of $A^\vee$. Since A is G-prime, GU_n must be dense in $A^\vee$ for each n. But $A^\vee$ is a Baire space, and so $\bigcap GU_n$ is dense, containing at least one point, say J. Since $J \in GU_n$ for all n, it follows that GJ meets each U_n, so that GJ is dense in $A^\vee$. Since GJ is finite (and primitive ideals are prime), this means that for each $K \in A^\vee$ there exists $t \in G$ such that $K \supseteqq tJ$. Note however that if $J \supseteqq tJ$ for some $t \in G$, then, applying t successively, we find that $J \supseteqq tJ \supseteqq t^n J$ for all $n \geqq 1$. Since G is finite, $t^n J = J$ for some n, so that $J = tJ$. From this it follows that if $sJ \supseteqq tJ$ for $s, t \in G$, then $sJ = tJ$.

Now let $J_1, \ldots, J_n$ be an enumeration of the distinct elements of GJ. Then $J_i \supseteqq J_j$ only if $i = j$, and $\bigcap J_i = 0$. Let $I_i = \bigcap \{J_j : j \neq i\}$, so that $I_i I_j = 0$ if $i \neq j$. Then $J_i \not\supseteqq I_i$, since otherwise $J_i \supseteqq J_j$ for some $i \neq j$. Thus $I_i \neq 0$, and also $J_i \cap I_i \in (I_i)^\vee$. But $J_i \cap I_i = \bigcap J_j = 0$, so that I_i is prime as an algebra. It is also clear that G permutes the I_i, so that if $L = \oplus I_i$, then L is a G-invariant ideal. Since A is G-prime, it follows that L is an essential ideal in A. The above analysis can be summerized as follows:

2.1. PROPOSITION. *Let A be G-prime and assume that $A^\vee$ is second countable. Then there is an (essential) G-invariant ideal L in A which is the sum of* *orthogonal ideals which are permuted transitively among each other by the action of G, and each of which is a prime C*-algebra.*

In order to apply this result to analyse the primeness of crossed products, we note first that if an algebra B contains an essential ideal I, then B is prime if and only if I is. Furthermore:

2.2. PROPOSITION. *Let I be an essential G-invariant ideal of A. Then $I \times_\alpha G$ is an essential ideal in $A \times_\alpha G$.*

PROOF. A simple computation shows that if $f \in A \times_\alpha G$ and if $f * (I \times_\alpha G) = 0$, then $f = 0$.

Thus, with notation as in Proposition 2.1, $A \times_\alpha G$ will be prime iff $L \times_\alpha G$ is prime. And so, in some sense, we have reduced the question to the case in which A is a sum of orthogonal ideals which as C*-algebras are prime, and which are permuted transitively among each other by the action of G. Such a system of transitively permuted orthogonal ideals is a very special case of the situation treated in theorem 17 of [12], and that theorem gives the following result in our special case. We include a sketch of a proof for the reader's convenience.

2.3. PROPOSITION. *Suppose that A is the sum of orthogonal ideals which are permuted transitively by the action of G. Let I be one of these ideals, and let H be the subgroup of elements of G which carry I into itself, so that H acts on I. Then $A \times_\alpha G$ is Morita equivalent to $I \times_\alpha H$.*

SKETCH OF PROOF. We define an equivalence bimodule X (i.e. "imprimitivity bimodule" as in Definition 6.10 of [29]) between $A \times_\alpha G$ and $I \times_\alpha H$ as follows. Let X be the vector space of those functions Φ from G into A such that $\Phi(t) \in \alpha_t(I)$ for $t \in G$. Define a right action of $I \times_\alpha H$ on X by

$$(\Phi\varphi)(t) = \sum \{\Phi(tp)\alpha_{tp}(\varphi(p^{-1})) : p \in H\}$$

for $\Phi \in X$ and $\varphi \in I \times_\alpha H$, and define an $I \times_\alpha H$-valued inner product $\langle\, , \rangle_B$, by

$$\langle \Phi, \Psi \rangle_B(p) = \sum \{\alpha_t(\Phi(t^{-1})^* \Psi(t^{-1}p)) : t \in G\}$$

for Φ, $\Psi \in X$ and $p \in H$. Define a left action of $A \times_\alpha G$ on X by

$$(f\Phi)(t) = \sum \{f(s)\alpha_s(\Phi(s^{-1}t)) : t \in G\}$$

for $f \in A \times_\alpha G$ and $\Phi \in X$, and define an $A \times_\alpha G$-valued inner product, $\langle\, , \rangle_E$ on X by

