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ON CONFIGURATIONS IN 3-SPACE WITHOUT ELEMENTARY PLANES AND ON THE NUMBER OF ORDINARY PLANES

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1. Introduction.

In the article [1] Bonnice and Kelly give an interesting example of a class of configurations in ordered projective 3-space not containing elementary planes, [1, p. 46], (i.e. planes containing exactly three points of the configuration). As proved by Motzkin for 3-space [4, p. 454] and by myself in general for d-space [2], any configuration of a finite point set not having all its points lying in one $(d-1)$-plane must contain at least one ordinary $(d-1)$-plane, i.e. a $(d-1)$-plane, such that all but one of the points of the configuration in the $(d-1)$-plane are contained in one $(d-2)$-plane.

The term configuration is used here in the sense defined in [2, p. 176].

Already Motzkin in [4, p. 452] observed, that there are configurations in 3-space without elementary planes, for example configurations consisting of points placed exclusively on two skew lines with at least 3 points on each line. Obviously there are no elementary planes. Further he noted that the Desargues configuration in 3-space with 10 points is also without elementary planes. As mentioned Bonnice and Kelly give a further class of examples.

The first aim of the present paper is to give still more examples of configurations in 3-space not containing elementary planes.

In the same article Bonnice and Kelly develop a method for estimating a lower bound on the number of ordinary planes for configurations with n points, and they prove, that this number must be at least $\frac{3}{11}\cdot n$, and that if the condition, that there is no elementary plane in the configuration, be added, then the number of ordinary planes must be at least $\frac{3}{7}n$.

The second aim of the present paper is to give some improvement of these results.

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2. Configurations in 3-space without elementary planes.

The Bonnice–Kelly example, which can be regarded as a generalized Desargues configuration, is as follows:

Consider in ordinary 3-space (completed with a plane at infinity) a prism whose bases are oddsided regular polygons. Let the number of sides be p. Take the 2 p vertices of the prism, the p points at infinity which are the intersections of the p pairs of parallel polygon sides and the plane at infinity, and the point common to all the sidelines of the prism and the plane at infinity. The configuration thus constructed has no elementary planes. For $p=3$ the configuration is the Desargues configuration.

We shall now give some further examples as promised:

2.1. As in the example just described we start with a prism in E, whose parallel bases are regular polygons with $p=4k$, $k \in \mathbf{Z}$, vertices. Let $\{P_i\}$ and $\{Q_i\}$, $i=1,2,\ldots,p$, be the respective vertex sets of these polygons labelled in cyclic order in such a way that the lines P_iQ_i are sidelines of the prism.

Let B_i be the point at infinity on the line P_iP_{i+1} and let A_i be the point at infinity on the line $P_{i-1}P_{i+1}$, all indices on the vertices taken mod p and on the line at infinity mod $p/2$. Note, that a line P_iA_i or $P_{i+p/2}A_i$ has precisely the point P_i, resp. $P_{i+p/2}$, in common with the P-polygon. Such lines will be labelled "tangents" to the polygons. Take the points P_i, Q_i, A_i, B_i and adjoin to this collection the center, O, of the P-polygon, the center, C, of the prism and the point, D, at infinity common to the lines P_iQ_i.

This collection of $3\cdot p+3$ points defines a configuration $\Gamma \equiv (\Gamma_0, \Gamma_1, \Gamma_2)$ of points, lines and planes such that no plane in Γ_2 is elementary. This can be seen as follows:

Obviously there is no elementary plane containing D. The same applies to C, since these two points play projectively symmetric roles.

There can be no elementary planes through any B_i, since all ordinary lines through B_i pass through C or D.

Planes containing any A_i can only be elementary, if they contain exactly two ordinary lines through A_i, and as lines through C or D are excluded, these should be lines P_iA_i and Q_iA_i. This is true, since line OA_i contains $P_{i+p/4}$. But two tangents through the same A_i either belong to the same polygon, or they span a plane containing C (e.g. A_iP_i and $A_iQ_{i+p/2}$) or D (e.g. A_iP_i and A_iQ_i), and in neither case can the plane be elementary.

