MATH. SCAND. 47 (1980), 195–231

SK_1 FOR FINITE GROUP RINGS: II

ROBERT OLIVER*

Let π be a p-group, and let A be a *p-ring*: the ring of integers in some finite extension F of the p-adic rationals $\hat{\mathbf{Q}}_p$. Our goal is to study the groups

$$SK_1(A\pi) = \operatorname{Ker}[K_1(A\pi) \to K_1(F\pi)]$$

$$\operatorname{Wh}(A\pi) = K_1(A\pi)/(A^* \times \pi^{ab}) \text{ (where } \pi^{ab} = \pi/[\pi,\pi]\text{), and}$$

$$\operatorname{Wh}'(A\pi) = \operatorname{Wh}(A\pi)/SK_1(A\pi) .$$

The main result is the following computation of $SK_1(A\pi)$:

THEOREM 3. *For any p-ring A and p-group* π,

$$SK_1(A\pi) \cong H_2(\pi)/H_2^{ab}(\pi) ,$$

where $H_2^{ab}(\pi)$ is the subgroup of $H_2(\pi)$ generated by all $H_2(\varrho)$ for abelian $\varrho \subseteq \pi$.

In particular, this can be applied when computing $SK_1(\mathbf{Z}\pi)$, shown by Wall [10] to be the torsion subgroup of $\operatorname{Wh}(\pi)$. There are short exact sequences

$$0 \to \operatorname{Cl}_1(\mathbf{Z}\pi) \to SK_1(\mathbf{Z}\pi) \to SK_1(\hat{\mathbf{Z}}_p\pi) \to 0$$

where $\operatorname{Cl}_1(\mathbf{Z}\pi)$ can be studied using K_2 (see [6]).

By a result of Wall's [10], $\operatorname{Wh}'(A\pi)$ is torsion-free for any p-ring A and p-group π. Hence

$$K_1(A\pi) \cong SK_1(A\pi) \times A^* \times \pi^{ab} \times \operatorname{Wh}'(A\pi)$$

is completely described (as an abstract group) by Theorem 3 and results in [8] and [10]. Additional control over the structure of $\operatorname{Wh}'(A\pi)$ is given by:

THEOREM 2. *For any* unramified *(over $\hat{\mathbf{Z}}_p$) p-ring A and any p-group* π, *set*

$$I(A\pi) = \operatorname{Ker}[A\pi \to A]; \quad \overline{I(A\pi)} = I(A\pi)/\langle x - gxg^{-1} \mid x \in I(A\pi),\ g \in \pi\rangle .$$

* The author was supported by a Sloan fellowship while part of this work was being done.

Received March 4, 1980.

Then there is a short exact sequence (natural in π and A)

$$0 \to \mathrm{Wh}'(A\pi) \xrightarrow{\Gamma} \overline{I(A\pi)} \xrightarrow{\omega} \pi^{\mathrm{ab}} \to 0\,,$$

where $\omega(\sum \lambda_i g_i) = \prod [g_i]^{\mathrm{Tr}(\lambda_i)}$ (Tr: $A \to \hat{\mathbf{Z}}_p$ *the trace map*).

One consequence of Theorem 2 is a simple formula for $H^1(\mathbf{Z}_2;\ \mathrm{Wh}'(\hat{\mathbf{Z}}_2\pi))$ (Proposition 13) when π is a 2-group and $\mathrm{Wh}'(\hat{\mathbf{Z}}_2\pi)$ has the involution induced by $g \to g^{-1}$ (answering a question of Wall's in [11]).

Note that the formula for $SK_1(A\pi)$ in Theorem 3 is independant of the p-ring A. If B/A is a totally ramified extension, the inclusion $A\pi \subseteq B\pi$ induces an isomorphism $SK_1(A\pi) \cong SK_1(B\pi)$. If, on the other hand, B/A is unramified, it is the transfer map

$$\mathrm{trf}\colon SK_1(B\pi) \to SK_1(A\pi)$$

which is an isomorphism.

Interestingly enough, $H_2(\pi)/H_2^{\mathrm{ab}}(\pi)$ has already been studied by Neumann [5], in relation to "cutting and pasting" ("Schneiden und Kleben") groups $SK_*(X)$ for topological spaces X. Upon combining his Theorem 2 with Theorem 3 here, one gets the amusing formulation

$$SK_1(A\pi) \cong SK_2(B\pi)\,.$$

A skerch of the organization of the paper now follows. In Section 1, generators and a partial set of relations are constructed for

$$\mathrm{Ker}\,[SK_1(A\tilde{\pi}) \to SK_1(A\pi)]$$

when $1 \to \mathbf{Z}_p \to \tilde{\pi} \to \pi \to 1$ is a short exact sequence of p-groups. They suffice, for instance, to show that $SK_1(A\pi) = 0$ when π has a normal abelian subgroup with cyclic quotient. Combining this with results in [6] gives a formula for $SK_1(\mathbf{Z}[\mathbf{Z}_n \times \pi])$ for any odd n and any dihedral, quaternionic, or semidihedral 2-group π.

Section 2 deals with the construction of the exact sequence

$$0 \to \mathrm{Wh}'(A\pi) \xrightarrow{\Gamma} \overline{I(A\pi)} \xrightarrow{\omega} \pi^{\mathrm{ab}} \to 0$$

of Theorem 2. Γ is defined by composing the p-adic logarithm

$$\log\colon \mathrm{Wh}'(A\pi) \to \overline{I(A\pi)} \otimes \mathbf{Q}$$

with a linear automorphism of $\overline{I(A\pi)} \otimes \mathbf{Q}$.

This is then used in Section 3 to calculate $SK_1(A\pi)$. For any extension

$$1 \to \varrho \to \tilde{\pi} \to \pi \to 1$$

of p-groups, subgroups $\varrho_1 \subseteq \varrho_0 \subseteq \varrho$ are defined: $\varrho_0 = \varrho \cap [\tilde{\pi}, \tilde{\pi}]$, and

$$\varrho_1 = \langle z \in \varrho \mid z = ghg^{-1}h^{-1}, \text{ some } g, h \in \tilde{\pi} \rangle .$$

The exact sequence of Theorem 2 (applied to ϱ, $\tilde{\pi}$, and π) is combined with some diagram chasing to produce a natural isomorphism

$$\operatorname{Coker}[SK_1(A\tilde{\pi}) \to SK_1(A\pi)] \cong \varrho_0/\varrho_1 .$$

$H_2(\pi)/H_2^{ab}(\pi)$ is then shown to be the "universal" ϱ_0/ϱ_1 for extensions of π, yielding a natural surjection

$$\Theta_{A\pi}: SK_1(A\pi) \to H_2(\pi)/H_2^{ab}(\pi) .$$

Finally, $\Theta_{A\pi}$ is shown to be an isomorphism for all A and π.

This is applied in Section 4 to calculate $SK_1(A\pi)$ in a number of cases. "Minimal" p-groups π with $SK_1(A\pi) \neq 0$ are constructed: these occur for $|\pi| = 64$ if $p=2$, or $|\pi| = p^5$ if p is odd. Relations

$$SK_1(A[\pi_1 \times \pi_2]) \cong SK_1(A\pi_1) \oplus SK_1(A\pi_2), \quad SK_1(A[\pi \wr \mathbf{Z}_p]) \cong SK_1(A\pi)$$

($\pi \wr \mathbf{Z}_p$ the wreath product) are also shown.

In a later paper, these results will be applied and extended to get similar formulas for $SK_1(A\pi)$ when π is an arbitrary finite group (and A still a p-ring). In particular, the results here in Section 2 apply to show that $SK_1(A\pi)$ is generated by induction from elementary subgroups of π. Combining that with Theorem 1 in [6] will show that induction from elementary subgroups also holds for $SK_1(\mathbf{Z}\pi)$.

Some final comments on notation. $G \rtimes H$ always denotes a semidirect product where G is normal. By a *commutator* will always be meant a group element of the form $ghg^{-1}h^{-1}$; *not* an arbitrary product of such elements, Brackets $\langle\ \rangle$ are used for "subgroup generated by". And "Tr" is used to denote trace maps, "trf" to denote transfer maps.

ACKNOWLEDGEMENT. I would like to thank the Math. Department at M.I.T., and especially Frank Peterson and Ed Miller, for their hospitality while much of this work was being done. I also thank the referee for his many helpful comments.

1.

The following lemma, consisting of results of Wall and Swan, is the starting point for studying $K_1(\hat{\mathbf{Z}}_p\pi)$:

LEMMA 1. *Let $\mathfrak{A}$ be any $\hat{\mathbf{Z}}_p$-order in a finite-dimensional semisimple $\hat{\mathbf{Q}}_p$-algebra, and $J \subseteq \mathfrak{A}$ any two-sided ideal.*

(i) *For any n,* $\operatorname{Ker}[\mathrm{GL}_n(J) \to K_1(\mathfrak{A}, J)]$ *is closed in* $\mathrm{GL}_n(J)$ *(in the p-adic topology). In particular,*

$$K_1(\mathfrak{A}, J) = \varprojlim K_1(\mathfrak{A}/p^n\mathfrak{A}, J/(J \cap p^n\mathfrak{A})) .$$

$K_1(\mathfrak{A}, J)$ *is the product of a pro-p-group with a group of order prime to p, and so* $K_1(\mathfrak{A}, J)_{(p)}$ *can be regarded as a* $\hat{\mathbf{Z}}_p$*-module.*

(ii) *If J has finite index in* $\mathfrak{A}$*, then*

$$\mathrm{rk}_{\hat{\mathbf{Z}}_p}(K_1(\mathfrak{A}, J)) = \mathrm{rk}_{\hat{\mathbf{Z}}_p}(K_1(\mathfrak{A})) = \mathrm{rk}_{\hat{\mathbf{Z}}_p}(Z(\mathfrak{A})) ,$$

where $Z(\mathfrak{A})$ *denotes the center of* $\mathfrak{A}$.

PROOF. (i) See Theorem 10 (and its proof) in [9]. The relative case follows from the absolute case by definition:

$$K_1(\mathfrak{A}, J) = \operatorname{Ker}[K_1(\mathfrak{B}) \to K_1(\mathfrak{A})]$$

for some p-adic order $\mathfrak{B} \subseteq \mathfrak{A} \times \mathfrak{A}$.

(ii) Let $\mathfrak{M} \supseteq \mathfrak{A}$ be a maximal order, and $Z(\mathfrak{M})$ its center. By ([8], Theorem 8.10 and Proposition 8.11), the maps

$$K_1(\mathfrak{A}, J) \to K_1(\mathfrak{A}) \to K_1(\mathfrak{M}) \to K_1(Z(\mathfrak{M}))$$

all have torsion kernel and cokernel; and the p-adic logarithm shows that $K_1(Z(\mathfrak{M}))$ and $Z(\mathfrak{M})$ have the same rank (see [10], Proposition 1.6).

Now let π be a p-group, and A a p-ring with maximal ideal $\mathfrak{p}_A$. By [1, Corollary XI.1.4], $A\pi$ is a local ring with maximal ideal

$$M_{A\pi} = \{\sum \lambda_i g_i \in A\pi \mid \sum \lambda_i \in \mathfrak{p}_A\} = \operatorname{Ker}[A\pi \xrightarrow{\varepsilon} A \to A/\mathfrak{p}_A] .$$

In particular, any element not in $M_{A\pi}$ is a unit; so matrices over $A\pi$ are diagonalizable, and $\mathrm{Wh}(A\pi)$ is generated by units.

Let

$$I = I(A\pi) = \operatorname{Ker}[\varepsilon\colon A\pi \to A]$$

denote the augmentation ideal. Consider the exponential and logarithmic functions

$$\operatorname{Exp}(x) = 1 + x + x^2/2! + x^3/3! + \dots$$

$$\operatorname{Log}(1+y) = y - y^2/2 + y^3/3 - \dots$$

for any $x, y \in I$ for which the series converge. They are clearly inverses where defined. There will be reason to distinguish two types of function: Exp and Log will be used to denote the functions between (subgroups of) $I(A\pi)$ and

$1+I(A\pi)$, and exp and log to denote induced homomorphisms involving subgroups of $K_1(A\pi, I)$.

For any A and π, set

$$\overline{I(A\pi)} = I(A\pi)/\text{conjugation} = I(A\pi)/\langle x-gxg^{-1} \mid x \in I,\ g \in \pi\rangle$$

(dividing out by the sub*group* generated by such $x-gxg^{-1}$). $\overline{I(A\pi)}$ can be regarded as the free A-module with basis

$$\{1-g_i \mid i=1,\ldots,k\}\ .$$

where $g_1,\ldots,g_k$ are conjugagy class representatives for non-identity elements $g \in \pi$.

PROPOSITION 2. *Let A be a p-ring, π a p-group, and $I=I(A\pi)$. Then the exponential map induces an isomorphism*

$$\exp\colon p\overline{I(A\pi)} \to K_1(A\pi, pI)\ .$$

PROOF. Let $\{e=g_0, g_1,\ldots,g_k\}$ be conjugagy class representatives for π, and $\{\lambda_1,\ldots,\lambda_l\}$ a $\hat{\mathbf{Z}}_p$-basis for A. The set

$$\{1+p\lambda_i(1-g_j) \mid 1 \leqq i \leqq l,\ 1 \leqq j \leqq k\}$$

is easily seen to generate $(1+pI(A\pi))/(1+p^nI(A\pi))$ (mod commutators) for any n; so by Lemma 1 (i) it generates $K_1(A\pi, pI)$. Since

$$K_1(A\pi, pI) \times (1+pA) = K_1(A\pi, p)\ ,$$

$K_1(A\pi, pI)$ has rank kl by Lemma 1 (ii); and so the above set must be a $\hat{\mathbf{Z}}_p$-basis. In particular, $K_1(A\pi, pI)$ is torsion free.

Now consider the composite

$$E\colon pI \xrightarrow{\text{Exp}} 1+pI \twoheadrightarrow K_1(A\pi, pI)\ .$$

Exp is a bijection (with Log as inverse), so E is onto. For any $x \in pI$,

$$\lim_{n\to\infty}(1+p^nx)^{1/p^n} = \lim_{n\to\infty}\left[1+x\frac{1(1-p^n)}{2!}x^2+\frac{1(1-p^n)(1-2p^n)}{3!}x^3+\ldots\right]$$
$$= \text{Exp}\,(x)\ .$$

For any $x, y \in pI$ and $n \geqq 1$,

$$(1+p^nx)(1+p^ny) \equiv 1+p^n(x+y) \pmod{p^{2n}}$$
$$(1+p^nx)^{1/p^n}(1+p^ny)^{1/p^n} \equiv (1+p^n(x+y))^{1/p^n} \pmod{p^n} \quad \text{in } K_1(A\pi, pI)$$

(using here that $K_1(A\pi, pI)$ is torsion free); and in the limit

$$E(x)\cdot E(y) = E(x+y)\,.$$

So E is a homomorphism, and thus induces an epimorphism

$$\exp\colon \overline{pI(A\pi)} \to K_1(A\pi, pI)\,.$$

($E(gxg^{-1})=E(x)$ for $g \in \pi$ and $x \in pI$, since $\mathrm{Exp}\,(gxg^{-1})=g\cdot \mathrm{Exp}\,(x)\cdot g^{-1}$). Both groups are free of rank kl, so exp is an isomorphism.