$$\langle \Phi, \Psi \rangle_E(t) = \sum \{\Phi(s)\alpha_t(\Psi(t^{-1}s)^*) : s \in G\}$$

for $\Phi, \Psi \in X$. Then with these definitions it is straightforward to verify that X becomes an equivalence bimodule. The crucial place at which one uses the fact that H is the full stability subgroup of I is in verifying the identity

$$\langle \Phi, \Psi \rangle_E \theta = \Phi \langle \Psi, \theta \rangle_B .$$

Specifically, one first calculates that

$$(\langle \Phi, \Psi \rangle_E \theta)(t) = \sum \{(\Phi(s)\alpha_r(\Psi(r^{-1}s)^*))\alpha_r(\theta(r^{-1}t)) : r, s \in G\} .$$

But $\alpha_r(\theta(r^{-1}t)) \in \alpha_t(I)$, so terms of the sum can be non-zero only if $\Phi(s) \in \alpha_t(I)$, that is, if $s \in tH$. Thus we can replace the summation over s by one over tp as p runs over H. Once this is done, it is easy to continue the calculation to obtain $\Phi\langle \Psi, \theta \rangle_B(t)$.

We remark that as a special case of Theorem 2.13 of [13], one can actually show that

$$A \times_\alpha G \cong (I \times_\alpha H) \otimes M_n$$

where n is the cardinality of G/H and M_n is the algebra of $n \times n$ matrices. But in general there will be no natural isomorphism—isomorphisms depend on a choice of coset representatives for H in G—so that calculations become slightly more complicated.

Anyway, Morita equivalence gives a natural isomorphism between the lattices of ideals (Theorem 3.1 of [31]), and in particular preserves primeness. Thus we have reduced the question of the primeness of $G \times_\alpha A$ to that of $I \times_\alpha H$, where now I itself is prime. It is to this situation that we can apply Theorem 1.1 to obtain a result similar to Proposition 2.6 of [20]. Let us change our notation to that of the first section, so that we now assume that G acts on A and that A is prime. We need to know when $A \times_\alpha G$ is prime. Let $N = P_\alpha$, the normal subgroup of G consisting of those t for which α_t is partly inner. As seen in the first section, any non-zero ideal of $A \times_\alpha G$ will have non-zero intersection with $A \times_\alpha N$. Thus if $A \times_\alpha N$ is prime, so is $A \times_\alpha G$. But the converse need not be true, for the ideals of $A \times_\alpha N$ which arise as intersections of $A \times_\alpha N$ with ideals of $A \times_\alpha G$ are of a special kind. This is true when N is any normal subgroup of G. Specifically, if I is an ideal of $A \times_\alpha G$ and if $J = I \cap (A \times_\alpha N)$, then it is clear that J will be invariant under conjugation by elements of G (viewed as unitaries in the double centralizer algebra of $A \times_\alpha G$). Put in other words, G acts as a group of automorphisms of $A \times_\alpha N$, where the automorphism, β_t, corresponding to $t \in G$ is defined by

$$(\beta_t(f))(s) = \alpha_t(f(t^{-1}st)) .$$

Thus we can talk about G-invariant ideals in $A \times_\alpha N$. Then the intersection with $A \times_\alpha N$ of any ideal of $A \times_\alpha G$ will be G-invariant. Suppose conversely that J is a G-invariant ideal of $A \times_\alpha N$. Because A is already contained in $A \times_\alpha N$ in the form of functions supported at the identity element of N, it is easily seen that $(A \times_\alpha G)J = J(A \times_\alpha G)$. (That is, J is $(A \times_\alpha G)$-invariant, as defined in 2.1 of [32]. In fact, $A \times_\alpha N$ will be a normal subalgebra of $A \times_\alpha G$ in the spirit of [32], though not according to the specific definitions given there.) Thus the two-sided ideal in $A \times_\alpha G$ generated by J is just $(A \times_\alpha G)J$. But again, since A is contained in $A \times_\alpha N$, this is just the span of the $\delta_t * J$ for $t \in G$. Now the elements of J are supported on N, so that those of $\delta_t * J$ are supported on the coset tN. From this it is clear that the intersection of $(A \times_\alpha G)J$ with $A \times_\alpha N$ is just J again. Thus:

2.4. Proposition. *Let N be some normal subgroup of G, so that G acts as a group of automorphisms of $A \times_\alpha N$. Then the ideals of $A \times_\alpha N$ which occur as intersections with $A \times_\alpha N$ of ideals of $A \times_\alpha G$ are exactly the G-invariant ideals of $A \times_\alpha N$.*

Combining this with Theorem 1.1, we obtain:

2.5. Proposition. *Let A be prime and let $N = P_\alpha$. Then $A \times_\alpha G$ is prime iff $A \times_\alpha N$ is G-prime.*

We remark next that if L is a C*-algebra, then any automorphism, α, of L lifts uniquely in an evident way to an automorphism of $M(L)$, which we still denote by α. If L is an essential ideal in a C*-algebra A, so that $A \subseteq M(L)$, and if α is the restriction to L of an automorphism α of A, then α on A will agree with the restriction to A of α on $M(L)$. If α now denotes an action of G on A, and if L is a G-invariant essential ideal of A, then α lifts to an action α of G on $M(L)$ which on $A \subseteq M(L)$ agrees with the original action. The proof of the following is trivial.