Planes not passing through C, D or any B_i or A_i must have two points either from $\{Q_i \mid i=1,\ldots,p\}$ or from $\{O\} \cup \{P_i \mid i=1,\ldots,p\}$, but as two such points never span an ordinary line, such a plane cannot be elementary. This concludes the proof.

2.2. Now add to the configuration just described the center, T, of the polygon $Q_1 \ldots Q_p$. This new configuration with $3p+4$ points has no elementary planes.

2.3. Take again the configuration from 2.1 for the case $p \geqq 8$, and remove the points B_i, $i=1,\ldots,p/2$, as well as O. Again we have a configuration without elementary planes. The number of points is $\frac{5}{2}p+2$.

We shall leave the proofs for 2.2 and 2.3 to the reader.

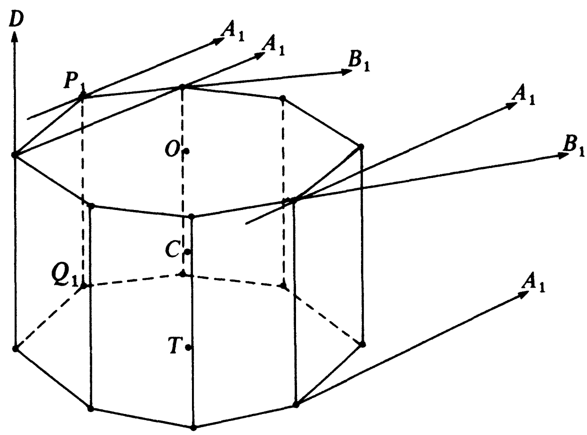

Fig. 1.

In order better to assess the symmetrical nature of the described configurations we will consider them in another projection: Let the plane containing $\{Q_1,\ldots,Q_p\}$ be the plane at infinity, and let C and D be placed as top points of a double pyramid with the polygon $P_1,\ldots,P_p$ as the common base. Then the Q_i's become intersections between the plane at infinity and the sidelines of the pyramids. The points A_i are intersections between the plane at infinity and those polygon diagonals, which are not parallel to any of the polygon sides, and the points B_i are intersections between the plane at infinity and the polygon sides and those diagonals, which are parallel to them. O is the center of the double pyramid, and T is the intersection of its axis and the plane at infinity.

In connection with the case $p=4$ we shall make some further observations. Consider the figure in fig. 2. This configuration, Γ, can be viewed as consisting of two parts, namely a cube $P_1 \ldots P_4 Q_1 \ldots Q_4$ with its center C and the 3 points D, B_1 and B_2, where its edge lines intersect the plane at infinity, added, and an octahedron, $M_1N_1M_2N_2OT$, inscribed in the cube with the 6 points A_1, A_2, E_1, E_2, F_1 and F_2, where its edge lines intersect the plane at infinity, added.