From this we get

PROPOSITION 3. *Let A be a p-ring, and π a p-group. For any $x \in I(A\pi)$, $\mathrm{Log}\,(1+x)$ converges in $I(A\pi)\otimes\mathbf{Q}$. The resulting map*

$$\mathrm{Log}\colon 1+I(A\pi) \to I(A\pi)\otimes\mathbf{Q}$$

is a homomorphism, and induces an injection

$$\log\colon \mathrm{Wh}'(A\pi) \to \overline{I(A\pi)}\otimes\mathbf{Q}\,.$$

In particular, for $u \in 1+I(A\pi)$, $[u]$ lies in $SK_1(A\pi)\subseteq \mathrm{Wh}\,(A\pi)$ if and only if $\mathrm{Log}\,(u)=0$ (in $\overline{I(A\pi)}\otimes\mathbf{Q}$).

PROOF. $A/p[\pi]$ being a finite local ring [1, Corollary XI.1.4], any proper ideal is nilpotent. In particular, $I(A\pi)^k\subseteq pA\pi$ for some k, and then convergence of $\mathrm{Log}\,(1+x)$ follows directly. Furthermore, this says that $(1+x)^{p^n} \in 1+pI$ for any $x \in I$ and $p^n>k$; and since

$$\mathrm{Log}\,(1+x) = \frac{1}{p^n}\mathrm{Log}\,(1+x)^{p^n}$$

by algebra, Proposition 2 shows that the induced map is a homomorphism to $\overline{I(A\pi)}\otimes\mathbf{Q}$. $\mathrm{Wh}'\,(A\pi)$ is torsion free by [10, Theorem 4.1], and thus injects by Proposition 2.

The following simple arithmetic facts are shown here for future reference.

LEMMA 4. (i) *Let A be a p-ring, $\mathfrak{p}\subseteq A$ the maximal ideal, $\bar{A}=A/\mathfrak{p}$ the residue field, and τ the composite*

$$\tau\colon A \to \bar{A} \xrightarrow{\mathrm{Tr}} \mathbf{F}_p\,.$$

Then $\mathrm{Ker}\,(\tau)=\{\lambda-\lambda^p \mid \lambda \in A\}$.

(ii) *Let $z \in \mathbf{Z}_p$ be a generator; then in $\mathbf{Z}[\mathbf{Z}_p]$,*

$$(1-z)^p \equiv -p(1-z)\ \ (\mathrm{mod}\, p(1-z)^2)\,.$$

PROOF. (i) Define $\psi: A \to A$ by $\psi(\lambda)=\lambda-\lambda^p$; and let $\bar{\psi}$ be its reduction in $\bar{A}$. For any $r \geqq 1$, $\lambda \in A$, and $\mu \in \mathfrak{p}^r$,

$$\psi(\lambda+\mu) = \psi(\lambda)+\mu-p\lambda^{p-1}\cdot\mu-\ldots-\mu^p \equiv \psi(\lambda)+\mu \pmod{\mathfrak{p}^{r+1}};$$

and from this it easily follows that $\operatorname{Im}(\psi)$ is a union of cosets of $\mathfrak{p}$. In the residue field we have

$$\operatorname{Ker}(\bar{\psi}) = \{\lambda=\lambda^p\} = \boldsymbol{F}_p, \text{ and } \operatorname{Tr}\circ\bar{\psi}(\lambda) = \operatorname{Tr}(\lambda-\lambda^p) = 0$$

(λ and λ^p being Galois conjugate). So $\operatorname{Im}(\bar{\psi})=\operatorname{Ker}[\operatorname{Tr}: \bar{A} \to \boldsymbol{F}_p]$ by counting; and hence $\operatorname{Im}(\psi)=\operatorname{Ker}(\tau)$.

(ii) This is trivial for $p=2$. If p is odd, just note that

$$1 = [1-(1-z)]^p \equiv 1-p(1-x)-(1-z)^p \pmod{p(1-z)^2}.$$

(My thanks to the referee for pointing out this much shorter proof).

One more technical lemma is needed before starting the computation of SK_1.

LEMMA 5. *Let A be a p-ring, $\tilde{\pi}$ a p-group, $z \in \tilde{\pi}$ a central element of order p, and set $\pi=\tilde{\pi}/z$. Then*

(i) *The sequence*

$$1+(1-z)A\tilde{\pi} \to \operatorname{Wh}(A\tilde{\pi}) \to \operatorname{Wh}(A\pi) \to 0$$

is exact.

(ii) *Let $\varepsilon: A\tilde{\pi} \to A$ denote the augmentation map, and let $\tau: A \to \boldsymbol{F}_p$ be as above. Then the logarithm map*

$$\operatorname{Log}: 1+(1-z)A\tilde{\pi} \to (1-z)A\tilde{\pi}$$

is defined, and has image

$$\operatorname{Log}(1+(1-z)A\tilde{\pi}) = \{(1-z)\xi \mid \xi \in A\tilde{\pi},\ \tau\circ\varepsilon(\xi)=0\}.$$

PROOF. (i) The map $A\tilde{\pi} \to A\pi$ is a surjection of local rings, and hence induces a surjection on K_1 ([1, Corollary III.2.9]). All elements in $1+(1-z)A\tilde{\pi}$ are units (again because $A\tilde{\pi}$ is local), and so the natural map

$$1+(1-z)A\tilde{\pi} \to K_1(A\tilde{\pi}, (1-z)A\tilde{\pi})$$

is onto ([1, Theorem V.9.1]). Hence the exactness of

$$1+(1-z)A\tilde{\pi} \to K_1(A\tilde{\pi}) \to K_1(A\pi) \to 0$$

follows from the exact sequence for an ideal. Exactness of the corresponding sequence with Whitehead groups follows immediately.

(ii) Since $p(1-z) \mid (1-z)^p$ (Lemma 4), it follows that $n!(1-z) \mid (1-z)^n$ for all $n \geqq 1$. In particular, $\operatorname{Log}(1+(1-z)\xi)$ converges in $(1-z)A\tilde{\pi}$ for any $\xi \in A\tilde{\pi}$ (z being central). Furthermore, for any $\xi \in I(A\tilde{\pi})$, the series

$$\operatorname{Exp}((1-z)\xi) = 1+(1-z)\xi+\frac{(1-z)^2}{2!}\xi^2+\frac{(1-z)^3}{3!}\xi^3+\dots$$

converges in $1+(1-z)A\tilde{\pi}$ ($\xi^n \to 0$ as $n \to \infty$: see the proof of Proposition 3.)

It follows that

$$\operatorname{Log}(1+(1-z)A\tilde{\pi}) \supseteqq (1-z)I(A\tilde{\pi}) .$$

For any $\lambda \in A$,

$$\operatorname{Log}(1+\lambda(1-z)) \equiv (1-z)(\lambda-\lambda^p) \pmod{(1-z)^2}$$

by Lemma 4. Since $1+\lambda(1-z)$ is central, this shows that

$$\begin{aligned}\operatorname{Log}(1+(1-z)A\tilde{\pi}) &= \{(1-z)\xi \mid \xi \in A\tilde{\pi},\ \varepsilon(\xi)=\lambda-\lambda^p, \text{ some } \lambda \in A\} \\ &= \{(1-z)\xi \mid \tau\circ\varepsilon(\xi)=0\} \quad \text{(Lemma 4)} .\end{aligned}$$

From now until Theorem 1, we consider a fixed p-ring A and a fixed extension

$$1 \to \mathbf{Z}_p \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1$$

of p-groups. Let $z \in \operatorname{Ker}(\alpha)$ be a generator: note that z is central in $\tilde{\pi}$. The goal now is to study

$$\operatorname{Ker}[SK_1(A\tilde{\pi}) \to SK_1(A\pi)] .$$

Define subsets $\mathscr{S} \subseteqq \pi$ and $\tilde{\mathscr{S}} = \alpha^{-1}(\mathscr{S}) \subseteqq \tilde{\pi}$:

$$\mathscr{S} = \mathscr{S}(\tilde{\pi}, z) = \{g \in \pi \mid \text{the elements in } \alpha^{-1}g \text{ are conjugate}\} ,$$

and

$$\tilde{\mathscr{S}} = \tilde{\mathscr{S}}(\tilde{\pi}, z) = \{g \in \tilde{\pi} \mid g \text{ is conjugate to } zg\} .$$

Note that $\mathscr{S} \neq \varnothing$ if and only if z is a commutator.

Let $A/p[\mathscr{S}] \subseteqq A/p[\pi]$ denote the subgroup of linear combinations of elements in $\mathscr{S}$, and set

$$\overline{A/p[\mathscr{S}]} = A/p[\mathscr{S}]/\text{conjugation}, \quad \overline{A/p[\pi]} = A/p[\pi]/\text{conjugation} .$$

Let $\alpha_*: A\tilde{\pi} \to \overline{A/p[\pi]}$ be the obvious projection.

Proposition 6. *Define*

$$C = \{\alpha_*(\xi-\xi^p) \in \overline{A/p[\pi]} \mid \xi \in A\tilde{\pi},\ [1+(1-z)\xi]=1 \textit{ in } \operatorname{Wh}(A\tilde{\pi})\} .$$

Then $C \subseteq \overline{A/p[\mathscr{S}]}$ *and is a subgroup. Furthermore, there is a short exact sequence*

$$0 \to C \to \mathrm{Ker}\ [\overline{A/p[\mathscr{S}]} \xrightarrow{\tau\circ\varepsilon} \boldsymbol{F}_p] \xrightarrow{\Omega} \mathrm{Ker}\,[SK_1(A\tilde{\pi}) \to SK_1(A\pi)] \to 0;$$

where for any $\xi \in \mathrm{Ker}\,[\overline{A/p[\mathscr{S}]} \to \boldsymbol{F}_p]$ *and any lifting to* $\tilde{\xi} \in A[\tilde{\mathscr{S}}]$,

$$\Omega(\xi) = [\mathrm{Log}^{-1}((1-z)\tilde{\xi})] \in SK_1(A\tilde{\pi})\,.$$

PROOF. Define subgroups $S \subseteq T \subseteq 1+(1-z)A\tilde{\pi}$:

$$T = \mathrm{Ker}\,[1+(1-z)A\tilde{\pi} \to \mathrm{Wh}'(A\tilde{\pi})]$$

$$S = \mathrm{Ker}\,[1+(1-z)A\tilde{\pi} \to \mathrm{Wh}(A\tilde{\pi})]\,.$$

By Lemma 5(i), there is a short exact sequence

$$0 \to \frac{1+(1-z)A\tilde{\pi}}{S} \to \mathrm{Wh}(A\tilde{\pi}) \to \mathrm{Wh}(A\pi) \to 0\,.$$

It follows that

$$\begin{aligned}
&\mathrm{Ker}\,[SK_1(A\tilde{\pi}) \to SK_1(A\pi)] = \mathrm{Ker}\,[SK_1(A\tilde{\pi}) \to \mathrm{Wh}(A\pi)]\\
&\cong \mathrm{Ker}\,[\mathrm{Wh}(A\tilde{\pi}) \to \mathrm{Wh}(A\pi)\oplus \mathrm{Wh}'(A\tilde{\pi})]\\
&\cong \mathrm{Ker}\left[\frac{1+(1-z)A\tilde{\pi}}{S} \to \mathrm{Wh}(A\tilde{\pi}) \to \mathrm{Wh}'(A\tilde{\pi})\right]\\
&= T/S\,.
\end{aligned}$$

By Proposition 3,

$$\begin{aligned}
T &= \{u \in 1+(1-z)A\tilde{\pi} \mid \mathrm{Log}\,(u)=0 \text{ in } \overline{I(A\tilde{\pi})}\}\\
&= \{u \in 1+(1-z)A\tilde{\pi} \mid \mathrm{Log}\,(u)=(1-z)\xi,\ \xi \text{ a sum of elements}\\
&\qquad \lambda g (g \in \tilde{\mathscr{S}}) \text{ and } \lambda(g-hgh^{-1})\ (g,h \in \tilde{\pi})\}\,.
\end{aligned}$$

We first show that

$$S \cap (1+(1-z)^r A\tilde{\pi}) = T \cap (1+(1-z)^r A\tilde{\pi})$$

for $r \geqq 2$, by downwards induction. Note that $p(1-z) \mid (1-z)^p$, so

$$S \cap (1+(1-z)^p A\tilde{\pi}) = T \cap (1+(1-z)^p A\tilde{\pi})$$

by Proposition 2 ($K_1(A\tilde{\pi}, pI)$ is torsion free and thus injects into $\mathrm{Wh}'(A\tilde{\pi})$). Now assume $2 \leqq r \leqq p-1$; then

$$\operatorname{Log}(1+(1-z)^r\xi) \equiv (1-z)^r\xi \pmod{(1-z)^{r+1}},$$

and it thus suffices to check that $1+(1-z)^r\xi$ is a product of commutators $(\bmod (1-z)^{r+1})$ when

$$\xi = \lambda(g-hgh^{-1}) \ (g,h \in \tilde{\pi}) \quad \text{or} \quad \xi = \lambda g \ (g \in \tilde{\mathcal{S}}).$$

In the first case:

$$\begin{aligned}[h, 1-\lambda(1-z)^r g] &= 1+\lambda(1-z)^r(g-hgh^{-1})(1-\lambda(1-z)^r g)^{-1} \\ &\equiv 1+\lambda(1-z)^r(g-hgh^{-1}) \pmod{(1-z)^{r+1}}.\end{aligned}$$

If $g \in \tilde{\mathcal{S}}$, choose $h \in \tilde{\pi}$ such that $zg = hgh^{-1}$; then

$$\begin{aligned}[h, 1-\lambda(1-z)^{r-1}g] &= 1+\lambda(1-z)^r g(1-\lambda(1-z)^{r-1}g)^{-1} \\ &\equiv 1+\lambda(1-z)^r g \pmod{(1-z)^{r+1}}.\end{aligned}$$

Thus, letting $\bar{S} \subseteq \bar{T}$ be the images of S, T in $(1+(1-z)A\tilde{\pi})/(1+(1-z)^2A\tilde{\pi})$, we have $\bar{T}/\bar{S} = T/S$. Now consider the induced map

$$\operatorname{Log}: \frac{1+(1-z)A\tilde{\pi}}{1+(1-z)^2A\tilde{\pi}} \to \frac{(1-z)A\tilde{\pi}}{(1-z)^2A\tilde{\pi}}$$

where

$$\operatorname{Log}(1+(1-z)\xi) \equiv (1-z)(\xi-\xi^p) \pmod{(1-z)^2}$$

by Lemma 4. Both groups are isomorphic to $A/p[\pi]$ (under addition: note again that $(1-z)^2 \mid p(1-z)$); so everything can be translated to the map

$$\psi: A/p[\pi] \to A/p[\pi], \quad \psi(\xi) = \xi - \xi^p.$$

Let $\mathrm{pr}: A/p[\pi] \to \overline{A/p[\pi]}$ denote the projection. Then $\mathrm{pr}\circ\psi$ is a homomorphism, and regarding $\bar{S}$ and $\bar{T}$ as subgroups of $A/p[\pi]$ we have

$$\bar{T} = (\mathrm{pr}\circ\psi)^{-1}(\overline{A/p[\mathcal{S}]}) \quad \text{and} \quad C = \mathrm{pr}\circ\psi(\bar{S}).$$

In particular, C is a subgroup of $\overline{A/p[\mathcal{S}]}$. Furthermore,

$$\mathrm{pr}\circ\psi(\bar{T}) = \operatorname{Ker}[\overline{A/p[\mathcal{S}]} \to \boldsymbol{F}_p]$$

by Lemma 4.