2.6. Proposition. *Let L be a G-invariant essential ideal of A. Then A is G-prime iff L is G-prime.*

2.7. Proposition. *Let L be a G-invariant essential ideal of A, so that G acts on $M(L)$, and let N be a normal subgroup of G, so that G acts on $M(L) \times_\alpha N$ and on $A \times_\alpha N$. Then $A \times_\alpha N$ is G-prime iff $M(L) \times_\alpha N$ is G-prime.*

Proof. $L \times_\alpha N$ is a G-invariant essential ideal of both $A \times_\alpha N$ and $M(L) \times_\alpha N$, so we can apply Proposition 2.6 twice.

Combining this with Proposition 2.5 we obtain:

2.8. Proposition. *Let A be prime, and let $N=P_\alpha$. Let L be a G-invariant ideal of A on which N acts by generalized inner automorphisms, so that N acts by inner automorphisms on $M(L)$. Then $A\times_\alpha G$ is prime iff $M(L)\times_\alpha N$ is G-prime.*

Now crossed products by actions given by inner automorphisms are well-known to have a fairly simple description. Perhaps the most general statement of this in the literature is Theorem 18 of [12]. What we must do here is to understand how the G-action relates to this description. For ease of notation we will denote $M(L)$ by A. Thus we assume that A is a prime algebra with identity element on which G acts, and that N is a normal subgroup which acts by inner automorphisms on A. We ask when $A\times_\alpha N$ will be G-prime. Now for each $t\in N$ let u_t be a unitary in A such that $\alpha_t(a)=u_t^*au_t$ for $a\in A$. Let C be the subspace of $A\times_\alpha N$ having as basis the $u_t\delta_t$. Since u_su_t is a scalar multiple of u_{st} for $s,t\in N$, it is clear that C is a C*-subalgebra of $A\times_\alpha N$. Furthermore, for $a\in A$

$$(u_t\delta_t)a \;=\; u_t\alpha_t(a)\delta_t \;=\; u_t(u_t^*au_t)\delta_t \;=\; a(u_t\delta_t)\,,$$

so each $u_t\delta_t$ commutes with A in $A\times_\alpha N$. Thus C is in the commutant of A in $A\times_\alpha N$. Finally, since the $u_t\delta_t$ clearly form an A-basis for $A\times_\alpha N$, it is clear that $A\otimes C\cong A\times_\alpha N$, the isomorphism being given by $a\otimes h\to(a\delta_e)*h$ for $a\in A$, $h\in C$. Since the center of A is the scalars, it follows that C is exactly the commutant of A in $A\times_\alpha N$. (Compare with Lemma 2.3 of [20].)

Now A is G-invariant in $A\times_\alpha N$, and so C must be also. Actually this can easily be seen directly, using the fact that $\alpha_r(u_t)$ will be a scalar multiple of $u_{rtr^{-1}}$ for $r\in G$, $t\in N$. It follows that under the isomorphism with $A\otimes C$ the action of G on $A\times_\alpha N$ is just the product of its actions on A and C.

Let I be a G-invariant ideal in $A\times_\alpha N$. Since C is a finite-dimensional C*-algebra, it is generated by its minimal projections. Choose a minimal projection, p, such that $pIp\neq 0$. Then pIp in $A\otimes C$ will look like $M\otimes p$ where M is some ideal of A. Since A is prime, M will contain a non-zero G-invariant ideal, say J, so that $J\otimes p\subseteq I$. But if K is the G-invariant ideal in C generated by p, it follows that $J\otimes K\subseteq I$. We have thus shown:

2.9 Lemma. *Every G-invariant ideal I of $A\times_\alpha N$ contains an ideal of form $J\otimes K$ where J and K are G-invariant ideals of A and C respectively.*

From this lemma we immediately obtain the following analogue of Proposition 2.6 of [20]:

2.10. Proposition. *Let A be prime with identity element, and let $N=P_\alpha$. Then $A\times_\alpha N$ is G-prime if and only if C is G-prime, or equivalently, G-simple.*

The algebra C is of the familiar type associated with projective representations [32], namely, the group algebra of N twisted by a cocycle. The action of G on C seems less familiar, but comes from the action of G on N together with a function on $G \times N$ with values in the circle and having cocyclelike properties with respect to G, N and the cocycle on N. All this data can be extracted from A and then examined independently of A, to determine if C is G-prime. It is easy to see that up to isomorphism this data does not depend on the choice of the ideal L in Proposition 2.8. However, we will not explicitly summarize all the results of this section, as the full statement is somewhat cumbersome.