We call each of these parts a "basic system", and we note, that any basic

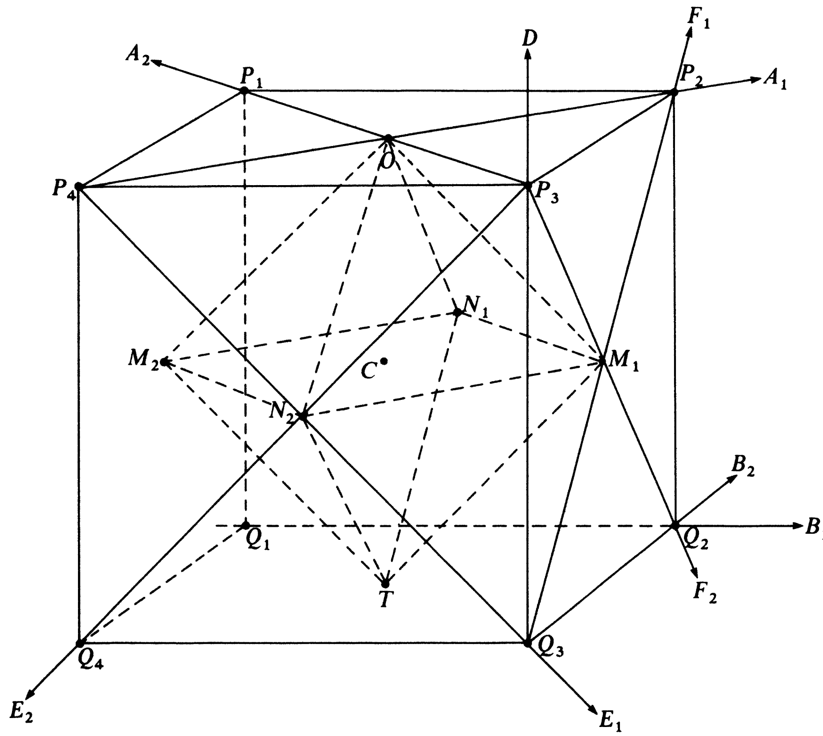

Fig. 2.

system can be construed in several ways as an octahedron with 6 points added as well as a cube with 4 points added as above. If in a given projection a basic system is pictured in the cubic way, then the complementary system in the same projection is pictured in the octahedronic way.

We shall not here discuss the structure of the configuration in detail, but we will name some properties, which might be of importance for the study of elementary planes:

All points in the total system play analogous roles. The same applies to the points of each basic system alone. All the points of one basic system sit in equivalent positions in relation to the other basic system.

Let us for a moment consider the lines of the configuration through a point of the configuration. We need only consider lines through one point, say D, and we easily observe the following:

The lines of Γ_1 through D are of 3 types: 1) 4-point-lines so as $DOCT$ containing one pair of points from one of the basic systems and one pair from the other so situated, that the two pairs separate each other on the line: $DC \parallel OT$. 2) 3-point-lines so as DP_1Q_1 containing points exclusively from one basic system, and 3) ordinary lines so as DA_1 containing a point from each basic system.

It follows from these observations, that we can determine, to which of the two basic systems any given point, X, belongs, by considering the line DX and the position of X on DX. If for instance DX is a line of type 1), containing the points D, X, Y and U, then X belongs to the same system as D, if $DX \parallel YU$ and to the other, if $DY \parallel XU$ or $DU \parallel XY$.

From this we conclude, that the total system can be divided into two basic systems in one way only.

2.4. The configuration described has no elementary planes. This follows from the fact, that if any such plane did exists, it would contain 3 points of Γ_0 spanning 3 ordinary lines, and thus at least one of these lines would contain two points from the same basic system. This however establishes a contradiction to the observations above.

Obviously the examples 2.1. and 2.2. for the case $p=4$ can be obtained by adding 3 or 4 points to a basic system.

For instance if we take the system, which on figure 2. is pictured in the cubic way, and add the points A_1, A_2 and O, we have 2.1., and if we further add T, we have 2.2.

We shall now in 2.5. and 2.6. give two further examples of configurations without elementary planes. In both cases we will leave the verification to the reader.

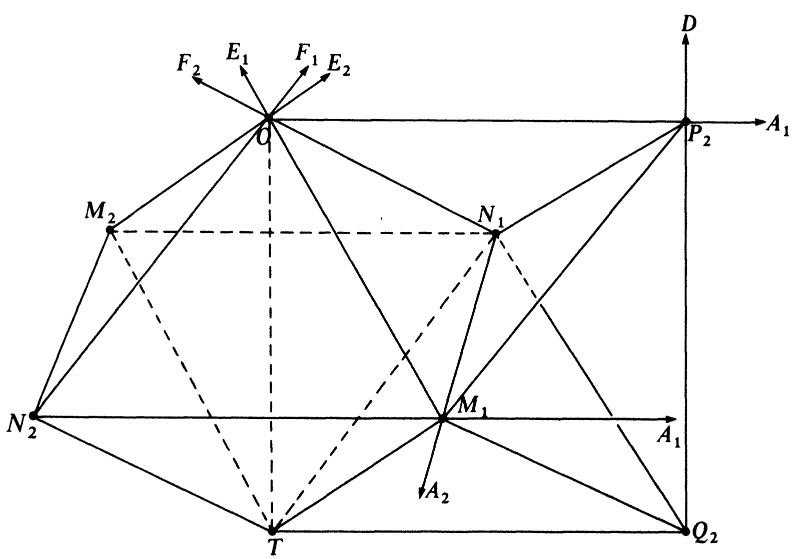

Fig. 3. A picture of 2.5.