It remains to show that $\bar{S} \supseteq \operatorname{Ker}(\mathrm{pr}\circ\psi)$. Upon writing $\mathrm{pr}\circ\psi$ as a composite

$$A/p[\pi] \xrightarrow{\mathrm{pr}} \overline{A/p[\pi]} \xrightarrow{\bar{\psi}} \overline{A/p[\pi]},$$

we see that $\operatorname{Ker}(\mathrm{pr}\circ\psi)$ is generated by elements $\lambda(g-hgh^{-1})$ and by $\mathbf{Z}_p \subseteq A/p$. For $r \in \mathbf{Z}_p$,

$$1+r(1-z) \equiv z^{-r} \in S \pmod{(1-z)^2};$$

and as in earlier computations

$$1+\lambda(1-z)(g-hgh^{-1}) \equiv [h, 1-\lambda(1-z)g] \pmod{(1-z)^2}.$$

This shows that

$$\operatorname{Ker}[SK_1(A\tilde{\pi}) \to SK_1(A\pi)] \cong \bar{T}/\bar{S} \cong \operatorname{Ker}[\overline{A/p[\mathscr{S}]} \to \mathbf{F}_p]/C.$$

Retracing the proof shows that the isomorphism is actually given by the map Ω defined above.

When $\mathscr{S}=\varnothing$, this gives immediately:

PROPOSITION 7. *If z is not a commutator, then*

$$SK_1(A\tilde{\pi}) \to SK_1(A\pi)$$

is one-to-one.

When $\mathscr{S} \neq \varnothing$, we want to find as many elements as possible in C. For convenience, for $g, h \in \pi$, set

$$\{g,h\} = [\bar{g}, \bar{h}] \in \tilde{\pi}$$

for any $\bar{g} \in \alpha^{-1}g$ and $\bar{h} \in \alpha^{-1}h$ (the extension is central, so this is uniquely defined).

LEMMA 8. *Assume z is a commutator, and let $C \subseteq \overline{A/p[\mathscr{S}]}$ be defined as above. Then, for any $\lambda \in A/p$ and $g \in \mathscr{S}$:*

(i) $\lambda(g-g^k) \in C$ *if* $p \nmid k$
(ii) $(\lambda-\lambda^p)g \in C$
(iii) $\lambda(g-h) \in C$ *if* $h \in \mathscr{S}$ *with* $\{h,g\}=z$.

PROOF. Fix $h \in \pi$ such that $\{h,g\}=z$. By restricting to the subgroup generated by g and h, we may assume π is abelian. In particular, $\mathscr{S}$ contains no pth powers in this case.

For convenience, let $\hat{C} \subseteq \overline{A/p[\pi]}$ be the subgroup generated by C, and elements λg for $g \notin \mathscr{S}$. Since $C \subseteq \overline{A/p[\mathscr{S}]}$, it will suffice to prove that the above elements lie in $\hat{C}$. Note that for any $\xi \in \overline{A/p[\pi]}$, $\xi^p \in \hat{C}$.

Fix $\bar{g} \in \alpha^{-1}(g)$ and $\bar{h} \in \alpha^{-1}(h)$.

(iii) We have $[\bar{h}, \bar{g}]=z$. So for any $t>0$,

$$[\bar{g}^{-1}\bar{h}, 1-\lambda(\bar{g}-\bar{h})^t] = 1+\lambda(1-z^t)(\bar{g}-\bar{h})^t[1-\lambda(\bar{g}-\bar{h})^t]^{-1}$$

$$\equiv 1+t(1-z)[\lambda(\bar{g}-\bar{h})^t+\lambda^2(\bar{g}-\bar{h})^{2t}+\lambda^3(\bar{g}-\bar{h})^{3t}+\dots] \pmod{(1-z)^2}.$$

Dropping pth powers, this gives

$$t[\lambda(g-h)^t+\lambda^2(g-h)^{2t}+\dots]\in\hat{C}\qquad(\text{all } t\geqq 1)\,.$$

Induction downwards shows that $\lambda(g-h)^t\in\hat{C}$ for all $t\geqq 1$ (this holds trivially when $p\mid t$). In particular, $\lambda(g-h)\in\hat{C}$.

(i) Note that whenever $p\nmid k$, $\{h^m,g^k\}=\{h^m,g^{p+k}\}=z$ for some m. By (iii) above,

$$(1)\qquad \lambda g^k(1-g^p)=\lambda(g^k-h^m)-\lambda(g^{p+k}-h^m)\in\hat{C}\qquad\text{for any } k\geqq 0$$

(noting that $\lambda g^k\in\hat{C}$ when $p\mid k$). This in turn implies that

$$(2)\qquad \lambda g(1-g)^r=\lambda g(1-g)^{r-p}(1-g^p)\in\hat{C}\qquad\text{for any } r\geqq p.$$

For any $r\geqq 1$:

$$\begin{aligned}[\bar{h},1+\lambda(1-\bar{g})^r] &= 1+\lambda[(1-z\bar{g})^r-(1-\bar{g})^r][1+\lambda(1-\bar{g})^r]^{-1}\\ &=1+\left[\sum_{i=1}^{r}\lambda\binom{r}{i}(1-\bar{g})^{r-i}(1-z)^i\bar{g}^i\right][1-\lambda(1-\bar{g})^r+\lambda^2(1-\bar{g})^{2r}-\dots]\\ &\equiv 1+\lambda r(1-z)(1-\bar{g})^{r-1}\bar{g}[1-\lambda(1-\bar{g})^r+\lambda^2(1-\bar{g})^{2r}-\dots]\ (\mathrm{mod}\ (1-z)^2)\,.\end{aligned}$$

Again dropping pth powers, this yields

$$(3)\quad \lambda g(1-g)^{r-1}-\lambda^2 g(1-g)^{2r-1}+\lambda^3 g(1-g)^{3r-1}-\dots\in\hat{C}\qquad\text{when } p\nmid r\,.$$

Since $\lambda^i g(1-g)^{ir-1}\in\hat{C}$ when $ir-1\geqq p$ (by (2)), a downward induction now gives

$$(4)\qquad \lambda g(1-g)^{r-1}=\lambda g(1-g)^{r-2}(1-g)\in\hat{C},\qquad\text{any } 2\leqq r\leqq p-1\,.$$

This in turn implies $\lambda g^k(1-g)\in\hat{C}$ for $1\leqq k\leqq p-2$ (by induction on k); and hence

$$\lambda g\equiv\lambda g^2\equiv\dots\equiv\lambda g^{p-1}\ (\mathrm{mod}\,\hat{C})\,.$$

Combining this with (1) shows that $\lambda(g-g^k)\in\hat{C}$ whenever $p\nmid k$.

(ii) When $r=1$, (3) takes the form

$$\lambda g-\lambda^2 g(1-g)+\lambda^3 g(1-g)^2-\dots\in\hat{C}\,.$$

By (2) and (4), all but two terms drop out, leaving

$$\lambda g+\lambda^p g(1-g)^{p-1}=\lambda g+\lambda^p(g+g^2+\dots+g^p)\in\hat{C}\,.$$

Since $\lambda^p g^i\equiv\lambda^p(g)\ (\mathrm{mod}\,\hat{C})$ for $1\leqq i\leqq p-1$, and $\lambda^p g^p\in\hat{C}$, this gives

$$(\lambda-\lambda^p)g\in\hat{C}\,.$$

Since $A/\{\lambda-\lambda^p\}\cong\mathbf{Z}_p$ by Lemma 4, Propositions 6 and 7 and Lemma 8 now give:

THEOREM 1. *Let* $1 \to \mathbf{Z}_p \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1$ *be any extension of p-groups, let* $z \in \mathbf{Z}_p$ *generate the kernel, and let A be any p-ring. Define*

$$\mathscr{S} = \mathscr{S}(\tilde{\pi}, z) = \{g \in \pi \mid \text{the elements in } \alpha^{-1}(g) \text{ are conjugate}\} .$$

Let $\sim$ *be the equivalence relation on* $\mathscr{S}$ *generated by:*

$$g \sim h \text{ if } g, h \text{ are conjugate },$$
$$\text{if } g = h^k \ (p \nmid k), \text{ or}$$
$$\text{if } [\bar{g}, \bar{h}] = z \text{ for any } \bar{g} \in \alpha^{-1}(g),\ \bar{h} \in \alpha^{-1}(h) .$$

Then $\operatorname{Ker}[SK_1(A\tilde{\pi}) \to SK_1(A\pi)] \cong \mathbf{Z}_p^k$, *where*

(i) $k = 0$ *if* $\mathscr{S} = \varnothing$

(ii) $k \leqq |\mathscr{S}/\sim| - 1$ *if* $\mathscr{S} \neq \varnothing$.

More precisely, if $\mathscr{S} \neq \varnothing$, *let* $\{g_0, \ldots, g_r\}$ *be* $\sim$*-equivalence class representatives in* $\mathscr{S}$ *lifted to* $\tilde{\pi}$, *and let* $\lambda \in A$ *be any element with* $\tau(\lambda) \neq 0$ *in* $\boldsymbol{F}_p$. *Then the elements*

$$\operatorname{Exp}(\lambda(1-z)(g_0 - g_i)), \quad 1 \leqq i \leqq r ,$$

generate $\operatorname{Ker}[SK_1(A\tilde{\pi}) \to SK_1(A\pi)]$.

We end the section with an example, showing that Theorem 1 easily implies $SK_1(A\pi) = 0$ for many familiar p-groups π.

PROPOSITION 9. *Assume* π *is a p-group, and* $\varrho \lhd \pi$ *an abelian normal subgroup such that* π/ϱ *is cyclic. Then for any p-ring A,* $SK_1(A\pi) = 0$.

PROOF. Choose $z \in \varrho \cap Z(\pi)$ of order p, and set $\pi' = \pi/z$. We may assume inductively that $SK_1(A\pi') = 0$. If z is not a commutator, we are done (Proposition 7); otherwise let $\mathscr{S}$ and $\sim$ be defined as before. Set $\varrho' = \varrho/z$, fix $x \in \pi'$ generating π'/ϱ', and define $\{g, h\}$ $(g, h \in \pi')$ by

$$\{g, h\} = [\bar{g}, \bar{h}] \in \pi, \quad \text{any } \bar{g}, \bar{h} \in \pi \text{ covering } g, h .$$

For any $g \in \mathscr{S}$, choose $h \in \mathscr{S}$ such that $\{g, h\} = z$. Either gh^i or $g^i h$ lies in ϱ' for some i (π'/ϱ' being cyclic); we may assume by symmetry that $gh^i = a \in \varrho'$. Writing $h = bx^j$ for some $b \in \varrho'$ we get

$$z = \{g, h\} = \{gh^i, h\} = \{a, bx^j\} = \{a, x^j\} = \{ax, x^j\}$$
$$= \{ax, x^j(ax)^{-j}\} = \{x, x^j(ax)^{-j}\}$$

since $x^j(ax)^{-j} \in \varrho'$. So

$$g \sim h \sim gh^i = a \sim ax \sim x^j(ax)^{-j} \sim x \quad \text{in } \mathscr{S},$$

the relation is transitive, and $SK_1(A\pi)=0$ by Theorem 1 (and the induction hypothesis).

In particular, $SK_1(A\pi)=0$ for any 2-ring A when π is a dihedral, quaternionic, or semidihedral 2-group. Combining this with results in [6] gives:

COROLLARY. *Let π be a dihedral, quaternionic, or semidihedral 2-group, $n \geqq 1$ an odd integer, and $\tau(n)$ the number of divisors of n. Then*

$$SK_1(\mathbf{Z}[\mathbf{Z}_n \times \pi]) \cong (\mathbf{Z}_2)^{\tau(n)-1}.$$

PROOF. Since n is odd, we can write $\hat{\mathbf{Z}}_2[\mathbf{Z}_n] \cong \sum A_i$, a sum of 2-rings. Then

$$SK_1(\hat{\mathbf{Z}}_2[\mathbf{Z}_n \times \pi]) \cong \sum SK_1(A_i[\pi]) = 0$$

by Proposition 9. Since $SK_1(\hat{\mathbf{Z}}_p[\mathbf{Z}_n \times \pi])=0$ for odd p ([10, Theorem 9.1]),

$$SK_1(\mathbf{Z}[\mathbf{Z}_n \times \pi]) \cong \mathrm{Cl}_1(\mathbf{Z}[\mathbf{Z}_n \times \pi]) \cong \mathbf{Z}_2^{\tau(n)-1}$$

by Theorem 4 in [6].

2.

Again consider the injection

$$\log\colon \mathrm{Wh}'(A\pi) \to \overline{I(A\pi)} \otimes \mathbf{Q}$$

described in Proposition 3. For any unramified (over $\hat{\mathbf{Z}}_p$) p-ring A and any p-group π, define

$$\varphi = \varphi_A \in \mathrm{Gal}(A/\hat{\mathbf{Z}}_p)\colon \varphi(\lambda) \equiv \lambda^p \pmod p \text{ for } \lambda \in \mathrm{A};$$

$$\Phi\colon I(A\pi) \to I(A\pi)\colon \Phi(\textstyle\sum \lambda_i g_i) = \sum \varphi(\lambda_i) g_i^p;$$

and

$$\Gamma = \Gamma_{A\pi}\colon \mathrm{Wh}'(A\pi) \to \overline{I(A\pi)} \otimes \mathbf{Q}\colon \Gamma(u) = \log(u) - \frac{1}{p}\Phi(\log u).$$

Since Φ is nilpotent ($\Phi^m=0$ if $p^m=\exp(\pi)$), an inverse can easily be constructed to show that Γ is one-to-one (and an isomorphism rationally).

We first check that Γ is integral-valued and natural.

PROPOSITION 10. (i) $\Gamma(\mathrm{Wh}'(A\pi)) \subseteq \overline{I(A\pi)}$.

(ii) *Γ is natural with respect to maps induced by group homomorphisms and Galois automorphisms of A. For any p-group π and any pair $B \supseteq A$ of unramified p-rings, the squares*

$$\begin{array}{ccc} \mathrm{Wh}'(A\pi) & \xrightarrow{\Gamma} & \overline{I(A\pi)} \\ \downarrow{\scriptstyle \mathrm{incl}} & & \downarrow{\scriptstyle \mathrm{incl}} \\ \mathrm{Wh}'(B\pi) & \xrightarrow{\Gamma} & \overline{I(B\pi)} \end{array} \qquad \begin{array}{ccc} \mathrm{Wh}'(B\pi) & \xrightarrow{\Gamma} & \overline{I(B\pi)} \\ \downarrow{\scriptstyle \mathrm{trf}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ \mathrm{Wh}'(A\pi) & \xrightarrow{\Gamma} & \overline{I(A\pi)} \end{array}$$

commute.

Proof. (i) For any $x \in I(A\pi)$,

$$\begin{aligned} \Gamma(1+x) &= [x-\tfrac{1}{2}x^2+\ldots]-\left[\frac{1}{p}\Phi(x)-\frac{1}{2p}\Phi(x^2)+\ldots\right] \\ &\equiv \sum_{k=1}^{\infty} (-1)^{pk+1}\frac{1}{pk}[x^{pk}-\Phi(x^k)] \pmod{\overline{I(A\pi)}}\,. \end{aligned}$$

To see this for $p=2$, note that

$$\left[\frac{1}{p}\Phi(x)+\frac{1}{3p}\Phi(x^3)+\ldots\right] \equiv -\left[\frac{1}{p}\Phi(x)+\frac{1}{3p}\Phi(x^3)+\ldots\right] \pmod{\overline{I(A\pi)}}\,.$$

So it suffices to show that $pk \mid [x^{pk}-\Phi(x^k)]$ for all k; or that

$$p^n \mid [x^{p^n}-\Phi(x^{p^{n-1}})]$$

(in $\overline{I(A\pi)}$) for all $n \geqq 1$ and $x \in I(A\pi)$.