3. Simple crossed products.

In this section we make an analysis, similar to that of the previous section, of how the crossed product of a C*-algebra by a finite group can be simple. We also obtain some related results about the structure of crossed products when the C*-algebra involved is simple.

To begin with, it is clear that in order for $A \times_\alpha G$ to be simple, it is necessary for A to be G-simple, in the sense that A have no proper G-invariant ideals. This means that $\check{A}$ can have no proper open G-invariant subsets. Since $\check{A}$ is a T_0 space and orbits will be relatively discrete, it follows that $\check{A}$ consists of exactly one orbit. By the arguments preceeding Proposition 2.1 it follows that A must be the sum of orthogonal simple ideals which are permuted transitively among each other by the action of G. If I is one of these simple ideals and H is its stability subgroup, then we can apply Proposition 2.3 to conclude that $A \times_\alpha G$ is Morita equivalent to $I \times_\alpha H$, so that $A \times_\alpha G$ is simple iff $I \times_\alpha H$ is. Thus we must analyse the structure of a crossed product when the C*-algebra involved is itself simple. For this, we again use the results of the first section. Let us change our notation to that of the first section, so that now G acts on A and A is simple. Let $N = P_\alpha$, the normal subgroup of partly inner elements. Then according to Theorem 1.1 any ideal of $A \times_\alpha G$ will meet $A \times_\alpha N$ in a non-zero ideal, which will be G-invariant for the reasons given before Proposition 2.4. By Proposition 2.4 itself, it is clear that $A \times_\alpha G$ will be simple iff $A \times_\alpha N$ is G-simple. Since A is now simple, each element of N will act on A by a generalized inner automorphism coming from a unitary in $M(A)$. Then the algebra C can be defined as after Proposition 2.8. It will be a subalgebra of $M(A) \times_\alpha N$, but as in Section 2, it is easily seen that $A \times_\alpha N$ is isomorphic to $A \otimes C$, and that the action of G on $A \times_\alpha N$ is the product of its actions on A and C. By Lemma 2.9 it then follows that the G-invariant ideals of $A \times_\alpha N$ correspond exactly to the G-invariant ideals of C. Since C is a finite dimensional C*-algebra, it is the direct sum of its G-simple ideals. Let $C_1, \ldots, C_k$ be the G-simple ideals of C, and for each i let $I_i = (A \times_\alpha G)(A \otimes C_i)$, which is a two-sided ideal by the arguments before Proposition 2.4. It is then clear that the I_i are orthogonal. Furthermore, each I_i is a simple ideal, since any proper ideal contained in I_i must intersect $A \times_\alpha N$ in a non-zero G-invariant ideal contained in $A \otimes C_i$, and so equal to $A \otimes C_i$. Clearly the sum of the I_i contains $A \times_\alpha N$, and so is $A \times_\alpha G$. We summarize the above:

3.1. THEOREM. *Suppose that A is simple, and let $N = P_\alpha$. Then $A \times_\alpha G$ is the finite sum of simple algebras. The number of simple algebras involved is the number of G-simple ideals in C, where C is the commutant of A in $A \times_\alpha N$ (so that the dimension of C is the order of N).*

3.2. THEOREM. *Let A be G-simple so that A is a sum of simple ideals which are permuted transitively by G. Let B be one of these simple ideals and let H be the stability subgroup of B. Let N be the normal subgroup of H consisting of elements whose action on B is by generalized inner automorphisms. Let C be the commutant of B in $B \times_\alpha N$, so that H acts on C. Then $A \times_\alpha G$ is simple iff C is H-simple.*

Consider now the situation in which A need not be G-simple, but in which A is nevertheless the sum of a finite number of simple C^*-algebras. Then A will be the sum of algebras each of which is a sum of simple algebras which are permuted transitively by G, and so to which the earlier analysis of this section applies. Clearly $A \times_\alpha G$ will then be the sum of the corresponding crossed products, and each of these crossed products is a sum of simple algebras. Making only crude estimates, we obtain:

3.3. PROPOSITION. *Let A be the direct sum of n simple algebras. Then $A \times_\alpha G$ is the direct sum of no more than $n|G|$ simple algebras, where $|G|$ is the order of G.*

That this bound cannot be improved in general can be seen by considering trivial actions of commutative groups.