2.5. Take a basic system. Add the points of any 3-point-line from the complementary basic system. The resulting configuration is without elementary planes.

All configurations created in this way are isomorphic, since all 3-point-lines from the complementary system clearly play similar roles relative to the given basic system.

2.6. In order to explain our next example we will make an examination of the planes of a basic system. Obviously there are two types: elementary planes, of which there are 3 through any point, and 6-point-planes containing a complete quadrangle with 2 diagonal points. There are, through each point, 6 of these planes. To see this, look for instance at the center of the cube in a basic system pictured in the cubic way.

All the elementary planes play similar roles in relation to the basic system, and the same applies to all the 6-point-planes.

All points on a 6-point-plane play similar roles in relation to the whole point set on the 6-point-plane.

Any 6-point-plane can be construed as the union of (the point sets on) 4 3-point-lines, which all play similar roles in relation to the 6-point-plane.

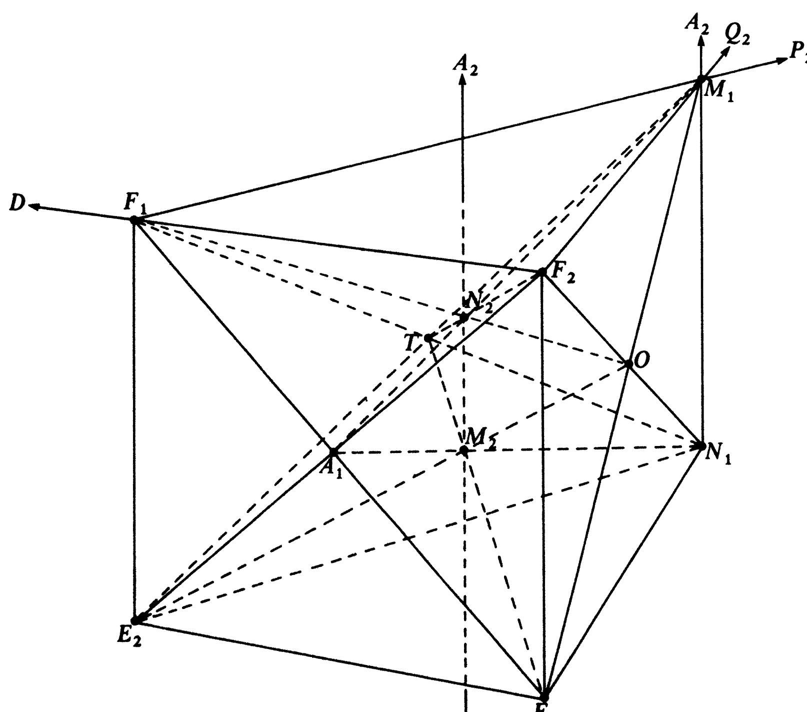

Fig. 4. Another picture of 2.5.

Now for our example: take a basic system and add 6 points from one 6-point-plane from the complementary system. Again the resulting system is without elementary planes and all such systems are isomorphic.

2.7. In our next example, we shall consider a regular icosahedron.

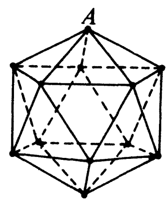

Fig.5.

Take the regular icosahedron, add to its 12 points its center, and add further the 15 points where the 30 edges, pairwise parallel, meet the plane at infinity. The resulting configuration of 28 points has no elementary planes.

2.8. Consider a regular dodecahedron. Add also here its center and the 15 points, where the 30 edges meet the plane at infinity. The resulting configuration with 35 points has no elementary planes.

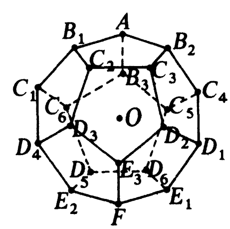

Fig. 6.

The proofs for 2.7 and 2.8 are quite long, so we shall omit them.

