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A CLASS OF SYMMETRIC 2-BASES

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A set $A=\{a_1<a_2<\ldots<a_k\}$ of positive integers is called a 2-basis for n if every positive integer $\leqq n$ either belongs to A or is the sum of two elements of A (not necessarily distinct). E.g. $\{1, 3, 4\}$ is a 2-basis for 8. The *2-range* of A, denoted by $n_2(A)$, is the largest n for which A is a 2-basis. Further, the *extremal 2-range*, $n_2(k)$, is the maximal value of $n_2(A)$ for A a k-element set.

It is clear that $n_2(A)\leqq 2a_k$. If $n_2(A)=2a_k$, A is called *restricted*, $n_2^*(k)$ is the maximal value of $n_2(A)$ for A a restricted k-element set. Finally, a set $A=\{a_1<\ldots<a_k\}$ is *symmetric* if $a_i+a_{k-i}=a_k$ for $0<i<k$.

Rohrbach [6] proved that $0.25k^2\leqq n_2^*(k)\leqq n_2(k)$ and

$$n_2(k) \leqq 0.4992k^2 \quad \text{for } k\gg 0\,,$$

$$n_2^*(k) \leqq 0.4654k^2 \quad \text{for } k\gg 0\,.$$

He conjectured that both $n_2(k)$ and $n_2^*(k)$ are asymptotic to $0.25k^2$. For $n_2(k)$ this was refuted by Hämmerer and Hofmeister [1] who proved that $n_2(k) \geqq \frac{5}{18}k^2$. This was improved by Mrose [5] who proved that $n_2(k)\geqq\frac{2}{7}k^2$. A simpler construction which gives the same bound was given by Kløve and Mossige [3]. The upper bounds have been improved by Klotz [2] who showed that $n_2(k)\leqq 0.4802k^2$ for $k\gg 0$ and Moser, Pounder and Riddell [4] who showed

$$n_2^*(k) < 0.424346k^2 \quad \text{for } k\gg 0\,.$$

For a more detailed survey, see Wagstaff [7].

In this paper I show the following.

THEOREM. *For* $k\geqq 1$ *we have*

$$n_2^*(k) \geqq \tfrac{6}{23}k^2+O(k)\,.$$

We prove the theorem by constructing a set of k elements which has the stated 2-range.

We use the following notations: Let a, b and c be integers, $c>0$.

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Then

$$[a(0)a] = \{a\},$$

$$[a(c)b] = \{a+ic \mid 0 \leqq i \leqq (b-a)/c\}, \quad \text{if } b \geqq a \text{ and } c \mid (b-a)$$

$$= \emptyset \quad \text{if } b < a.$$

$$[a,b] = [a(1)b].$$

Let x, y and z be integers where $y \geqq x \geqq 1$ and let

$$S_1(z) = [z, z+x-1],$$

$$S_2(z) = [z+x-1(x)z+yx-1],$$

$$S_3(z) = [z+yx-1, z+yx+x-2],$$

$$T(z) = S_1(z) \cup S_2(z) \cup S_3(z),$$

$$B(z,d) = [z(d)z+d^2].$$

If U and V are any sets of integers, then

$$U+V = U \cup V \cup \{u+v \mid u \in U, v \in V\}.$$

With this notation $n_2(A)$ is the maximal n such that $[1,n] \subset A+A$.

LEMMA 1. *For any integer z we have*

$$T(0)+T(z) \supset [z, z+2yx+2x-4].$$

PROOF. We divide the proof of lemma 1 into four parts, the verification of which is straightforward:

$$[z, z+2x-2] \subset S_1(0)+S_1(z),$$

$$[z+2x-1, z+yx+x-2] \subset S_1(0)+S_2(z),$$

$$[z+yx+x-2, z+2yx+x-3] \subset S_2(0)+S_3(z),$$

$$[z+2yx+x-2, z+2yx+2x-4] \subset S_3(0)+S_3(z).$$

LEMMA 2. *For any integer z we have*

$$T(0)+B(z,x-1) \supset [z, z+yx+x^2-x-1].$$

PROOF. We divide the proof of lemma 2 into three parts:

$$[z, z+x^2-x] \subset S_1(0)+B(z,x-1),$$

$$[z+x^2-x, z+yx-1] \subset S_2(0)+B(z,x-1),$$

$$[z+yx, z+yx+x^2-x-1] \subset S_3(0)+B(z,x-1).$$

The first and last of these are straightforward. To prove the second, let $u \in [z+x^2-x, z+yx-1]$. Then $u = z+lx-m$ where $1 \leqq m \leqq x$ and $x \leqq l \leqq y$. Hence

$$u = ((l-m+1)x-1)+(z+(m-1)(x-1)) \in S_2(0)+B(z,x-1)\,.$$

LEMMA 3. *Let x, y and l be positive integers, $y \geqq x \geqq 1$ and $l \geqq 1$. Let $u_i = 2yx + 2x - 3 + i(yx+x^2-x)$ for $i = 0, 1, \ldots, l$.* Let

$$A = T(0) \cup T(u_l) \cup \bigcup_{i=0}^{l-1} B(u_i, x-1) - \{0\}\,.$$

Then
$$|A| = (l+4)x+2y-5$$
and
$$n_2(A) = 2((l+3)yx+lx^2-(l-3)x-5)\,.$$

PROOF. Since $|T(z)| = 2x+y-2$, $|B(z,x-1)| = x$, and the sets defining A are disjoint,

$$|A| = 2(2x+y-2)+lx-1 = (l+4)x+2y-5\,.$$

By lemmata 1 and 2,

$$[1, u_0-1] \subset T(0)+T(0)$$
$$[u_i, u_{i+1}-1] \subset T(0)+B(u_i, x-1) \qquad \text{for } i = 0, 1, \ldots, l-1$$
$$[u_l, w] \subset T(0)+T(u_l)$$

where $w = u_l + yx + x - 2 = (l+3)yx + lx^2 - (l-3) - 5$ is the largest element of A. Since A is symmetric (which may easily be checked), it is well known that $[1, w] \subset A+A$ implies that $n_2(A) = 2w$. The proof is easy: let $b \in [0, w]$, then $b = 0$, $b \in A$, or $b = a_1 + a_2$ where $a_1, a_2 \in A$. Hence $2w - b = w + w, w + (w-b)$, or $(w-a_1)+(w-a_2)$ respectively. Since A is symmetric, $w - a \in A$ when $a \in A$. Hence $2w - b \in A + A$. This completes the proof of lemma 3.

We can now prove the theorem. Let $k \geqq 19$ be given. In lemma 3 let $l = 9$ and x be the integer with the opposite parity of k which is closest to $(k+4)/23$. Finally let $y = \frac{1}{2}(k-13x+5)$. Then $|A| = k$. Further, if $x = ((k+4)/23)+\theta$, then $|\theta| \leqq 1$,

$$y = \frac{5k}{23}+\frac{63}{46}-\frac{13}{2}\theta \quad \text{and} \quad n_2(A) = \frac{6}{23}k^2+\frac{48}{23}k-\left(138\theta^2+\frac{134}{23}\right).$$

This proves the theorem. A closer look at the expression for $n_2(A)$ shows that $n_2(A) \geqq \frac{6}{23}k^2$ for $k \geqq 46$. Using other bases we can show that $n_2^*(k) > \frac{6}{23}k^2$ for $k \leqq 45$ as well. Hence this is true for all k.

For any positive value of l, if x is close to $(l+3)k/(2l^2+10l+24)$ and y is chosen such that $|A|=k$, then we get

$$n_2(A) = \frac{l^2+6l+9}{4(l^2+5l+12)}k^2+O(k)\ .$$

The coefficient for k^2 takes its largest value for $l=9$.

The construction used in [3] to prove that $n_2(k)>\frac{2}{7}k^2$ is closely related to the construction in this paper. In the notation of lemma 3, if $A=T(0) \cup B(u_0,x-1)\cup B(u_1,x-1)$, where x is close to $k/7$ and y close to $3k/7$ we get $n_2(A)=\frac{2}{7}k^2+O(k)$.

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