We can use Proposition 3.3 to obtain some information about fixed-point algebras, once we recall the following observation of Rosenberg [33], which we will also need later. (See also 2.2 of [18].)

3.4. PROPOSITION. *Let p denote the constant function on G with value $|G|^{-1}$, so that p is a projection in $M(A \times_\alpha G)$. Then $p(A \times_\alpha G)p$ is naturally isomorphic to A^G. Thus A^G is Morita equivalent to the ideal of $A \times_\alpha G$ generated by p.*

3.5. COROLLARY. *Let A be the direct sum of n simple algebras. Then A^G is the direct sum of no more than $n|G|$ simple algebras.*

We remark that Corollary 3.5 was obtained by Kharchenko [14] in the algebraic case, by quite different methods, and with a much worse bound. It seems to me that the methods used above can also be used in the algebraic case to obtain some kind of corresponding theorem with better bound, but I have not checked the details.

In Section 5 we will need the converse of Corollary 3.5, so we give a proof here. The algebraic case is treated in Theorem 4.3 of [10], but the proof there depends on the existence of an identity element in A, so the proof given below is fairly different.

3.6. PROPOSITION. *Suppose that A^G is the direct sum of n simple algebras. Then A is the direct sum of no more than $n|G|$ simple algebras.*

PROOF. Recall [32] that if B is any subalgebra of A, then the ideal J of B is said to be A-invariant if $AJ=JA$. Now for any G-invariant ideal I of A, the ideal $I\cap A^G$ of A^G will contain an approximate identity for I, so that

$$A(I \cap A^G) = I = (I \cap A^G)A .$$

In particular, $I\cap A^G$ is A-invariant. Conversely, if J is any A-invariant ideal of A^G, then AJ is a G-invariant ideal of A, and by averaging one sees that $(AJ)\cap(A^G)=J$. We obtain in this way a bijection between the G-invariant ideals of A and the A-invariant ideals of A^G. Furthermore, if two A-invariant ideals of A^G are orthogonal, it is clear that the corresponding ideals of A will be orthogonal.

Assume now that A^G is the direct sum of n simple algebras. Note then that if J is an A-invariant ideal in A^G, then the complementary ideal, say K, must also be A-invariant, for an approximate identity in A^G for A will decompose as e_m+f_m with $e_m \in J$ and $f_m \in K$, so that for $a \in A$, $k \in K$ we have

$$ak = \lim (e_m ak + f_m ak) .$$

But $e_m ak \in (AJ)k=0$, and the remainder is in KA, so that $ak \in KA$. Note also that the intersection of A-invariant ideals is also A-invariant. Thus A^G will be the direct sum of no more than n minimal (non-zero) A-invariant ideals. Then A will be the direct sum of the corresponding G-invariant ideals.

Thus it suffices to prove the proposition under the assumption that A^G contains no proper A-invariant ideals, that is, A contains no proper G-invariant ideals and so is G-simple. But we saw at the beginning of this section that in this case A is the sum of no more than $|G|$ simple algebras.

We remark that by considering the action of G by left translation on the algebra of functions on G with pointwise multiplication, we find that the bound given in the above proposition is the best possible.

4. Algebras of type I.

We recall that a C*-algebra is said to be of type I (or postliminal, or GCR) if its image under every irreducible *-representation contains the compact operators [3]. The purpose of this section is to show how the results of Section 1 enable us to prove:

4.1. THEOREM. *Let the finite group G act on the C*-algebra A, and let A^G and $A \times_\alpha G$ denote the fixed point algebra and crossed product. Then the following are equivalent*:

1. A^G *is of type* I.
2. A *is of type* I.
3. $A \times_\alpha G$ *is of type* I.

The most difficult part of the proof is showing that 1) implies 2). This is not surprising since this implication is false if G is only assumed to be compact. For example, it is shown in [16, 23] that there are actions of the torus on simple non type I algebras such that the fixed point algebra is one-dimensional (i.e. ergodic actions). Nor is it true that 3) implies 2) if G is only assumed compact, as shown by Takesaki in [35]. We remark also that the argument given below for the fact that 3) implies 1) works also for compact groups, but Dorte Olesen has pointed out to me that it seems to be unknown whether the reverse implication is true for compact groups. The fact that 2) implies 3) is a special case of Corollary 6.6 of [36] or Theorem 6.1 of [34], but we will indicate an alternative proof below.