3. On the number of ordinary planes in 3-space.

We shall in the following again consider a configuration in ordered projective 3-space with the property, that the points of the configuration do not belong to one plane. The set of points from the configuration will be named Γ_0, the set of lines belonging to the configuration, i.e. the lines spanned by points of Γ_0, will be named Γ_1, and the set of planes will be named Γ_2. The configuration itself will be called Γ. Points, lines, planes from Γ will usually be denoted A_i, B_i etc. or similar, where i denotes the dimension.

It is our aim to prove the following two theorems:

THEOREM 3.1. *The number of ordinary planes of a non-planar configuration with n points is at least* $\frac{2}{5} \cdot n$.

THEOREM 3.2. *The number of ordinary planes in a non-planar configuration with n points in which all the ordinary planes are non-elementary is at least* $\frac{6}{11} \cdot n$.

DEFINITION 3.3. A plane A_2 of Γ_2 is called a "first-met plane" for the point A_0 of Γ_0, if there exists a point P in A_2, but not in any other plane of Γ_2, such that one of the open segments A_0P intersects no plane of Γ_2.

A_2 is said to be first-met "in P".

The lines of Γ_1 contained in A_2 divide A_2 in 2-cells. A_2 is said to be first-met for A_0 "in the cell δ_2" of A_2, if and only if A_2 is first-met in a point P of (the interior of) δ_2.

Note, that we cannot exclude the possibility, that A_2 is first-met for A_0 in more than one of its cells.

It will be convenient to adopt some of the terminology of [1] and [3]. In particular the concept of polyhedral residence of a point A_0 of Γ_0 is useful. This concept is defined [1; p. 47] in the following way: The set of planes of Γ_2 not containing A_0 will generally partition the space into polyhedral domains. The polyhedral residence of A_0 is that polyhedral domain which contains A_0. The only cases in which such a partitioning is not affected are: 1) Γ_0 is subset of two skew lines, and 2) all points of Γ_0 with one exception are on a plane. The theorems are easily checked in these special cases [1, p. 52], so henceforth we will assume that each point A_0 of Γ_0 has a polyhedral residence, which we will denote $\mathscr{P}(A_0)$.

DEFINITION 3.4. If A_2 is an ordinary plane of Γ_2 with all its points of Γ_0 except one, A_0, on a line A_1, then A_0 is called "the leader on A_2 for A_1", and A_1 is "the follower on A_2 for A_0".

Note, that if A_2 is elementary, then each vertex of the triangle formed by the three points of Γ_0 in A_2 is a leader on A_2 for the opposite side line, and each side line is the follower on A_2 for the opposite vertex.

We will associate ordinary planes and points of Γ_0 to each other by the following definitions:

DEFINITION 3.5. An ordinary plane A_2 is an "association plane" for a point A_0 of Γ_0 if either:

1) A_2 is an ordinary face plane of $\mathscr{P}(A_0)$, or

2) A_0 is a leader on A_2, whose follower A_1 on A_2 is in a non-ordinary face plane of $\mathscr{P}(A_0)$.

In both cases the face plane mentioned is said to "give rise to" the association of A_2.

If 1) holds, A_2 is called a "neighbour" or "neighbour plane" for A_0, and if 2) holds, A_2 is called a "member" for A_0.

DEFINITION 3.6. A point A_0 of Γ_0 is an "association point" for a plane A_2 of Γ_0, if and only if A_2 is an association plane for A_0. If A_2 is neighbour for A_0, A_0 is called "neighbour" or "neighbour point" for A_2.

Now let $m(A_0)$ be the number of members for A_0, $l(A_0)$ the number of neighbour planes for A_0 and $o(A_0)$ the number of association planes for A_0. Then of course $o(A_0)=m(A_0)+l(A_0)$. In the same way the number of association points for A_2 shall be named $o(A_2)$.

DEFINITION 3.7. The number $o(A_0)$ shall be called "the order" or "association order" for the point A_0 of Γ_0. The number $o(A_2)$ shall in the same way be called the "order" or "association order" for the plane A_2 of Γ_2.

NOTE. The use of the term order is slightly at variance with its use in [3].