Write $x=\sum \lambda_i g_i$, and consider a typical term in x^{p^n}:

$$\lambda_{i_1}\ldots\lambda_{i_q}g_{i_1}\ldots g_{i_q} \qquad (q=p^n)\,.$$

Letting $\mathbf{Z}_{p^n}$ act by cyclically permuting the g_i's, we get a total of p^{n-t} conjugate terms, where p^t is the number of cyclic permutations leaving each term invariant. Then $g_{i_1}\ldots g_{i_q}$ is a p^t-power, and the sum of the conjugate terms has the form

$$p^{n-t}\bar{\lambda}^{p^t}\bar{g}^{p^t} \in \overline{I(A\pi)} \qquad \left(\bar{\lambda} = \prod_{j=1}^{p^{n-t}} \lambda_{i_j},\ \bar{g} = \prod_{j=1}^{p^{n-t}} g_{i_j}\right).$$

If $t=0$, we are done. If $t>0$, there is a corresponding term $p^{n-t}\bar{\lambda}^{p^{t-1}}\bar{g}^{p^{t-1}}$ in the expansion of $x^{p^{n-1}}$. It remains only to show that

$$p^{n-t}\bar{\lambda}^{p^t}\bar{g}^{p^t} \equiv p^{n-t}\Phi(\bar{\lambda}^{p^{t-1}}\bar{g}^{p^{t-1}}) = p^{n-t}\varphi(\bar{\lambda}^{p^{t-1}})\bar{g}^{p^t} \pmod{p^n}\,.$$

But $p^t \mid [\bar{\lambda}^{p^t}-\varphi(\bar{\lambda}^{p^{t-1}})]$, since $p \mid [\bar{\lambda}^p-\varphi(\bar{\lambda})]$.

(ii) Naturality with respect to group homomorphisms is clear. Naturality with respect to Galois automorphisms holds since they all commute with the Frobenius automorphism φ. If $B \supseteqq A$, then Γ commutes with the inclusion maps since $\varphi_B | A = \varphi_A$.

In checking naturality with respect to the trace and transfer maps, first note that Φ commutes with the trace (since it commutes with Galois automorphisms). It suffices therefore to show naturality of the logarithm; or equivalently that the following diagram commutes:

$$\begin{array}{ccc} pI(B\pi) & \xrightarrow{\mathrm{Tr}_{B/A}} & pI(A\pi) \\ \downarrow\exp & & \downarrow\exp \\ \mathrm{Wh}'(B\pi) & \xrightarrow{\mathrm{trf}} & \mathrm{Wh}'(A\pi)\,. \end{array}$$

Recalling that for $b \in B$, $\mathrm{Tr}_{B/A}(b)$ is the trace of the matrix for multiplication by b as an A-linear map ([4, Theorem I.5.3]); we get, for $x \in pI(B\pi)$ and $n>0$,

$$\mathrm{trf}\,(1+p^n x) \equiv 1+p^n\,\mathrm{Tr}_{B/A}(x) \pmod{p^{2n}}\,.$$

Hence

$$\mathrm{trf}\,(1+p^n x)^{1/p^n} \equiv (1+p^n\,\mathrm{Tr}_{B/A}(x))^{1/p^n} \pmod{p^n}\,,$$

and in the limit (the transfer being continuous)

$$\mathrm{trf}\,(\exp(x)) = \exp(\mathrm{Tr}_{B/A}(x))\,.$$

The following lemma supplies the main induction step for proving Theorem 2. For later use, it is shown here for arbitrary p-rings.

LEMMA 11. *Let A be a p-ring, $1 \to \mathbf{Z}_p \to \tilde{\pi} \to \pi \to 1$ an extension of p-groups, and $z \in \mathbf{Z}_p$ a generator. Let*

$$\mathrm{lg}\colon \mathrm{Ker}\,[\mathrm{Wh}\,(A\tilde{\pi}) \to \mathrm{Wh}\,(A\pi)] \to \mathrm{Ker}\,[\overline{I(A\tilde{\pi})} \to \overline{I(A\pi)}]$$

be the map induced by the logarithm. Define

$$\varepsilon\colon A\tilde{\pi} \to A \quad \text{and} \quad \tau\colon A \to \boldsymbol{F}_p$$

as in Lemma 4 and 5. Then

(i) *if z is a commutator,* lg *is onto.*

(ii) *if z is not a commutator, the sequence*

$$\mathrm{Ker}\,[\mathrm{Wh}\,(A\tilde{\pi}) \to \mathrm{Wh}\,(A\pi)] \xrightarrow{\mathrm{lg}} \mathrm{Ker}\,[\overline{I(A\tilde{\pi})} \to \overline{I(A\pi)}] \xrightarrow{\nu} \boldsymbol{F}_p \to 0$$

is exact, where $\nu((1-z)\xi) = \tau\circ\varepsilon(\xi)$.

PROOF. First note that $\mathrm{Ker}\,[\mathrm{Wh}\,(A\tilde{\pi}) \to \mathrm{Wh}\,(A\pi)]$ is generated by $1+(1-z)A\tilde{\pi}$ by Lemma 5 (i). So lg is integral valued by Lemma 5 (ii) and a well defined homomorphism by Proposition 3. Furthermore,

$$\mathrm{Im}\,(\mathrm{lg}) = \mathrm{Im}\,[1+(1-z)A\tilde{\pi} \xrightarrow{\mathrm{Log}} (1-z)A\pi \twoheadrightarrow \mathrm{Ker}\,[\overline{I(A\tilde{\pi})} \to \overline{I(A\pi)}]]\,.$$

If $z=[h,g]$, choose $\lambda \in A$ with $\tau(\lambda) \neq 0$. Then

$$\lambda(1-z)g \notin \mathrm{Log}\,(1+(1-z)A\tilde{\pi})$$

by Lemma 5, but $\lambda(1-z)g=0$ in $\overline{I(A\tilde{\pi})}$. Since Im (Log) is a subgroup of index p (Lemma 5), this shows that lg is onto.

If z is not a commutator, then g and zg are never conjugate for $g \in \tilde{\pi}$. It follows that $(1-z)\xi=0$ in $\overline{I(A\tilde{\pi})}$ if and only if ξ is a sum of terms of the form

$$\lambda(g-hgh^{-1}) \quad \text{and} \quad \lambda(g+zg+\ldots+z^{p-1}g)\,.$$

These both lie in Ker $(\tau\circ\varepsilon)$, so ν is well defined, and the above sequence is exact by Lemma 5.

The following lemma in its full generality will not be needed until Section 3.

LEMMA 12. *Let π be a p-group, and $\varrho \lhd \pi$ a non-trivial normal subgroup generated by commutators in π. Then ϱ contains a commutator $z \in Z(\pi)$ of order p. In particular, any non-abelian π contains a central commutator of order p.*

PROOF. Note that $[g,h]^n=[g^n,h]$ whenever $[g,h] \in Z(\pi)$; it thus suffices to find some non-trivial central commutator in ϱ. So we may assume $\varrho \nsubseteq Z(\pi)$; choose any

$$e \neq x' \in Z(\pi/(\varrho \cap Z(\pi))) \cap \varrho/(\varrho \cap Z(\pi));$$

and fix some $x \in \pi$ sitting over x'. Then $x \notin Z(\pi)$, and there is $g \in \pi$ with

$$e \neq z = [x,g] \in \varrho \cap Z(\pi)\,.$$

The last statement follows upon setting $\varrho=[\pi,\pi]$.

We can now prove:

THEOREM 2. *Let π be any p-group, A an unramified p-ring, and $\Gamma=\Gamma_{A\pi}$ as defined above. Then the sequence*

$$0 \to \mathrm{Wh}'(A\pi) \xrightarrow{\Gamma} \overline{I(A\pi)} \xrightarrow{\omega} \pi^{\mathrm{ab}} \to 0$$

is exact, where

$$\omega(\sum \lambda_i g_i) = \prod\,[g_i]^{\mathrm{Tr}(\lambda_i)} \in \pi^{\mathrm{ab}} \quad (\mathrm{Tr}=\mathrm{Tr}_{A/\hat{\mathbb{Z}}_p})\,.$$

PROOF. We first show that $\omega\circ\Gamma=0$. It clearly suffices to do this when $\pi=\pi^{\mathrm{ab}}$ is abelian; and since Γ is natural with respect to the trace and transfer maps

(Proposition 10), it suffices to do it when $A=\hat{\mathbf{Z}}_p$. Set $I=I(A\pi)=\overline{I(A\pi)}$, for short.

First note that $I^2 \subseteqq \operatorname{Ker}(\omega)$:

$$\omega(\lambda g(1-a)(1-b)) = g^{\lambda}(ag)^{-\lambda}(bg)^{-\lambda}(abg)^{\lambda} = e \qquad \text{for } \lambda \in A,\ a,b,g \in \pi .$$

For any $u=1+\sum \lambda_i g_i(1-a_i)(1-b_i) \in 1+I^2$,

$$u^p \equiv 1+\sum \lambda_i^p g_i^p(1-a_i)^p(1-b_i)^p \equiv 1+\sum \varphi(\lambda_i)g_i^p(1-a_i^p)(1-b_i^p) = \Phi(u) \pmod{pI^2} .$$

So $u^p/\Phi(u) \in 1+pI^2$, and

$$\Gamma(u) = \log(u)-\frac{1}{p}\Phi(\log(u)) = \frac{1}{p}\log(u^p/\Phi(u)) \in I^2 \subseteqq \operatorname{Ker}(\omega)$$

(Φ commutes with log, since π is abelian). But $1+I^2$ generates $\mathrm{Wh}(\hat{\mathbf{Z}}_p\pi)$, since for $u \in 1+I$,

$$\begin{aligned} u &= 1-\sum \lambda_i(1-g_i) \qquad (\text{some } \lambda_i \in A,\ g_i \in \pi) \\ &\equiv \prod[1-\lambda_i(1-g_i)] \equiv \prod[1-(1-g_i)]^{\lambda_i} = \prod g_i^{\lambda_i} \pmod{1+I^2} . \end{aligned}$$

Exactness of the sequence

$$0 \to \mathrm{Wh}'(A\pi) \to \overline{I(A\pi)} \to \pi^{\mathrm{ab}} \to 0$$

can now be checked. Fix $z \in Z(\pi)$ of order p, and assume z is a commutator if π is not abelian. Assume inductively that the corresponding sequence for $\pi'=\pi/z$ is exact. Since $\omega\circ\Gamma=0$, and the maps

$$\mathrm{Wh}'(A\pi) \to \mathrm{Wh}'(A\pi'), \qquad \overline{I(A\pi)} \to \overline{I(A\pi')}, \qquad \text{and} \qquad \pi^{\mathrm{ab}} \to (\pi')^{\mathrm{ab}}$$

are all surjective (Lemma 5(i)); it remains only to check exactness for the corresponding sequence of kernels:

$$0 \to \operatorname{Ker}[\mathrm{Wh}'(A\pi) \to \mathrm{Wh}'(A\pi')] \xrightarrow{\Gamma_0} \operatorname{Ker}[\overline{I(A\pi)} \to \overline{I(A\pi')}] \xrightarrow{\omega_0} \operatorname{Ker}[\pi^{\mathrm{ab}} \to (\pi')^{\mathrm{ab}}] \to 0 .$$

Clearly ω_0 is onto (and we have seen that Γ_0 is one-to-one).

For any $\xi \in A\pi\otimes\mathbf{Q}$, $\Phi((1-z)\xi)=0$ ($z^p=e$ and z is central). So Γ_0 is just the map induced by the logarithm, and the composite

$$\operatorname{Ker}[\mathrm{Wh}(A\pi) \to \mathrm{Wh}(A\pi')] \xrightarrow{\beta} \operatorname{Ker}[\mathrm{Wh}'(A\pi) \to \mathrm{Wh}'(A\pi')] \xrightarrow{\Gamma_0} \operatorname{Ker}[\overline{I(A\pi)} \to \overline{I(A\pi')}]$$

is the map lg of Proposition 11. If π is abelian, then

$$\operatorname{Im}(\Gamma_0) \supseteqq \operatorname{Im}(\lg) = \{(1-z)\xi \mid \operatorname{Tr}(\varepsilon(\xi)) \equiv 0 \pmod p\} = \operatorname{Ker}(\omega_0)$$

by Proposition 11 (note that $\omega((1-z)\xi) = z^{-\operatorname{Tr}(\varepsilon(\xi))}$ in this case). If π is non-abelian, then z is a commutator, so $\Gamma_0 \circ \beta$ is onto by Proposition 11, Γ_0 is an isomorphism, and we are done.

One application of Theorem 2 is the following answer to a conjecture of Wall's, related to calculating L-groups ([11, Conjecture 5.1.3]).

PROPOSITION 13. *Let π be a 2-group, and consider the involution on* $\mathrm{Wh}'(\hat{\mathbf{Z}}_2\pi)$ *induced by $g \to g^{-1}$. Then*

$$H^1(\mathbf{Z}_2;\ \mathrm{Wh}'(\hat{\mathbf{Z}}_2\pi)) = \frac{\{[g] \in \pi^{\mathrm{ab}} \mid [g^2] = e\}}{\langle [g] \mid g \textit{ conjugate to } g^{-1}\rangle}$$

(*where for $g \in \pi$, $[g]$ denotes its class in π^{ab}*).

PROOF. Give $I(\hat{\mathbf{Z}}_2\pi)$ and π^{ab} the obvious involutions, so that all maps in the short exact sequence of Theorem 2:

$$0 \to \mathrm{Wh}'(\hat{\mathbf{Z}}_2\pi) \xrightarrow{\Gamma} \overline{I(\hat{\mathbf{Z}}_2\pi)} \xrightarrow{\omega} \pi^{\mathrm{ab}} \to 0$$

are $\mathbf{Z}_2$-linear. Then

$$H^1(\mathbf{Z}_2;\ \overline{I(\hat{\mathbf{Z}}_2\pi)}) = 0$$

(the involution permutes a $\hat{\mathbf{Z}}_2$-basis of $\overline{I(\hat{\mathbf{Z}}_2\pi)}$); and so

$$H^1(\mathrm{Wh}'(\hat{\mathbf{Z}}_2\pi)) \cong \operatorname{Coker}[H^0(\mathbf{Z}_2;\ \overline{I(\hat{\mathbf{Z}}_2\pi)}) \to H^0(\mathbf{Z}_2;\ \pi^{\mathrm{ab}})]$$

$$\cong \frac{\{[g] \in \pi^{\mathrm{ab}} \mid [g^2] = e\}}{\langle [g] \mid g \text{ conjugate } g^{-1}\rangle}.$$

EXAMPLES. The only 2-groups π of order $\leqq 32$ with $H^1(\mathbf{Z}_2;\ \mathrm{Wh}'(\hat{\mathbf{Z}}_2\pi)) \neq 0$ are the two semidirect products

$$\mathbf{Z}_8 \rtimes \mathbf{Z}_4 = \langle a, b \mid a^8 = b^4 = e,\ bab^{-1} = a^r\rangle$$

for $r = 3$ or 5. For an arbitrary metacyclic 2-group

$$\pi = \langle a, b \mid a^{2^n} = e,\ b^{2^m} = a^s,\ bab^{-1} = a^r\rangle,$$

Proposition 13 applies to show that

$$H_1(\mathbf{Z}_2;\ \mathrm{Wh}'(\hat{\mathbf{Z}}_2\pi)) \cong 0 \quad \text{if } m = 1, \text{ or } 4 \nmid s, \text{ or } r \equiv \pm 1 \pmod{2^n}$$

$$\cong \mathbf{Z}_2 \quad \text{otherwise}.$$

3.