The fact that 3) implies 1) follows immediately from Proposition 3.4 together with the fact that type I-ness is preserved under Morita equivalence (Proposition 2.5 of [30] and Theorem 6.23 of [29]).

To show that 1) implies 2) we use the results of Section 1. Specifically, this is done by combining them with Proposition 3.4 to prove:

4.2. LEMMA. *Let the action on A of the finite group G be by purely outer automorphisms. If A^G is of type* I, *then A contains a non-zero ideal of type* I.

PROOF. Let J be the ideal of $A \times_\alpha G$ which is Morita equivalent to A^G. Then J is of type I. But it follows from Theorem 1.1 that J will meet A in a non-zero ideal, which must, of course, also be of type I.

To prove, that 1) implies 2) we proceed first by induction on the order of G. The implication is obvious if G has only one element. We now assume that the implication is known to be true for all groups of order strictly less than that of

G. Suppose G has a proper normal subgroup, N. Then it is easily verified that A^N is G-invariant, and that the action of G on A^N drops to an action of G/N. Furthermore $(A^N)^{G/N} = A^G$, which is of type I by assumption. From the induction hypothesis, A^N must then be of type I. But then, again by the induction hypothesis, A must be of type I as desired. Thus it remains to prove the theorem in the case in which G is simple.

Suppose now that G is simple, and let π be an irreducible representation of A with kernel $J \in \check{A}$. We must show that $\pi(A)$ contains the compact operators. Let $K = \bigcap \{tJ : t \in G\}$, so that K is a G-invariant ideal in the kernel of π, and let $B = A/K$. Then π drops to an irreducible representation of B, and $\pi(A) = \pi(B)$. Furthermore $B^G = A^G/(A^G \cap K)$, which is of type I. Thus it suffices to consider the case in which $K = 0$. We assume that this is now true in A. Set $I = \bigcap \{tJ : t \in G, tJ \neq J\}$. Then, as in the discussion preceeding Proposition 2.1, I is a prime algebra, and for any $t \in G$ either $tI = I$ or tI is orthogonal to I. Since I is not contained in J, it is clear that the restriction of π to I is irreducable. Let H be the stability subgroup of I, and let $a \in I^H$. If $s, t \in G$ and $sI = tI$, then $s^{-1}t \in H$ so that $\alpha_s(a) = \alpha_t(a)$. From this it is easily seen that the evident projection from A to I gives an isomorphism of A^G with I^H, so that I^H is type I. Thus it suffices to consider the case in which A is prime and π is faithful.

Now by assumption G is simple and A is prime, so that P_α is either all of G or just e_G. That is, the action is either entirely purely outer or entirely partly inner. Suppose first that it is entirely purely outer. Then according to Lemma 4.2, A will contain a non-zero type I ideal. Since π is faithful, its restrictions to I will still be irreducible, and so $\pi(I)$ will contain the compact operators. Thus the proof is complete in this case.

Suppose instead that the action is entirely purely inner. As before, we can find a G-invariant ideal on which each element of G acts as a generalized inner automorphism. The fixed point algebra of this ideal will of course be type I since A^G is. Since π is faithful, it suffices to show that the restriction of π to this ideal contains the compact operators. Thus we can assume that, in fact, the action of G on A is by generalized inner automorphisms.

Let E be the ideal in $A \times_\alpha G$ which is Morita equivalent to A^G, and so is of type I. Then E is an ideal in $M(A) \times_\alpha G$. But as in Section 2, $M(A) \times_\alpha G \cong M(A) \otimes C$ where C is the commutant of $M(A)$ in $M(A) \times_\alpha G$. Since C is a finite dimensional C*-algebra, it follows that $M(A)$ contains a non-zero type I ideal, and so A must also. This concludes the proof that 1) implies 2).

The fact that 2) implies 3) is an immediate consequence of the following proposition, which will be needed in the next section. Since we will be working with a number of automorphisms in the next paragraphs, we will denote fixed-point subalgebras by superscripting with the automorphism involved.

4.3. PROPOSITION. *Let B denote the C^*-algebra of all operators on the finite dimensional Hilbert space $L^2(G)$, and let γ denote the action of G on B by conjugation by the action of right translation on $L^2(G)$. Let $\alpha\otimes\gamma$ denote the corresponding action on $A\otimes B$. Then $A\times_\alpha G$ is naturally isomorphic to the fixed-point algebra $(A\otimes B)^{\alpha\otimes\gamma}$.*

We remark that this result (and the proof given below) holds also for compact groups if B is taken to consist only of the compact operators. For compact Abelian groups this is 5.3 of [15], though with different proof.