SUBTHEOREM 3.8. *If A_2 is a non-ordinary plane of Γ_2 and δ_2 is a 2-cell* [2, p. 176] *in the partitioning of A_2 by lines of Γ_1 in A_2, then there is a line from Γ_1 in A_2 which has nothing in common with δ_2.*

PROOF. Since the plane is not ordinary, it can contain at most one "forkpoint", i.e. a point of Γ_0 on two lines of Γ_1, whose union contains all points of $\Gamma_0 \cap A_2$. So there must be at least 3 non-forkpoints of Γ_0 in the plane forming the vertices of a proper triangle. Let σ_2 be that domain of the triangle which includes δ_2. Now if $\delta_2=\sigma_2$, all points of Γ_0 apart from σ_2's vertices must lie on the extensions of σ_2's sides, and they cannot be confined to one side line, since the plane is not ordinary. Clearly any two such points taken from different side lines must span a line as required. If on the other hand $\sigma_2 \supset \delta_2$, then there is a vertex of σ_2, not in δ_2, and as it is not a forkpoint, there must be at least 3 lines of Γ_1 through it, and thus at least one which is not a side line of σ_2. This line meets our requirements.

For another proof see [1, p. 50].

Now the proof of the theorem will be based on the following subtheorem 3.9., which in turn is based on Lemma 2 of [2, p. 179], which asserts, that every first-met plane for a point A_0 from Γ_0 gives rise to an association plane of A_0. If specifically the first-met plane, A_2, is ordinary, then it is a neighbour for A_0. And if it is not ordinary, then subtheorem 3.8. guarantees a line A_1 of Γ_1 in A_2, running outside a 2-cell δ_2, in which A_2 is first-met, and then lemma 2 [2, p. 179] assures us that A_0A_1 is an ordinary plane with A_0 as leader and with its follower, A_1, on A_2, i.e. A_0A_1 is a member for A_0. Note, that if A_2 gives rise to an association of a member, A_0B_1, of A_0, where B_1 touches δ_2, A_2 gives rise to at least two members of A_0.

SUBTHEOREM 3.9. *The order of any given point A_0 of Γ_0 is either at least* 4 *or A_0 has (exactly)* 3 *members.*

PROOF. First we make the following observation: If A_2 is an association plane for A_0 given rise to by more than one face plane of $\mathscr{P}(A_0)$, then A_2 must be a member of A_0, and the follower, A_1, for A_0 on A_2 must be the intersection of two face planes, which must both be non-ordinary. Obviously no third plane can give rise to the association of A_2 in this case.

Now let $p(A_0)$ be the number of face planes of $\mathscr{P}(A_0)$. It follows from the observation just made, that

$$o(A_0) \geqq \frac{p(A_0)}{2}$$

with equality only if $o(A_0)=m(A_0)$. So if $p(A_0)>6$, $o(A_0)\geqq 4$, while if $p(A_0)=6$, either $o(A_0)\geqq 4$ or A_0 has 3 members.

This leaves the cases $p(A_0)=5$ or $p(A_0)=4$ to dispose of.

In case $p(A_0)=4$, $\mathscr{P}(A_0)$ is a tetrahedron, so all the faces are adjacent. As one plane can cut away at most one edge of a tetrahedron, there can be at most one pair of non-adjacent faces.

We cannot exclude the possibility, that a non-ordinary face plane of $\mathscr{P}(A_0)$ gives rise only to such members of A_0, for which the follower contains points of the face of $\mathscr{P}(A_0)$. (The face can contain more than one cell). So even when two faces are adjacent, they can give rise to the same member, and we must count the association only half for each of the face planes. But from the remark made just before 3.9. it follows that in that case each of them gives rise to one more member.

Thus all face planes count for at least one association (perhaps two half, if the face plane is non-ordinary) except in case $p(A_0)=5$, where at most two face planes count for only one half each. It is now easy to see, that in any case we have $o(A_0)\geqq 4$.

We now have a way of counting some of the ordinary planes of Γ_2. However the planes in this count are not necessarily pairwise different, so we must now study the possibilities for repetitions in our count. That is, we must estimate the maximal order of a plane A_2 of Γ_2, $o(A_2)$.

First we make the following observation:

If A_1 is a line of Γ_1 in the plane A_2 of Γ_2, then there are two planes, A_2' and A_2'', which might collapse into one, both belonging to Γ_2 and containing A_1 and such, that A_2 and A_2', resp. A_2 and A_2'', partition the space into two components (wedges or projective halfspaces), one of which contains no points of Γ_0. A_2' and A_2'' collapse into one, if and only if Γ_0 is a subset of two planes.