The results in Section 2 will now be applied to compute $SK_1(A\pi)$. This will be based to a large extent on using the snake lemma exact sequence

$$\operatorname{Ker}[\operatorname{Wh}(A\tilde{\pi}) \to \operatorname{Wh}(A\pi)] \xrightarrow{\beta} \operatorname{Ker}[\operatorname{Wh}'(A\tilde{\pi}) \to \operatorname{Wh}'(A\pi)]$$
$$\xrightarrow{\partial} \operatorname{Coker}[SK_1(A\tilde{\pi}) \to SK_1(A\pi)] \to 0$$

to study $\operatorname{Coker}(SK_1)$ when $\tilde{\pi} \to \pi$ is a surjection of p-groups.

LEMMA 14. *Let $1 \to \varrho \to \tilde{\pi} \to \pi \to 1$ be an extension of p-groups, such that ϱ is generated by commutators in $\tilde{\pi}$. Then for any unramified p-ring A, the induced map*

$$SK_1(A\tilde{\pi}) \to SK_1(A\pi)$$

is onto.

PROOF. Lemma 12 (on the existence of central commutators) allows us to reduce this to the case where $\varrho \cong \mathbf{Z}_p$. Let $z \in \varrho$ be a generator, and consider the maps

$$\operatorname{Ker}[\operatorname{Wh}(A\tilde{\pi}) \to \operatorname{Wh}(A\pi)] \xrightarrow{\beta} \operatorname{Ker}[\operatorname{Wh}'(A\pi) \to \operatorname{Wh}'(A\pi)]$$
$$\xrightarrow[\cong]{\Gamma_0} \operatorname{Ker}[\overline{I(A\tilde{\pi})} \to \overline{I(A\pi)}]\ .$$

The composite is surjective by Proposition 11 ($\Gamma_0 = \log$ in this case). By the snake lemma,

$$\operatorname{Coker}[SK_1(A\tilde{\pi}) \to SK_1(A\pi)] \cong \operatorname{Coker}(\beta) = 0\ .$$

This can now be applied to the next proposition, which reduces the computation of $SK_1(A\pi)$ to the case when A is unramified over $\hat{\mathbf{Z}}_p$.

PROPOSITION 15. *Let $A \subseteq B$ be any totally ramified extension of p-rings. Then, for any p-group π, the inclusion $A \subseteq B$ induces an isomorphism*

$$i_*\colon SK_1(A\pi) \to SK_1(B\pi)\ .$$

PROOF. Replacing A, if necessary, by its maximal sub-p-ring unramified over $\hat{\mathbf{Z}}_p$, we may assume that it is unramified. Then A/p is the residue field for both A and B, so the map

$$K_1(A/p[\pi]) \to K_1(B/p[\pi])$$

is split injective. Since

$$SK_1(A\pi) \subseteq K_1(A/p[\pi]) \quad \text{and} \quad SK_1(B\pi) \subseteq K_1(B/p[\pi])$$

($K_1(A\pi, pI)$ and $K_1(B\pi, pI)$ being torsion free by Proposition 2), i_* is injective.

In proving surjectivity, we may clearly assume that π is non-abelian. Let $z \in \pi$ be a central commutator of order p, and assume inductively the proposition holds for $\pi' = \pi/z$. Since $SK_1(A\pi)$ maps onto $SK_1(A\pi')$ (Lemma 14), it suffices to show that i_* restricts to a surjection

$$i_0\colon \operatorname{Ker}[SK_1(A\pi) \to SK_1(A\pi')] \to \operatorname{Ker}[SK_1(B\pi) \to SK_1(B\pi')]\,.$$

Letting

$$\tau_A\colon A \to \boldsymbol{F}_p \quad \text{and} \quad \tau_B\colon B \to \boldsymbol{F}_p$$

be the maps used in Section 1, note that $\tau_A = \tau_B | A$: The inclusion $A \subseteq B$ induces an isomorphism of residue fields. So by Theorem 1, i_0 sends a generating set for the first kernel onto a generating set for the second, and is thus onto.

In particular, this result implies, for any p-ring A with residue field $\bar{A}$ and any p-group π, that $SK_1(A\pi)$ is a subgroup of $K_1(\bar{A}\pi)$ depending only on $\bar{A}$ (not on A itself).

We next apply Theorem 2 to compute

$$\operatorname{Coker}[SK_1(A\tilde{\pi}) \to SK_1(A\pi)]$$

when A is an unramified p-ring, and

$$1 \to \varrho \xrightarrow{\iota} \tilde{\pi} \xrightarrow{\alpha} \pi \to 1$$

an arbitrary extension of p-groups. Consider the diagram

$$\begin{array}{ccccccccc}
0 \to & SK_1(A\varrho) & \to & \mathrm{Wh}(A\varrho) & \xrightarrow{\Gamma} & I(A\varrho) & \xrightarrow{\omega} & \varrho^{ab} & \to 0 \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle \overline{I(\iota)}} & & \downarrow{\scriptstyle \iota^{ab}} & \\
0 \to & SK_1(A\tilde{\pi}) & \to & \mathrm{Wh}(A\tilde{\pi}) & \xrightarrow{\Gamma} & \overline{I(A\tilde{\pi})} & \xrightarrow{\omega} & \tilde{\pi}^{ab} & \to 0 \\
& {\scriptstyle \alpha_*}\downarrow & & \downarrow{\scriptstyle \mathrm{Wh}(\alpha)} & & \downarrow & & \downarrow & \\
0 \to & SK_1(A\pi) & \to & \mathrm{Wh}(A\pi) & \xrightarrow{\Gamma} & \overline{I(A\pi)} & \xrightarrow{\omega} & \pi^{ab} & \to 0
\end{array}$$

The rows are all exact by Theorem 2, and the composite in any column is zero. A little diagram chasing shows that there is a well defined "boundary" map, analogous to that in the snake lemma:

$$\Delta = [\mathrm{Wh}(\alpha) \circ \Gamma_{A\tilde{\pi}}^{-1} \circ \overline{I(\iota)} \circ \omega^{-1}]\colon \operatorname{Ker}(\iota^{ab}) \to \operatorname{Coker}(\alpha_*)\,.$$

Let $\tilde{\varkappa}_\alpha$ denote the composite

$$\tilde{\varkappa}_\alpha\colon \varrho_0 = \varrho \cap [\tilde{\pi},\tilde{\pi}] \twoheadrightarrow \operatorname{Ker}[\iota^{ab}\colon \varrho^{ab} \to \tilde{\pi}^{ab}] \xrightarrow{\Delta} \operatorname{Coker}[SK_1(A\tilde{\pi}) \to SK_1(A\pi)]\,.$$

PROPOSITION 16. *Let $1 \to \varrho \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1$ be an arbitrary extension of p-groups, and set*

$$\varrho_0 = \varrho \cap [\tilde{\pi},\tilde{\pi}]$$
$$\varrho_1 = \langle z \in \varrho \mid z \text{ a commutator in } \tilde{\pi}\rangle\,.$$

Then for any unramified p-ring A, $\tilde{\varkappa}_\alpha$ induces an isomorphism

$$\varkappa_\alpha\colon \varrho_0/\varrho_1 \xrightarrow{\cong} \operatorname{Coker}[SK_1(A\tilde{\pi}) \to SK_1(A\pi)]$$

which is natural with respect to maps between extensions.

PROOF. $\tilde{\varkappa}_\alpha$ is clearly natural, by construction. Consider the maps

$$SK_1(A\tilde{\pi}) \xrightarrow{\alpha_{1*}} SK_1(A[\tilde{\pi}/\varrho_1]) \xrightarrow{\alpha_{2*}} SK_1(A\pi)\,.$$

Since ϱ_1 is generated by commutators. α_{1*} is onto by Lemma 14. So $\tilde{\varkappa}_\alpha | \varrho_1 = 0$ by naturality, and $\tilde{\varkappa}_\alpha$ factors through ϱ_0/ϱ_1. Furthermore, α_{2*} is one-to-one by Proposition 7 (ϱ/ϱ_1 contains no commutators), and so it suffices (looking at subextensions) to show that $\varkappa_\alpha$ is an isomorphism when $\varrho \cong \mathbf{Z}_p$ and contains no commutators.

Consider the diagram

$$\begin{array}{ccccc}
\operatorname{Ker}[\operatorname{Wh}(A\tilde{\pi}) \to \operatorname{Wh}(A\pi)] & \xrightarrow{\lg} & \operatorname{Ker}[\overline{I(A\tilde{\pi})} \to \overline{I(A\pi)}] & \xrightarrow{\nu} & \mathbf{Z}_p \to 0 \\
\downarrow\beta & & \downarrow\mathrm{id} & & \\
0 \to \operatorname{Ker}[\operatorname{Wh}'(A\tilde{\pi}) \to \operatorname{Wh}'(A\pi)] & \xrightarrow{\Gamma} & \operatorname{Ker}[\overline{I(A\tilde{\pi})} \to \overline{I(A\pi)}] & \xrightarrow{\omega} & \operatorname{Ker}[\tilde{\pi}^{ab} \to \pi^{ab}] \to 0 \\
\downarrow\partial & & & & \\
\operatorname{Coker}[SK_1(A\tilde{\pi}) \to SK_1(A\pi)] & & & & \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

The first row is exact by Proposition 11, the second by Theorem 2, and the column by the snake lemma. If $\varrho \nsubseteq [\tilde{\pi},\tilde{\pi}]$ (so $\varrho_0 = \varrho_1 = 1$), then $\operatorname{Ker}(\nu) = \operatorname{Ker}(\omega)$, β is onto, and

$$\varrho_0/\varrho_1 \cong 0 \cong \operatorname{Coker}(\beta) \cong \operatorname{Coker}[SK_1(A\tilde{\pi}) \to SK_1(A\pi)]\,.$$

If $\varrho \subseteq [\tilde{\pi},\tilde{\pi}]$, then $\varrho_0/\varrho_1 \cong \mathbf{Z}_p$ and $\tilde{\pi}^{ab} \cong \pi^{ab}$. So

$$\operatorname{Coker}[SK_1(A\tilde{\pi}) \to SK_1(A\pi)] \cong \operatorname{Coker}(\beta) \cong \mathbf{Z}_p\,,$$

generated by $\partial\circ\Gamma_{A\tilde{\pi}}^{-1}((1-z)\xi)$ for any $\xi \in A\tilde{\pi}$ with $\tau\circ\varepsilon(\xi)\neq 0$. In particular, we may take $\xi=\lambda$ for some $\lambda \in A$ with $\mathrm{Tr}(\lambda)=-1$; and referring to the diagram used to define $\tilde{\varkappa}_\alpha$, we get

$$\partial\circ\Gamma_{A\tilde{\pi}}^{-1}((1-z)\lambda) = \Delta(\omega((1-z)\lambda)) = \Delta(z^{-\mathrm{Tr}(\lambda)}) = \Delta(z).$$

So $\varkappa_\alpha$ is also an isomorphism in this case.

In order to get a "universal" ϱ_0/ϱ_1 for extensions of a p-group π, we now consider $H_2(\pi)$ using Hopf's definition:

$$H_2(\pi) = R\cap[F,F]/[R,F]$$

when $\pi=F/R$ and F is free (see, e.g. [2, Section VI.9]). Given any central extension

$$1\to\varrho\to\tilde{\pi}\xrightarrow{\alpha}\pi\to 1,$$

choose a lifting $\tilde{\nu}\colon F\to\tilde{\pi}$ of the projection $\nu\colon F\to F/R=\pi$. Then

$$\tilde{\nu}(R)\subseteq\varrho,\qquad \tilde{\nu}([R,F]) = 1,$$

and $\tilde{\nu}$ induces a well defined map (independant pf ν and $\tilde{\nu}$)

$$\delta^\alpha\colon H_2(\pi)\to\varrho.$$

Clearly, $\mathrm{Im}(\delta^\alpha)=\varrho\cap[\tilde{\pi},\tilde{\pi}]$. By naturality, δ^α is seen to be the same as the homomorphism δ_*^E in [7]; in particular, it is part of an exact sequence ([7, Theorem V.2.2]):

$$\varrho\otimes\tilde{\pi}^{\mathrm{ab}}\xrightarrow{\gamma} H_2(\tilde{\pi})\xrightarrow{\alpha_*} H_2(\pi)\xrightarrow{\delta^\alpha}\varrho\to\tilde{\pi}^{\mathrm{ab}}\to\pi^{\mathrm{ab}}.$$

Lemma 17. *Let π be a p-group.*

(i) *There is a central extension $1\to\sigma\to\tilde{\pi}\xrightarrow{\alpha}\pi\to 1$ such that δ^α is an isomorphism.*

(ii) *For any $\varrho\lhd\pi$ there is an extension $1\to\sigma\to\tilde{\pi}\xrightarrow{\bar{\alpha}}\pi\to 1$; such that upon setting $\tilde{\varrho}=\bar{\alpha}^{-1}(\varrho)$, the subextension*

$$1\to\sigma\to\tilde{\varrho}\xrightarrow{\alpha}\varrho\to 1$$

is central, and δ^α is one-to-one.

Proof. (i) See Proposition V.5.1 in [7].

(ii) Set $q=|\pi/\varrho|$, and regard π as a subgroup of the wreath product $\varrho\wr\Sigma_q=\varrho^q\rtimes\Sigma_q$. Fix a central extension

$$1\to\sigma\to\varrho'\xrightarrow{\alpha'}\varrho\to 1$$

of p-groups such that $\delta^{\alpha'}$ is an isomorphism; and set

$$\tilde{\varrho} = (\alpha' \wr \Sigma_q)^{-1}(\varrho) \quad \text{and} \quad \tilde{\pi} = (\alpha' \wr \Sigma_q)^{-1}(\pi)$$

(as subgroups of $\varrho' \wr \Sigma_q$). This gives extensions

$$1 \to \sigma^q \to \tilde{\varrho} \xrightarrow{\alpha} \varrho \to 1 \quad \text{and} \quad 1 \to \sigma^q \to \tilde{\pi} \xrightarrow{\bar{\alpha}} \pi \to 1;$$

δ^α is the composite of $\delta^{\alpha'}$ with the diagonal, and hence one-to-one.