PROOF. Let $C(G)$ denote the C^*-algebra of functions on G with pointwise multiplication, and let λ denote the action of G on $C(G)$ by left translation. It is well-known [28] and easily seen that $C(G)\times_\lambda G\cong B$ in a natural way. Let ϱ denote the action of G on $C(G)$ or $L^2(G)$ by right translation. From the fact that ϱ and λ commute it is easily seen that γ, when viewed on $C(G)\times_\lambda G$, just corresponds to ϱ acting on $C(G)$. Let i denote the trivial representation of G on A, so that the two actions $\alpha\otimes\varrho$ and $i\otimes\lambda$ of G on $A\otimes C(G)$ commute. Then $\alpha\otimes\varrho$ lifts to an action on $(A\otimes C(G))\times_{i\otimes\lambda}G$. The latter is isomorphic to $A\otimes(C(G)\times_\lambda G)$, and under this isomorphism $\alpha\otimes\varrho$ becomes $\alpha\otimes\gamma$. Now if σ and τ are two commuting actions of G on a C^*-algebra D, so that D^σ is τ-invariant, and also G acts on $D\times_\tau G$ via σ, then it is easily seen by an averaging argument that

$$(D\times_\tau G)^\sigma \cong (D^\sigma)\times_\tau G\ .$$

Applying all of the above to $A\otimes C(G)$, we find that

$$\begin{aligned}(A\otimes B)^{\alpha\otimes\gamma} &\cong (A\otimes(C(G)\times_\lambda G))^{\alpha\otimes\gamma}\\ &\cong ((A\otimes C(G))\times_{i\otimes\lambda}G)^{\alpha\otimes\gamma} \cong ((A\otimes C(G))^{\alpha\otimes\varrho})\otimes_{i\otimes\lambda}G\ .\end{aligned}$$

Now $A\otimes C(G)$ can be naturally identified with the space of A-valued functions on G, and under this identification $(A\otimes C(G))^{\alpha\otimes\varrho}$ corresponds to the space of functions f satisfying

$$f(ts^{-1}) = \alpha_s(f(t))$$

for all $s,\ t\in G$. But such functions are determined by their values at the identity element of G, so that $(A\otimes C(G))^{\alpha\otimes\varrho}$ can be naturally identified with A. Under this identification $i\otimes\lambda$ is easily seen to correspond to α. Thus the last crossed product written above is seen to be naturally isomorphic to $A\times_\alpha G$.

5. Liminal and T_1.

We recall that a C^*-algebra is said to be liminal (or CCR) if its image under every irreducible $*$-representation consists of exactly the compact operators

[3]. The purpose of this section is to prove two related theorems, the first of which is:

5.1. THEOREM. *Let the finite group G act on the C^*-algebra A. Then the following are equivalent*:

1. A_G *is liminal.*
2. A *is liminal.*
3. $A \times_\alpha G$ *is liminal.*

Now it is well-known [3] that an algebra is liminal if and only if it is of type I and every primitive ideal is maximal, that is, the topology of its primitive ideal space is T_1. In view of Theorem 4.1, the above theorem follows from:

5.2. THEOREM. *Let the finite group G act on the C^*-algebra A. Then the following are equivalent*:

1. $(A^G)^\vee$ *is* T_1.
2. $A^\vee$ *is* T_1.
3. $(A \times_\alpha G)^\vee$ *is* T_1.

In the purely algebraic situation the corresponding result was obtained by Lorenz [18], even for crossed products defined with cocycles. Some related results for prime ideals are discussed by Fisher and Osterburg in [10]. The proof given below that 1) implies 2) in Theorem 5.2 was suggested by half of the proof of Theorem 2.7 of [18], while the proof that 2) implies 1) was partly motivated by the proof of Theorem 4.2 of [10]. The techniques of Section 1 are not used in the proof of Theorem 5.2, though they are, of course, then used in the proof of Theorem 5.1 in the form of Theorem 4.1.

There is presumably some relation between Theorem 5.2 and Theorem 2 of Poguntke's paper [27]. In particular, these two theorems should have some common generalization to the setting of Fell's C^*-algebraic bundles over finite groups [6, 7, 8].

PROOF OF THEOREM 5.2. The fact that 3) implies 1) follows immediately from Proposition 3.4 and the fact that Morita equivalence preserves the T_1 property (Corollary 3.3 of [31]).