We call the planes A_2' and A_2'' "tangential planes of A_2 through A_1". The two planes are called a "tangential pair".

Clearly all neighbour points of A_2 must lie in any tangential pair through any line of Γ_1 in A_2.

SUBTHEOREM 3.10. *An ordinary plane has at most* 8 *neighbour points.*

PROOF. Let $A_0^{(i)}$, $i=1,2,3$, be three non-collinear points of Γ_0 in the ordinary plane A_2. Then for any i, j, the tangential pair of A_2 through $A_0^{(i)}A_0^{(j)}$, $i \neq j$, contains all neighbour points of A_2. A particular neighbour point is an intersection point of three planes, one from each of these tangential pairs. There are at most $2^3=8$ such points. This proves the subtheorem.

Note, that in case none of the tangential pairs collapse into one plane the said set of 8 points, which are possible sites of neighbour points of A_2, are the vertices of a projective cube such, that any two opposite face planes of the cube intersect each other either in a line not in A_2 or in one of the lines $A_0^{(i)}A_0^{(j)}$, $i \neq j$. The matter is discussed in detail by R. R. Rottenberg [5].

SUBTHEOREM 3.11. *If an ordinary plane, A_2, is not elementary, then the number of neighbour points of A_2 is at most* 6.

If the follower on A_2 contains more than three points of Γ_0, the number of neighbour points of A_2 is at most 4.

PROOF. Let $A_0^{(1)}$ be the leader of A_2, while its follower contains the points $A_0^{(i)}$, $i=2,\dots,p$. Now take any two points, $A_0^{(i)}$ and $A_0^{(j)}$, from the follower. Then the three points, $A_0^{(1)}$, $A_0^{(i)}$ and $A_0^{(j)}$, determine a cube as above. In case at least 7 vertices of this cube belong to Γ_0, they can be vertices of no other cube determined by $A_0^{(1)}$ and two other points of the follower. This proves the first part of the subtheorem. (All is trivial, if any pair collapses into one plane).

In case only 6 points of the cube belong to Γ_0, and provided they all lie on three of the four cube edges through $A_0^{(1)}$, they can be vertices of three cubes determined by $A_0^{(1)}$ and two points of the follower. In this case the follower can contain three points of Γ_0, $A_0^{(2)}$, $A_0^{(3)}$ and $A_0^{(4)}$. The three cubes are one for each of the $\binom{3}{2}$ pairs of points from this set. On figure 7 we have illustrated that by marking the intersections of an arbitrary plane not through $A_0^{(1)}$ with the cube edges through $A_0^{(1)}$ and with the "legs" on A_2, i.e. the lines connecting the leader with the points of its follower.

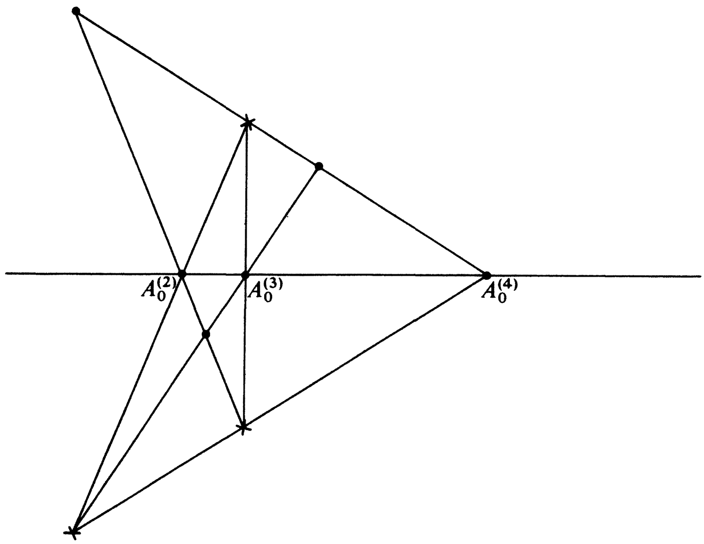

Fig. 7. The crosses are on the edges, where the six vertices of Γ_0 lie. The points $A_0^{(i)}$ are on the legs. The dots are the three possible sites for (the intersection with) the fourth cube edge.