Now define, for any p-group π,

$$H_2^{ab}(\pi) = \operatorname{Im}[\textstyle\sum \{H_2(\sigma) \mid \sigma \subseteq \pi, \sigma \text{ abelian}\} \to H_2(\pi)] .$$

Since $H_2(\pi) = \Lambda^2(\pi)$ for π abelian, $H_2^{ab}(\pi)$ is just the subgroup generated by elements

$$g \wedge h = [\bar{g}, \bar{h}] \in H_2(\pi) ,$$

where $g, h \in \pi$, $gh = hg$, and $\bar{g}, \bar{h}$ are liftings to some free group. So for any central extension

$$1 \to \varrho \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1$$

we get

$$\delta^\alpha(H_2^{ab}(\pi)) = \langle g \in \varrho \mid g \text{ a commutator in } \tilde{\pi} \rangle .$$

Now let

$$1 \to \varrho \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1$$

be an arbitrary extension, and define

$$\varrho_0 = \varrho \cap [\tilde{\pi}, \tilde{\pi}], \qquad \varrho_1 = \langle g \in \varrho \mid g \text{ a commutator in } \tilde{\pi} \rangle$$

as before. Then ϱ/ϱ_1 is central in $\tilde{\pi}/\varrho_1 ([\varrho, \tilde{\pi}] \subseteq \varrho_1)$, and for any unramified p-ring A there are maps

$$SK_1(A\pi) \twoheadrightarrow \operatorname{Coker}[SK_1(A\tilde{\pi}) \to SK_1(A\pi)] \xleftarrow[\cong]{\varkappa_\alpha} \varrho_0/\varrho_1 \twoheadleftarrow^{\bar{\delta}^\alpha} H_2(\pi)/H_2^{ab}(\pi) .$$

Define

$$\Theta_{A\pi}\colon SK_1(A\pi) \to H_2(\pi)/H_2^{ab}(\pi)$$

to be the composite of these maps for any extension α such that δ^α is one-to-one (so that $\bar{\delta}^\alpha$ is an isomorphism). Using the naturality of $\varkappa_\alpha$ and taking pullbacks of extension, one easily checks that $\Theta_{A\pi}$ is well defined independently of α.

PROPOSITION 18. *Let A be an unramified p-ring and π a p-group. Then the epimorphism*

$$\Theta_{A\pi}: SK_1(A\pi) \to H_2(\pi)/H_2^{ab}(\pi)$$

is natural in π. If $\Theta_{A\tilde{\pi}}$ is an isomorphism for any p-group $\tilde{\pi}$ surjecting onto π, then $\Theta_{A\pi}$ is also an isomorphism.

PROOF. Naturally follows from the naturality of $\varkappa_\alpha$ and δ^α. To check the last statement, it suffices to show for any π that $\Theta_{A\pi}$ is an isomorphism if and only if there is some $\tilde{\pi}$ surjecting onto π so that

$$SK_1(A\tilde{\pi}) \to SK_1(A\pi)$$

is the zero map.

If there is such an $\alpha: \tilde{\pi} \twoheadrightarrow \pi$, then the surjection

$$H_2(\pi)/H_2^{ab}(\pi) \twoheadrightarrow \varrho_0/\varrho_1 \cong \operatorname{Coker}[SK_1(A\tilde{\pi}) \to SK_1(A\pi)] = SK_1(A\pi)$$

($\varrho_1 \subseteq \varrho_0 \subseteq \varrho = \operatorname{Ker}(\alpha)$ as before) shows that $\Theta_{A\pi}$ is an isomorphism. Conversely, if $\Theta_{A\pi}$ is an isomorphism, choose any central extension

$$1 \to \varrho \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1$$

for which δ^α is one-to-one; then

$$SK_1(A\pi) \cong H_2(\pi)/H_2^{ab}(\pi) \cong \varrho_0/\varrho_1 \cong \operatorname{Coker}[\alpha_*: SK_1(A\tilde{\pi}) \to SK_1(A\pi)]\,,$$

and so $\alpha_* = 0$.

We now prove that $\Theta_{A\pi}$ always is an isomorphism by studying

$$\operatorname{Coker}[SK_1(A\varrho) \to SK_1(A\pi)]$$

when $\varrho \lhd \pi$ with cyclic quotient. This involves two lemmas.

LEMMA 19. *Let A be an unramified p-ring, and $\varrho \lhd \pi$ p-groups with π/ϱ cyclic. Then the boundary map*

$$\partial: \operatorname{Ker}[\mathrm{Wh}'(A\varrho) \to \mathrm{Wh}'(A\pi)] \to \operatorname{Coker}[SK_1(A\varrho) \to SK_1(A\pi)]$$

(induced by the snake lemma) is onto.

PROOF. This is clear if ϱ is abelian: $SK_1(A\pi) = 0$ by Proposition 9. If not, choose some central $z \in [\varrho, \varrho]$ which is a commutator in π (Lemma 12). Set $\pi' = \pi/z$, $\varrho' = \varrho/z$, and assume the lemma holds for $\varrho' \lhd \pi'$.

Consider the commutative diagram:

$$
\begin{array}{ccc}
 & & \operatorname{Ker}[SK_1(A\pi) \to SK_1(A\pi')] & \\
 & & \downarrow \eta & \\
\operatorname{Ker}[\mathrm{Wh}'(A\varrho) \to \mathrm{Wh}'(A\pi)] & \xrightarrow{\partial} & \operatorname{Coker}[SK_1(A\varrho) \to SK_1(A\pi)] & \xrightarrow{j_*} \\
\downarrow \nu_1 & & \downarrow \nu_2 & \operatorname{Coker}[\mathrm{Wh}(A\varrho) \to \mathrm{Wh}(A\pi)] \\
\operatorname{Ker}[\mathrm{Wh}'(A\varrho') \to \mathrm{Wh}'(A\pi')] & \xrightarrow{\partial'} & \operatorname{Coker}[SK_1(A\varrho') \to SK_1(A\pi')] \to 0 &
\end{array}
$$

where the rows are exact by the snake lemma (and the induction hypothesis). We will show

(i) $j_* \circ \eta = 0$, (ii) ν_1 is onto, and
(iii) $\operatorname{Ker}(\nu_2) \subseteq \operatorname{Im}(\eta) + \operatorname{Im}(\partial)$.

Surjectivity of ∂ then follows by diagram chasing.

(i) Showing $j_* \circ \eta = 0$ amounts to checking that every element in $\operatorname{Ker}[SK_1(A\pi) \to SK_1(A\pi')]$ has the form $1 + (1-z)\xi$ for some $\xi \in A\varrho$. By Theorem 1, it is enough to show that every equivalence class in $\mathscr{S} = \mathscr{S}(\pi, z)$ contains elements of ϱ'. Let $\alpha\colon \pi \to \pi'$ denote the projection. Then for any $g \in \alpha^{-1}(\mathscr{S})$, $[g, h] = z$ for some $h \in \pi$, all elements in

$$\{\alpha(gh^i), \alpha(g^i h) \mid i \in \mathbf{Z}\}$$

lie in the same equivalence class of $\mathscr{S}$, and at least one of them is in ϱ'.

(ii) Theorem 2 yields a commutative diagram

$$
\begin{array}{ccccccc}
0 \to & \operatorname{Ker}[\mathrm{Wh}'(A\varrho) \to \mathrm{Wh}'(A\pi)] & \to & \operatorname{Ker}[\overline{I(A\varrho)} \to \overline{I(A\pi)}] & \to & \operatorname{Ker}[\varrho^{\mathrm{ab}} \to \pi^{\mathrm{ab}}] \\
 & \downarrow \nu_1 & & \downarrow \bar{\nu}_1 & & \downarrow \cong \\
0 \to & \operatorname{Ker}[\mathrm{Wh}'(A\varrho') \to \mathrm{Wh}'(A\pi')] & \to & \operatorname{Ker}[\overline{I(A\varrho')} \to \overline{I(A\pi')}] & \to & \operatorname{Ker}[(\varrho')^{\mathrm{ab}} \to (\pi')^{\mathrm{ab}}] .
\end{array}
$$

One easily sees that $\bar{\nu}_1$ is onto, and so the same holds for ν_1.

(iii) Consider the following maps:

$$
\begin{array}{ccccccc}
 & 0 \to SK_1(A\varrho') & & \to \mathrm{Wh}(A\varrho') & & \to \mathrm{Wh}'(A\varrho') \to 0 & \\
 {}^{i_1'}\swarrow & & \nwarrow^{f_1} & & \nwarrow^{f_2} & & \nwarrow^{f_3} \\
SK_1(A\pi') & & 0 \to SK_1(A\varrho) & \to & \mathrm{Wh}(A\varrho) & \to & \mathrm{Wh}'(A\varrho) \to 0 \\
 {}^{f_1'}\nwarrow & & \swarrow_{i_1} & & \swarrow_{i_2} & & \swarrow_{i_3} \\
 & 0 \to SK_1(A\pi) & & \to \mathrm{Wh}(A\pi) & & \to \mathrm{Wh}'(A\pi) \to 0 &
\end{array}
$$

If z is a commutator in ϱ, then f_1 and f'_1 are both onto (Lemma 14), the sequence

$$\operatorname{Ker}(f'_1) \xrightarrow{\eta} \operatorname{Coker}(i_1) \xrightarrow{\nu_2} \operatorname{Coker}(i'_1)$$

is exact, and we are done.

If z is not a commutator in ϱ, then $\operatorname{Coker}(f_1) \cong \mathbf{Z}_p$ by Proposition 16. Choose $g \in \varrho$ such that g and zg are conjugate in π (z *is* a commutator in π); fix $\lambda \in A$ such that $\operatorname{Tr}(\lambda) \not\equiv 0 \pmod p$; and set

$$u = \Gamma^{-1}(\lambda(1-z)g) \in \operatorname{Wh}'(A\varrho).$$

Lift u to $\tilde{u} \in \operatorname{Wh}(A\varrho)$; then since $f_3(u) = 0 = i_3(u)$,

$$f_2(\tilde{u}) \in SK_1(A\varrho'),\ i_2(\tilde{u}) = \partial(u) \in SK_1(A\pi);$$

and $f_2(\tilde{u})$ generates $\operatorname{Coker}(f_1)$ by Proposition 16 and the definition of $\varkappa_\alpha$. Since $f_2(\tilde{u})$ and $i_2(\tilde{u})$ map to the same element of $SK_1(A\pi')$, diagram chasing shows that

$$\operatorname{Ker}[\nu_2 \colon \operatorname{Coker}(i_1) \to \operatorname{Coker}(i'_1)] = \langle \operatorname{Im}(\eta), i_2(\tilde{u}) \rangle \subseteq \operatorname{Im}(\eta) + \operatorname{Im}(\partial).$$

This gives in turn:

LEMMA 20. *Let $\varrho \lhd \pi$ be p-groups with $\pi/\varrho \cong \mathbf{Z}_p$. Then for any unramified p-ring A,*

$$\operatorname{Coker}[SK_1(A\varrho) \to SK_1(A\pi)] \cong \operatorname{Coker}[H_2(\varrho)/H_2^{ab}(\varrho) \to H_2(\pi)/H_2^{ab}(\pi)].$$

In particular, if $SK_1(A\varrho) = 0$, then $\Theta_{A\pi}$ is an isomorphism.

PROOF. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & SK_1(A\varrho) & \to & \operatorname{Wh}(A\varrho) & \to & \operatorname{Wh}'(A\varrho) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & SK_1(A\pi) & \to & \operatorname{Wh}(A\pi) & \to & \operatorname{Wh}'(A\pi) & \to & 0. \end{array}$$

By Lemma 19, there is an exact sequence

$$\operatorname{Ker}[\operatorname{Wh}(A\varrho) \to \operatorname{Wh}(A\pi)] \xrightarrow{\beta} \operatorname{Ker}[\operatorname{Wh}'(A\varrho) \to \operatorname{Wh}'(A\pi)] \to \operatorname{Coker}[SK_1(A\varrho) \to SK_1(A\pi)] \to 0.$$

The composite

$$\operatorname{Wh}'(A\varrho) \to \operatorname{Wh}'(A\pi) \xrightarrow{\operatorname{trf}} \operatorname{Wh}'(A\varrho)$$

is induced by tensoring with the bimodule ${}_{A\varrho}A\pi_{A\varrho}$, and is hence the norm map N for the action of π/ϱ on $\operatorname{Wh}'(A\varrho)$. In particular,

$$\mathrm{Ker}\,[\mathrm{Wh}'(A\varrho) \to \mathrm{Wh}'(A\pi)] \subseteq \mathrm{Ker}\,(N)\,.$$

Let $x \in \pi$ be any element generating π/ϱ and let ξ denote the action on $\mathrm{Wh}'(A\varrho)$ induced by conjugation by x. Elements of the form $a^{-1}\xi(a)$ for $a \in \mathrm{Wh}'(A\varrho)$ clearly lie in $\mathrm{Im}\,(\beta)$, and it follows that

$$\mathrm{Coker}\,(\beta) \cong \mathrm{Coker}\,[SK_1(A\varrho) \to SK_1(A\pi)]$$

is isomorphic to a subquotient of $H^1(\pi/\varrho;\ \mathrm{Wh}'(A\varrho))$ (π/ϱ being cyclic). Since $\Theta_{A\pi}$ induces a surjection

$$\mathrm{Coker}\,[SK_1(A\varrho) \to SK_1(A\pi)] \twoheadrightarrow \mathrm{Coker}\,[H_2(\varrho)/H_2^{\mathrm{ab}}(\varrho) \to H_2(\pi)/H_2^{\mathrm{ab}}(\pi)]\,,$$

we will be done upon showing that

$$\mathrm{Coker}\,[H_2(\varrho)/H^{\mathrm{ab}}(\varrho) \to H_2(\pi)/H_2^{\mathrm{ab}}(\pi)] \cong H^1(\pi/\varrho;\ \mathrm{Wh}'(A\varrho))$$

(both groups being finite).

First note that $H^1(\pi/\varrho;\ \overline{I(A\varrho)})=0$: $\overline{I(A\varrho)}$ has a $\hat{\mathbf{Z}}_p$-basis permuted by π/ϱ. So

$$H^1(\pi/\varrho;\ \mathrm{Wh}'(A\varrho)) \cong \mathrm{Coker}\,[H^0(\pi/\varrho;\ \overline{I(A\varrho)}) \to H^0(\pi/\varrho;\ \varrho^{\mathrm{ab}})]\,.$$

On the other hand, the spectral sequence for $\varrho \lhd \pi$ induces an exact sequence

$$H_3(\pi/\varrho) \xrightarrow{\partial} H_1(\pi/\varrho;\ \varrho^{\mathrm{ab}}) \to \mathrm{Coker}\,[H_2(\varrho) \to H_2(\pi)] \to 0\,.$$

Identifying $H_1(\pi/\varrho;\ \varrho^{\mathrm{ab}})$ with $H^0(\pi/\varrho;\ \varrho^{\mathrm{ab}})$, this gives

$$\mathrm{Coker}\,[H_2(\varrho)/H_2^{\mathrm{ab}}(\varrho) \to H_2(\pi)/H_2^{\mathrm{ab}}(\pi)] \cong H^0(\pi/\varrho;\ \varrho^{\mathrm{ab}})/J$$

for some $J \subseteq H^0(\pi/\varrho;\ \varrho^{\mathrm{ab}})$.

Let $x \in \pi$ again denote an element generating π/ϱ. By comparison with the spectral sequence for $\langle x\rangle$, we see that

$$\mathrm{Im}\,(\partial) = \langle x^p\rangle \in H^0(\pi/\varrho;\ \varrho^{\mathrm{ab}})\,.$$

Similarly, for any $g, h \in \varrho$ such that $[gx, h]=e$, a comparison with the spectral sequence for $\langle gx, h\rangle$ shows that the image of

$$H_2(\langle gx, h\rangle) \to H_2(\pi) \twoheadrightarrow H^0(\pi/\varrho;\ \varrho^{\mathrm{ab}})/\mathrm{Im}\,(\partial)$$

is generated by

$$[h] \in H^0(\pi/\varrho;\ \varrho^{\mathrm{ab}})/\mathrm{Im}\,(\partial)\,.$$

Since any rank-2 abelian subgroup of π not in ϱ has this form,

$$J = \langle [h] \in H^0(\pi/\varrho;\ \varrho^{\mathrm{ab}}) \mid h \in \varrho,\ h \text{ conjugate to } xhx^{-1} \text{ in } \varrho\rangle\,,$$

which is precisely the image of $H^0(\pi/\varrho;\ \overline{I(A\varrho)})$ in $H^0(\pi/\varrho;\ \varrho^{\mathrm{ab}})$.