We show next that 2) implies 1), with an eye to then invoking Proposition 4.3 to obtain the fact that 2) implies 3). Suppose now that $J \in A^\vee$. From the fact that any irreducible representation of a subalgebra is a subrepresentation of the restriction to the subalgebra of an irreducible representation of the containing algebra (2.10.2 of [3]), we can find $I \in A^\vee$ such that $I \cap A^G \subseteqq J$. Let $K = \bigcap \{tI : t \in G\}$. Since by assumption I is maximal, A/K will be a finite sum of simple algebras by the arguments preceeding Proposition 2.1. Now G acts on A/K, and so by Corollary 3.5, $(A/K)^G$ will be a finite sum of simple algebras. But

$$(A/K)^G \cong A^G/(K \cap A^G),$$

and $(K \cap A^G) \subseteq J$, so that $J/(K \cap A^G)$ is a primitive ideal of $(A/K)^G$ which must then be maximal. It follows that J is maximal, as was to be shown.

Now let B be the finite dimensional C*-algebra of operators on $L^2(G)$, as in Proposition 4.3. If $A^\vee$ is T_1, then it is easily seen that $(A \otimes B)^\vee$ is also. From Proposition 4.3 and the fact that 2) implies 1), it follows that $(A \times_\alpha G)^\vee$ is T_1. Thus 2) implies 3).

It remains to show that 1) implies 2). The proof which we give is close to half of the proof of Theorem 2.7 of [18]. To carry it out we first need some facts about lengths of modules. We will say that a Hermitian module (i.e. the Hilbert space of a non-degenerate *-representation) is of finite length if it is the direct sum of a finite number of simple (i.e. irreducible) modules. If V is a Hermitian A-module, we will let V_G denote V viewed as an A^G-module, while we will let ${}^GV$ denote the corresponding induced $(A \times_\alpha G)$-module [34]. That is, ${}^GV = (A \times_\alpha G) \otimes_A V$, which is the direct sum of the $|G|$ copies, $t \otimes V$ for $t \in G$, of the Hilbert space V.

5.3. Proposition. *Let V be a Hermitian A-module. Then the following are equivalent*:

1. *V is of finite length.*
2. *${}^GV$ is of finite length.*
3. *V_G is of finite length.*

Proof. It is clear that 3) implies 1). From the description of ${}^GV$ it is also easily seen that 1) implies 2). It thus remains to show that 2) implies 3). Our proof is motivated by the proof of Lemma 2.4 of [18]. Let p denote the idempotent of Proposition 3.4, and let $W = p({}^GV)$, which is a module over $D = p(A \times_\alpha G)p$. Now it is easily seen that the mapping

$$v \to |G|^{-1} \sum \{t \otimes v \,:\, t \in G\}$$

is an isometry of V onto W. From the fact that the isomorphism in Proposition 3.4 of A^G with D is given by

$$a \to |G|^{-1} \sum \{t \otimes a \,:\, t \in G\},$$

it is easily seen that the above isometry respects the algebra actions. Thus it suffices to show that W is of finite length. But if U is a D-submodule of W, and if Z is the $A \times_\alpha G$-submodule of ${}^GV$ generated by U, then it is easily seen that $Z = U \oplus (1-p)Z$. From this it follows that distinct D-submodules of W generate distinct $A \times_\alpha G$-submodule of ${}^GV$. Since ${}^GV$ is assumed to be of finite length, it follows that W, hence V_G, must be also.

5.4. COROLLARY. *Let the finite group G act on the* C*-*algebra A, and let* $I \in A^\vee$. *Then* $I \cap A^G$ *is the intersection of only a finite number of elements of* $(A^G)^\vee$.

PROOF. This follows from the fact that if V is a simple Hermitian A-module with kernel I, then V_G has finite length.

We now show that 1) implies 2) in Theorem 5.2. Let $I \in A^\vee$, let $K = \bigcap \{tI : t \in G\}$, and let $B = A/K$. Then G acts on B, $I/K \in B^\vee$, and $(I/K) \cap (B^G) = (0)$. We wish to show that I is maximal in $A^\vee$, and to do this it suffices to show that I/K is maximal in $B^\vee$. Now $B^G = A^G/(K \cap A^G)$, and since we are assuming that $(A^G)^\wedge$ is T_1, so will be $(B^G)^\wedge$. Thus we see that we have reduced the situation to that in which $I \cap A^G = 0$. But then 0 will be the intersection of a finite number of elements of $(B^G)^\wedge$. Since $(B^G)^\wedge$ is T_1, it follows that B^G is the direct sum of a finite number of simple algebras. Then from Proposition 3.6 it follows that B will be also, so that I/K is maximal.

ADDED IN PROOF. An analysis which is quite close to that of section 2 of this paper, but for actions on von Neumann algebras, is contained in [37, 38].

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