The second part of the subtheorem now follows immediately. We make no use of the second part of 3.11 in this paper.

Our final subtheorem concerns the number of possible neighbour points of an elementary plane, if that plane is also a member of one of its points:

SUBTHEOREM 3.12. *If A_2 is an elementary plane which is a member for at least one of its points A_0, then A_2 has at most* 6 *neighbour points.*

PROOF. Let A_0, B_0 and C_0 be the three points of Γ_0 in A_2. The neighbour points of A_2 are a subset of the set of vertices of a projective cube. By a projective transformation this cube can be taken into an euclidean cube, where A_0 is the center of the cube, while B_0 and C_0 are points at infinity on two sets of parallel edges.

By definition of member, there must through the line B_0C_0 be a plane of Γ_2, B_2, which is a first-met plane for A_0. The two face planes of the cube through B_0 and C_0 are a tangential pair through the line B_0C_0, so if such a B_2 exists, it must be one of these face planes. If we suppose, that A_2 has at least 7 neighbour points, these two opposite faces contain a pair of non-parallel diagonals, whose end points are in Γ_0. Now there is through each of these diagonals a set of two planes of Γ_2, one through each end point of the other diagonal. These two planes separate A_0 from the face plane containing their intersection line. Thus no plane through B_0C_0 is first-met for A_0 under the assumption of 7 neighbour points, and the subtheorem is proved.

SUBTHEOREM 3.13. *The order of an elementary plane is at most 9. The order of a non-elementary, ordinary plane is at most 7.*

PROOF. If an elementary plane is not a member for any of its points, then the order is at most 8. If it is member for any of its points, then the order is at most $6+3=9$.

As a non-elementary plane can be member only for its leader and have at most 6 neighbours, the order is at most 7.

PROOF OF THEOREM 3.1. The number of points in Γ_0 is n. Let the number of points of Γ_0, having 3 members be n_1. Then because of subtheorem 3.9, there must be $n_2=n-n_1$ points of Γ_2 having the order at least 4. The number of ordinary planes shall be called m.

Now we will count the number of pairs consisting of a point of Γ_0 and one of its members. Because of n's definition that number must be *at least* $n_1\cdot 3$. On the other hand it must be *at most* $m\cdot 3$, since for each ordinary plane there are at most 3 of its points for which it is a member. Thus we have

$$n_1\cdot 3 \leqq m\cdot 3 . \tag{1}$$

Then we will count the number of pairs consisting of a point of Γ_0 and a plane to which it is associated. That number must be *at least* $n_1\cdot 3+n_2\cdot 4$. On the other hand it must be *at most* $m\cdot 9$, since the order for any ordinary plane because of subtheorem 3.13 is at most 9. So we have:

$$n_1\cdot 3+n_2\cdot 4 \leqq m\cdot 9 . \tag{2}$$

Now we obtain by dividing in (1) with 3 and adding (1) and (2):

$$n_1\cdot 4+n_2\cdot 4 \leqq m\cdot 10$$

and as $n_1+n_2=n$, we have:

$$\tfrac{2}{5}n \leqq m .$$

That proves theorem 3.1.

PROOF OF THEOREM 3.2. We proceed in the same way as in the former proof. With similar notation we have, since a non-elementary plane can be a member only for its sole leader:

$$(3) \qquad n_1 \cdot 3 \leqq m$$

and further, because of subtheorem 3.13, we have:

$$(4) \qquad n_1 \cdot 3 + n_2 \cdot 4 \leqq m \cdot 7 .$$

Now we obtain:

$$(n_1 + n_2)4 \leqq m \cdot \tfrac{22}{3}$$

and so

$$\tfrac{6}{11} \cdot n \leqq m .$$

That proves theorem 3.2.

Obviously the result just obtained can be further improved, if we assume that there are no ordinary planes with less than 4 points of Γ_0 on its follower. However, as there for the present is no strong motive for doing assumptions of that sort, we shall refrain from stating such an improved result in a formal theorem.

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