We have thus shown that $\Theta_{A\pi}$ is an isomorphism if either (i) $\Theta_{A\tilde{\pi}}$ is an isomorphism for some $\tilde{\pi}$ surjecting onto π, or (ii) $SK_1(A\varrho)=0$ for some $\varrho \lhd \pi$ of index p. Proving that $\Theta_{A\pi}$ always is an isomorphism is now just a matter of choosing the right inductive argument.

THEOREM 3. *For any p-group π and unramified p-ring A,*

$$\Theta_{A\pi}\colon SK_1(A\pi) \to H_2(\pi)/H_2^{ab}(\pi)$$

is an isomorphism.

PROOF. Fix A, and let $\mathscr{P}$ be the family of all p-groups π such that $\Theta_{A\tilde{\pi}}$ is an isomorphism for any central extension $\tilde{\pi} \twoheadrightarrow \pi$. We will show inductively that $\mathscr{P}$ contains all p-groups.

Let π be an arbitrary p-group, $\varrho \lhd \pi$ a subgroup of index p, and assume $\varrho \in \mathscr{P}$. Now fix

(i) an extension $1 \to \sigma_0 \to \tilde{\pi}_0 \xrightarrow{\bar{\alpha}_0} \pi \to 1$, such that the subextension

$$1 \to \sigma_0 \to \tilde{\varrho}_0 \xrightarrow{\alpha_0} \varrho \to 1 \qquad (\tilde{\varrho}_0 = \bar{\alpha}_0^{-1}(\varrho))$$

is central and δ^{α_0} is one-to-one (Lemma 17).

(ii) an arbitrary central extension $1 \to \sigma_1 \to \tilde{\pi}_1 \xrightarrow{\bar{\alpha}_1} \pi \to 1$

(iii) $\bar{\alpha}\colon \tilde{\pi} \to \pi$ the pullback of $\tilde{\pi}_0$ and $\tilde{\pi}_1$ over π

(iv) $\alpha\colon \tilde{\varrho} \to \varrho$ the pullback of $\tilde{\varrho}_0$ and $\tilde{\varrho}_1 = \bar{\alpha}_1^{-1}(\varrho)$ over ϱ.

Then $\tilde{\pi}/\tilde{\varrho} \cong \mathbf{Z}_p$, and the extension

$$1 \to \sigma \to \tilde{\varrho} \xrightarrow{\alpha} \varrho \to 1$$

is central. Furthermore,

$$\delta^{\alpha}\colon H_2(\varrho) \to \sigma = \sigma_0 \times \sigma_1$$

is one-to-one, since δ^{α_0} is.

Consider the exact sequence ([7, V.2.2])

$$\sigma \otimes \tilde{\varrho}^{ab} \xrightarrow{\gamma} H_2(\tilde{\varrho}) \xrightarrow{\alpha_*} H_2(\varrho) \xrightarrow{\delta^{\alpha}} \sigma\,,$$

where $\gamma(x \otimes g) = x \wedge g$ for $x \in \sigma$ and $g \in \tilde{\varrho}$ ([7, V.2.1]). In particular, $\operatorname{Im}(\gamma) \subseteq H_2^{ab}(\tilde{\varrho})$. Since δ^{α} is one-to-one, $\varrho \in \mathscr{P}$, and $\tilde{\varrho}$ is a central extension of ϱ,

$$SK_1(A\tilde{\varrho}) \cong H_2(\tilde{\varrho})/H_2^{ab}(\tilde{\varrho}) = 0\,.$$

Thus, $\Theta_{A\tilde{\pi}}$ is an isomorphism by Lemma 20. Since $\tilde{\pi}$ maps onto $\tilde{\pi}_1$, $\Theta_{A\tilde{\pi}_1}$ is also an isomorphism (Proposition 18). But $\tilde{\pi}_1$ was an arbitrary central extension of π, so $\pi \in \mathscr{P}$, and we are done.

Combining this with Proposition 15 shows that for any p-group π, $SK_1(A\pi)$ is independent of the choice of p-ring A. In the case of a totally ramified extension $B \supseteqq A$, it was the inclusion which induced an isomorphism $SK_1(A\pi) \cong SK_1(B\pi)$. If B/A is unramified, we now show that the transfer map induces the isomorphism:

PROPOSITION 21. *Let $B \supseteqq A$ be p-rings, and π a p-group.*

(i) *If B and A are both unramified over $\hat{\mathbf{Z}}_p$, then the triangle*

$$\begin{array}{ccc} SK_1(B\pi) & \xrightarrow{\Theta_{B\pi}} & \\ \downarrow{\scriptstyle \mathrm{trf}} & & H_2(\pi)/H_2^{\mathrm{ab}}(\pi) \\ SK_1(A\pi) & \xrightarrow{\Theta_{A\pi}} & \end{array}$$

commutes.

(ii) *If B/A is any unramified extension, then the transfer map*

$$\mathrm{trf}\colon SK_1(B\pi) \to SK_1(A\pi)$$

is an isomorphism.

PROOF. (i) This reduces to showing, for any central extension

$$1 \to \varrho \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1\ ,$$

that the diagram used to define $\varkappa_\alpha$ commutes with the trace and transfer maps. This is clear for the vertical maps (transfers with respect to ring extension commute with maps induced by group homomorphisms). The diagram

$$\begin{array}{ccccccccc} 0 \to & SK_1(B\pi) & \to & \mathrm{Wh}(B\pi) & \xrightarrow{\Gamma} & \overline{I(B\pi)} & \xrightarrow{\omega} & \pi^{\mathrm{ab}} & \to 0 \\ & \downarrow{\scriptstyle \mathrm{trf}} & & \downarrow{\scriptstyle \mathrm{trf}} & & \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{id}} & \\ 0 \to & SK_1(A\pi) & \to & \mathrm{Wh}(A\pi) & \xrightarrow{\Gamma} & \overline{I(A\pi)} & \xrightarrow{\omega} & \pi^{\mathrm{ab}} & \to 0 \end{array}$$

commutes (and similarly for ϱ and $\tilde{\pi}$) by Proposition 10 and the definition of ω.

(ii) Let $A_0 \subseteqq A$ and $B_0 \subseteqq B$ be the maximal sub-p-rings unramified over $\hat{\mathbf{Z}}_p$ (so A/A_0 and B/B_0 are totally ramified). Consider the diagram

$$\begin{array}{ccc} SK_1(B_0\pi) & \xrightarrow{\mathrm{incl}} & SK_1(B\pi) \\ \downarrow{\scriptstyle \mathrm{trf}} & & \downarrow{\scriptstyle \mathrm{trf}} \\ SK_1(A_0\pi) & \xrightarrow{\mathrm{incl}} & SK_1(A\pi)\,. \end{array}$$

The inclusion maps are isomorphisms by Proposition 15, and the transfer map for B_0/A_0 is isomorphic by Theorem 3 and (i) above. It remains only to check that the square commutes; in other words that the natural map

$$\mu\colon A\pi \otimes_{A_0\pi} B_0\pi \to B\pi$$

(induced by multiplication) is an isomorphism. Let $t \in A$ be any generator of its maximal ideal $\mathfrak{p}_A$ (and thus also a generator of $\mathfrak{p}_B$). Then

$$\{1, t, \ldots, t^{e-1}\} \quad (e = [A\colon A_0] = [B\colon B_0])$$

is an $A_0\pi$-basis for $A\pi$ and a $B_0\pi$-basis for $B\pi$; and the result follows.

4.

Some specific examples of computing $SK_1(A\pi)$ will now be considered. They are all fairly straightforward, so many of them are stated with sketchy proofs or without proofs. Since $SK_1(A\pi)$ has been shown to be independent of A, most results will be stated only for $SK_1(\hat{\mathbf{Z}}_p\pi)$.

The first lemma shows, among other things, that $H_2(\pi)$ need not be calculated completely; it suffices to understand $H_2(\pi/Z(\pi))$. As before, for commuting elements $g, h \in \pi$, define

$$g \wedge h = [\bar{g}, \bar{h}] \in H_2(\pi)$$

for any liftings $\bar{g}, \bar{h}$ of g, h to some free group covering π. Define

$$\Lambda(\pi) = \{g \wedge h \in H_2(\pi) \mid g, h \in \pi,\ gh = hg\} .$$

Note that while this only is a *subset* of $H_2(\pi)$, it is closed under taking powers ($(g \wedge h)^i = g \wedge h^i$). The subgroup generated by $\Lambda(\pi)$ is, of course, just $H_2^{ab}(\pi)$.

LEMMA 22. (i) *For any central extension* $1 \to \varrho \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1$ *of p-groups,*

$$SK_1(\hat{\mathbf{Z}}_p\tilde{\pi}) \cong H_2(\tilde{\pi})/H_2^{ab}(\tilde{\pi}) \cong \operatorname{Ker}(\delta^\alpha)/\langle \Lambda(\pi) \cap \operatorname{Ker}(\delta^\alpha)\rangle .$$

(ii) *A p-group π has the property that $SK_1(\hat{\mathbf{Z}}_p\tilde{\pi}) = 0$ for all central extensions $\tilde{\pi}$ of π, if and only if $\Lambda(\pi) = H_2(\pi)$.*

(iii) *Let G be an abelian group, and $\pi = G \rtimes \mathbf{Z}_{p^n}$ some semidirect product. Then $H_2(\pi) = R_1 \oplus R_2$, where*

$$R_1 = \operatorname{Im}[H_2(G) \to H_2(\pi)] \cong H_0(\mathbf{Z}_{p^n}; H_2(G)) ,$$

and

$$R_2 = \{x \wedge g \mid g \in G,\ xg = gx\} \cong H_1(\mathbf{Z}_{p^n}; G) \quad (x \in \mathbf{Z}_{p^n} \textit{ any generator}) .$$

PROOF. (i) Consider again the exact sequence of ([7, V.2.2]):

$$\varrho \otimes \tilde{\pi}^{ab} \xrightarrow{\gamma} H_2(\tilde{\pi}) \xrightarrow{\alpha_*} H_2(\pi) \xrightarrow{\delta^\alpha} \varrho ,$$

where $\gamma(x \otimes g) = x \wedge g$ for $x \in \varrho$ and $g \in \tilde{\pi}$. So $\operatorname{Im}(\gamma) \subseteq H_2^{ab}(\tilde{\pi})$; and furthermore,

$$\begin{aligned} \alpha_*(H_2^{ab}(\tilde{\pi})) &= \langle g \wedge h \in H_2(\pi) \mid g, h \text{ lift to commuting elements of } \tilde{\pi}\rangle \\ &= \langle \Lambda(\pi) \cap \operatorname{Ker}(\delta^\alpha)\rangle . \end{aligned}$$

(ii) If $\Lambda(\pi)=H_2(\pi)$, the result follows directly from (i). If not, choose some

$$x \in H_2(\pi)-\Lambda(\pi)$$

and fix a central extension $1 \to \varrho \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1$ with δ^α an isomorphism (Lemma 17). Then by (i),

$$SK_1(\hat{\mathbf{Z}}_p[\tilde{\pi}/\delta^\alpha(x)]) \neq 0 .$$

(This, applied to $\pi \cong (\mathbf{Z}_{p^n})^4$, is the idea behind Neumann's example calculating $H_2(\pi)/H_2^{\mathrm{ab}}(\pi)$ in [5]).

(iii) This can be done by first showing

$$H_2(G \rtimes \mathbf{Z}) \cong H_0(\mathbf{Z}; H_2(G)) \oplus H_1(\mathbf{Z}; G) ,$$

using the obvious spectral sequence; and then applying the exact sequence ([7, V.2.2]):

$$p^n\mathbf{Z} \otimes (G \rtimes \mathbf{Z})^{\mathrm{ab}} \xrightarrow{\gamma} H_2(G \rtimes \mathbf{Z}) \to H_2(G \rtimes \mathbf{Z}_{p^n}) \xrightarrow{\delta} p^n\mathbf{Z} ;$$

noting that $\delta=0$.

In proposition 9, it was shown that $SK_1(\hat{\mathbf{Z}}_p\pi)=0$ if π is a p-group having an abelian normal subgroup with cyclic quotient. Some other examples where $SK_1(\hat{\mathbf{Z}}_p\pi)=0$ follow:

PROPOSITION 23. *$SK_1(\hat{\mathbf{Z}}_p\pi)=0$ if π is a p-group and*

(i) $|\pi/Z(\pi)| \leqq p^3$,

(ii) *$\pi/Z(\pi)$ is abelian of rank* $\leqq 3$,

(iii) *π/ϱ is abelian for some central cyclic $\varrho \lhd \pi$.*

(iv) $|\pi| \leqq p^4$, *or*

(v) $p=2$ *and* $|\pi| \leqq 32$.

PROOF. (i) This amounts to checking that $\Lambda(\pi)=H_2(\pi)$ if π is a p-group and $|\pi| \leqq p^3$. This is easily seen if π is abelian; in the other cases one checks (applying Lemma 22 (iii) for the first two):

$$H_2(\pi) = \langle a^p \wedge b\rangle \cong \mathbf{Z}_p \quad \text{if } \pi = \langle a,b \mid a^{p^2}=b^p=e,\ bab^{-1}=a^{p+1}\rangle$$

$$H_2(\pi) = \{a^ib^j \wedge c\} \cong \mathbf{Z}_p^2$$
$$\text{if } \pi = \langle a,b,c \mid a^p=b^p=c^p=[a,b]=[b,c]=e, [a,b]=c\rangle$$

$$H_2(\mathrm{Q}(8)) = 0 \quad \text{(quaternionic group of order 8)} .$$

(ii) $\Lambda(\pi)=H_2(\pi)$ if π is an abelian p-group of rank at most 3.

(iii) We prove this when $\varrho \cong \mathbf{Z}_p$. Set $\pi'=\pi/\varrho$, let $\alpha\colon \pi \to \pi'$ be the projection. and consider

$$\delta = \delta^\alpha : H_2(\pi') \to \mathbf{Z}_p .$$

Ker (δ) is generated by elements of the form $g_1 \wedge h_1 - g_2 \wedge h_2$ (π' is abelian, so $H_2(\pi') = H_2^{ab}(\pi')$); and we must show any such element lines in $\langle \text{Ker}(\delta) \cap \Lambda(\pi) \rangle$. If $\delta(g_1 \wedge h_2) \neq 0$, then

$$g_1 \wedge h_1 - g_2 \wedge h_2 = g_1 \wedge h_1 h_2^i - g_1^i g_2 \wedge h_2$$

and $\delta(g_1 \wedge h_1 h_2^i) = \delta(g_1^i g_2 \wedge h_2) = 0$ for some i. So we are done in this case, and similarly if $\delta(g_2 \wedge h_1) \neq 0$. But if $\delta(g_1 \wedge h_2) = \delta(g_2 \wedge h_1) = 0$, then

$$g_1 \wedge h_1 - g_2 \wedge h_2 = g_1 g_2^{-1} \wedge h_1 h_2 \in \Lambda(\pi') .$$

(iv) This is a special case of (i).

(v) Let π be a group of order 32, fix some $\mathbf{Z}_2 \lhd \pi$, and set $\pi' = \pi/\mathbf{Z}_2$. If π' is abelian, then $SK_1(\hat{\mathbf{Z}}_2\pi) = 0$ by (iii), and if π' is metacyclic then $SK_1(\hat{\mathbf{Z}}_2\pi) = 0$ by Proposition 9. This leaves four possibilities for π' (see [3, p. 349, Aufgabe 30]):

$$\pi' = \mathbf{Z}_2 \times \mathrm{D}(8),\ \mathbf{Z}_2 \times \mathrm{Q}(8),\ \mathbf{Z}_4 \underset{\mathbf{Z}_2}{\times} \mathrm{D}(8),\ \text{or}\ (\mathbf{Z}_2 \times \mathbf{Z}_4) \rtimes \mathbf{Z}_2;$$

which are easily eliminated using either Lemma 22 (iii) or the Künneth formula. ($H_2(\pi') = \Lambda(\pi')$ for all except $\pi' \cong \mathbf{Z}_2 \times \mathrm{D}(8)$; we come back to this particular case in the next proposition).

Some minimal examples where $SK_1(\hat{\mathbf{Z}}_p\pi) \neq 0$ will now be constructed:

PROPOSITION 24. (i) *If p is odd, then there exist π with $|\pi| = p^5$ and $SK_1(\hat{\mathbf{Z}}_p\pi) \neq 0$.*

(ii) *There exist π with $|\pi| = 64$ and $SK_1(\hat{\mathbf{Z}}_2\pi) \neq 0$.*

PROOF. (i) First assume $p > 3$, and set

$$\pi = \langle a_1, a_2, a_3, x \mid a_i^p = x^p = [a_i, a_j] = [a_1, x] = e ,\ xa_2x^{-1} = a_1a_2,\ xa_3x^{-1} = a_2a_3 \rangle .$$

$$H_2(\pi) = \langle a_2 \wedge a_3,\ a_1 \wedge x \rangle \cong \mathbf{Z}_p^2 \quad \text{(Lemma 22 (iii))}$$

$$\Lambda(\pi) = \langle a_2 \wedge a_3 \rangle \cup \langle a_1 \wedge x \rangle \quad (a_1 \wedge a_3 = a_1 \wedge a_2 = 0) .$$

Using Lemma 17, construct $1 \to \mathbf{Z}_p \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1$ such that

$$\text{Ker}(\delta^\alpha) = \langle a_2 \wedge a_3 + a_1 \wedge x \rangle;$$

then $SK_1(\hat{\mathbf{Z}}_p\tilde{\pi}) \cong \mathbf{Z}_p$ by Lemma 22 (i).

If $p = 3$, set

$$\pi = \langle a, b, x \mid a^3 = b^9 = x^3 = [a, b] = e,\ xax^{-1} = ab^6,\ xbx^{-1} = ab \rangle .$$

$$H_2(\pi) = \langle a \wedge b,\ x \wedge b^3 \rangle \cong \mathbf{Z}_3^2, \quad \Lambda(\pi) = \langle a \wedge b \rangle \cup \langle x \wedge b^3 \rangle .$$

Choose $1 \to \mathbf{Z}_3 \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1$ with $\operatorname{Ker}(\delta^\alpha) = \langle a \wedge b + x \wedge b^3 \rangle$; then $SK_1(\hat{\mathbf{Z}}_3\pi) \cong \mathbf{Z}_3$.

(ii) Set $\pi = \mathbf{Z}_2 \times \mathrm{D}(8)$, where $\mathrm{D}(8) = \langle a, b \mid a^4 = b^2 = (ab)^2 = e \rangle$, and c generates $\mathbf{Z}_2$. Then

$$H_2(\pi) \cong \mathbf{Z}_2^3 \text{ with basis } \{a^2 \wedge b,\ a \wedge c,\ b \wedge c\}$$

$$\Lambda(\pi) = H_2(\pi) - \{a^2 \wedge b + a \wedge c\} .$$

Choose $1 \to \mathbf{Z}_2^2 \to \tilde{\pi} \xrightarrow{\alpha} \mathbf{Z}_2 \times \mathrm{D}(8) \to 1$ with $\operatorname{Ker}(\delta^\alpha) = \langle a^2 \wedge b + a \wedge c \rangle$; then $SK_1(\hat{\mathbf{Z}}_2\pi) \cong \mathbf{Z}_2$.

Some more general relations between the $SK_1(\hat{\mathbf{Z}}_p\pi)$ will now be shown. For any π, $\pi \wr \mathbf{Z}_p$ denotes the wreath product:

$$\pi \wr \mathbf{Z}_p = \pi^p \rtimes \mathbf{Z}_p ,$$

where $\mathbf{Z}_p$ acts by permuting the factors.

PROPOSITION 25. (i) *For any p-groups π_1 and π_2,*

$$SK_1(\hat{\mathbf{Z}}_p[\pi_1 \times \pi_2]) \cong SK_1(\hat{\mathbf{Z}}_p[\pi_1]) \oplus SK_1(\hat{\mathbf{Z}}_p[\pi_2]) .$$

(ii) $SK_1(\hat{\mathbf{Z}}_p[\pi \wr \mathbf{Z}_p]) \cong SK_1(\hat{\mathbf{Z}}_p\pi)$ *for any p-group π.*

(iii) $SK_1(\hat{\mathbf{Z}}_p\pi) = 0$ *if π is a p-Sylow subgroup in some symmetric group Σ_n.*

PROOF. (i) This follows immediately from the Künneth formula:

$$H_2(\pi_1 \times \pi_2) \cong H_2(\pi_1) \oplus H_2(\pi_2) \oplus \pi_1^{\mathrm{ab}} \otimes \pi_2^{\mathrm{ab}};$$

and

$$\pi_1^{\mathrm{ab}} \otimes \pi_2^{\mathrm{ab}} = \langle g \wedge h \mid g \in \pi_1,\ h \in \pi_2 \rangle \subseteq H_2^{\mathrm{ab}}(\pi_1 \times \pi_2) .$$

(ii) Let $i\colon \pi \to \pi \wr \mathbf{Z}_p$ be inclusion onto one factor of π^p (*not* the diagonal). We prove only that the induced map

$$i_*\colon SK_1(\hat{\mathbf{Z}}_p\pi) \to SK_1(\hat{\mathbf{Z}}_p[\pi \wr \mathbf{Z}_p])$$

is onto (this is all that's needed for part (iii)). Proving that i_* is an isomorphism requires a more detailed study of "wedges" in $H_2(\pi \wr \mathbf{Z}_p)$.

Note first that the map $H_2(\pi^p) \to H_2(\pi \wr \mathbf{Z}_p)$ induced by inclusion is onto: $E_{1,1}^2 \cong E_{2,0}^2 \cong 0$ in the spectral sequence for $\pi \wr \mathbf{Z}_p$. Surjectivity of i_* follows upon decomposing it as a composite:

$$SK_1(\hat{\mathbf{Z}}_p\pi) \xrightarrow{\cong} H_0(\mathbf{Z}_p; SK_1(\hat{\mathbf{Z}}_p[\pi^p])) \twoheadrightarrow SK_1(\hat{\mathbf{Z}}_p[\pi \wr \mathbf{Z}_p]) .$$

(iii) This follows immediately from (i) and (ii), p-Sylow subgroups of symmetric groups being built up via products and wreath products.

The isomorphism $\Theta_{A\tilde{\pi}}$: $SK_1(A\pi) \cong H_2(\pi)/H_2^{ab}(\pi)$ is defined rather indirectly in general. We end by showing that in the situation of Theorem 1, however, Θ can be described quite explicitly on the elements described there.

PROPOSITION 26. *Let* $1 \to \mathbf{Z}_p \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1$ *be an extension of p-groups, and z a generator of* Ker (α). *Define*

(i) $$\mathcal{S} = \{g \in \pi \mid \text{elements of } \alpha^{-1}g \text{ are conjugate}\}$$
$$= \{g \in \pi \mid \delta^\alpha(g \wedge h) = z, \text{ some } h \in Z_\pi(g)\} .$$

(ii) $\chi\colon \mathcal{S} \to H_2(\pi)/\langle \mathrm{Ker}(\delta^\alpha) \cap \Lambda(\pi)\rangle$, *where*
$$\chi(g) = [g \wedge h], \text{ any } h \in Z_\pi(g) \text{ such that } \delta^\alpha(g \wedge h) = z .$$

(iii) $\alpha_*\colon H_2(\tilde{\pi})/H_2^{ab}(\tilde{\pi}) \to H_2(\pi)/\langle \mathrm{Ker}(\delta^\alpha) \cap \Lambda(\pi)\rangle$ *the map induced by* α *(an injection by Lemma* 22).

Then χ *factors through* $\mathcal{S}/\sim$ *(the relation defined in Theorem* 1); *and for any unramified p-ring A, g,* $h \in \alpha^{-1}(\mathcal{S})$, *and* $\lambda \in A$,
$$\Theta_{A\tilde{\pi}}[\mathrm{Exp}(\lambda(1-z)(g-h))] = \mathrm{Tr}(\lambda)\cdot\alpha_*^{-1}(\chi(\alpha g) - \chi(\alpha h)) .$$

PROOF. First note that χ is well defined:
$$g \wedge h - g \wedge h' = g \wedge h(h')^{-1} \in \mathrm{Ker}(\delta^\alpha) \cap \Lambda(\pi)$$
$$\text{if } \delta^\alpha(g \wedge h) = \delta^\alpha(g \wedge h') = z .$$

In a similar fashion, one checks that all elements in Im (χ) have order p, and that χ factors through $\mathcal{S}/\sim$.

Choose a central extension $1 \to \varrho_0 \to \bar{\pi} \xrightarrow{\tilde{\alpha}} \tilde{\pi} \to 1$ so that $\delta^{\tilde{\alpha}}$ is a surjection with kernel $H_2^{ab}(\tilde{\pi})$ (use Lemma 17). Set $\bar{\alpha} = \alpha \circ \tilde{\alpha}$ and $\varrho = \mathrm{Ker}(\bar{\alpha})$. The extension
$$1 \to \varrho \to \bar{\pi} \xrightarrow{\bar{\alpha}} \pi \to 1$$
is central: $[z, g] \in \delta^{\tilde{\alpha}}(H_2^{ab}(\tilde{\pi})) = 1$ for any $z \in \varrho$ and $g \in \bar{\pi}$. From Lemma 22 and the naturality of δ, $\bar{\alpha}$ is seen to be a central extension inducing a commutative square
$$\begin{array}{ccc} H_2(\tilde{\pi})/H_2^{ab}(\tilde{\pi}) & \xrightarrow[\cong]{\delta^{\tilde{\alpha}}} & \varrho_0 \\ \downarrow \alpha_* & & \downarrow \text{incl} \\ H_2(\pi)/\langle \mathrm{Ker}(\delta^\alpha) \cap \Lambda(\pi)\rangle & \xrightarrow[\cong]{\delta^{\bar{\alpha}}} & \varrho . \end{array}$$

Lift $g, h \in \tilde{\pi}$ to $\bar{g}, \bar{h} \in \bar{\pi}$, and set

$$x = \delta^{\tilde{\alpha}}(\chi(\alpha g)), \qquad y = \delta^{\tilde{\alpha}}(\chi(\alpha h)).$$

Note that $\bar{g}$ is conjugate to $x\bar{g}$ and $\bar{h}$ to $y\bar{h}$ in $\bar{\pi}$, and $\tilde{\alpha}(x)=\tilde{\alpha}(y)=z$.

Since $SK_1(A\bar{\pi})=0$ (Lemma 22), $\Theta_{A\bar{\pi}}$ is defined as the composite

$$SK_1(A\tilde{\pi}) \subseteq \mathrm{Wh}(A\tilde{\pi}) \leftarrow \mathrm{Wh}(A\bar{\pi}) \xrightarrow{\Gamma_{A\bar{\pi}}} \overline{I(A\bar{\pi})} \leftarrow I(A\varrho_0) \xrightarrow{\omega} \varrho_0 \xleftarrow[\cong]{\delta^{\tilde{\alpha}}} H(\tilde{\pi})/H_2^{\mathrm{ab}}(\tilde{\pi}).$$

$\mathrm{Exp}(\lambda(1-z)(g-h))$ lifts to

$$u = \mathrm{Exp}[\lambda(1-x)(\bar{g}-1)-\lambda(1-y)(\bar{h}-xy^{-1})] \in \mathrm{Wh}(A\bar{\pi}).$$

Noting that $\Gamma_{A\bar{\pi}}(u)=\log(u)$ (x and y having order p), we get

$$\begin{aligned}\Gamma_{A\bar{\pi}}(u) &= \lambda(1-x)(\bar{g}-1)-\lambda(1-y)(\bar{h}-xy^{-1})\\ &= -\lambda(1-x)+\lambda(1-y)xy^{-1} = \lambda(xy^{-1}-1) \in \mathrm{Im}[I(A\varrho_0) \to \overline{I(A\bar{\pi})}].\end{aligned}$$

Then

$$\begin{aligned}\Theta(\mathrm{Exp}(\lambda(1-z)(g-h))) &= (\delta^{\tilde{\alpha}})^{-1}\circ\omega(\lambda(xy^{-1}-1))\\ &= (\delta^{\tilde{\alpha}})^{-1}((xy^{-1})^{\mathrm{Tr}(\lambda)})\\ &= \mathrm{Tr}(\lambda)\cdot\alpha_*^{-1}(\chi(\alpha g)-\chi(\alpha h)).\end{aligned}$$

Using this formula, examples can be constructed to show that the upper bound for the rank of

$$\mathrm{Ker}[SK_1(A\tilde{\pi}) \to SK_1(A\pi)]$$

given in Theorem 1 is *not* always the best possible.

REFERENCES

1. H. Bass, *Algebraic K-theory*, W. A. Benjamin, New York, Amsterdam, 1968.
2. P. Hilton and U. Stammbach, *A course in homological algebra* (Graduate Texts in Mathematics 4), Springer-Verlag, Berlin - Heidelberg - New York, 1971.
3. B. Huppert, *Endliche Gruppen* (Grundlehren Math. Wissensch. 134), Springer-Verlag, Berlin - Heidelberg - New York, 1967.
4. G. Janusz, *Algebraic number fields* (Pure and Applied Mathematics 55), Academic Press, New York - London, 1973.
5. W. Neumann, *Manifold cutting and pasting groups*, Topology 15 (1975), 237–244.
6. R. Oliver, *SK_1 for finite group rings*: I, Invent. Math. 57 (1980), 183–204.
7. U. Stammbach, *Homology in group theory* (Lecture Notes in Math. 359), Springer-Verlag, Berlin - Heidelberg - New York, 1973.
8. R. Swan and E. Evans, *K-theory of finite groups and orders* (Lecture Notes in Math. 149), Springer-Verlag, Berlin - Heidelberg - New York, 1970.

9. C. T. C. Wall, *On the classification of hermetian forms*, III: *Complete semilocal rings*, Invent. Math. 19 (1973), 59–71.
10. C. T. C. Wall, *Norms of units in group rings*, Proc. London Math. Soc. 29 (1974), 593–632.
11. C. T. C. Wall, *Classification of hermetian forms*, VI: *Group rings*, Ann. of Math. 103 (1976), 1–80.

MATEMATISK INSTITUT
ÅRHUS UNIVERSITET
NY MUNKEGADE
8000 ÅRHUS C
DENMARK