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# ON THE FUNDAMENTAL GROUP OF A SPACE OF SECTIONS

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## Introduction.

Let $p\colon E \to X$ be a fibration over a connected C.W. complex $X$, with fiber $F$. Suppose that $p$ has a section, $s$, and denote by $\Gamma$ $(=\Gamma(E))$ the space of all sections of $p$. A general problem is to compute $\pi_k(\Gamma, s)$ for $k \geqq 0$. In this paper we calculate $\pi_1(\Gamma, s)$ if $F = S^n$ or $P^n$, $n \geqq 2$, $\dim X \leqq n$, provided $p$ is the spherical or projective fibration associated with a real vector bundle. In principal we can also compute (depending on knowledge of twisted cohomotopy) $\pi_k(\Gamma, s)$ if $F = S^n$ or $P^n$ and $\dim X \leqq 2n-k-1$, provided $p$ comes from a real vector bundle.

If the fibration is trivial, then $E = F \times X$ and $s$ is simply given by a map $f\colon X \to F$; thus $\pi_1(\Gamma, s) = \pi_1(\mathscr{M}, f)$, where $\mathscr{M}$ is the space of all maps from $X$ to $F$. An interesting special case is $F = S^2$, $X$ a closed surface $\Sigma$. In a recent paper [6] V. Hansen calculated $\pi_1(\mathscr{M}, f)$ in this case, up to a group extension. We generalize his results (in particular we compute the group extension, thus solving Problem 1 in [6]) by calculating $\pi_1(\Gamma, s)$, where $E$ is the sphere bundle of a 3-plane bundle $\xi$ over $\Sigma$. The section $s$ induces a splitting $\xi = \eta \oplus \varepsilon^1$ ($\varepsilon^i$ = trivial $i$-plane bundle), where $\eta$ is an oriented bundle if $\xi$ is. In this case we denote by $\chi(\eta) \in H^2(\Sigma; \mathbf{Z})$ the Euler class of $\eta$. Also, if $\Sigma$ is oriented, we let $A_i, B_i \in H^1(\Sigma; \mathbf{Z})$ $(1 \leqq i \leqq g = \operatorname{genus} \Sigma)$ denote the generators obtained by regarding $\Sigma$ as the connected sum of $g$ copies of $S^1 \times S^1$. We orient these so that $A_i \cup B_i = \sigma$, the generator of $H^2(\Sigma; \mathbf{Z})$, for $1 \leqq i \leqq g$. We write $H^1(\Sigma; \mathbf{Z}) = \mathbf{Z}^{2g}$, and let (for any integer $e$) $\mathscr{R}_{g,e}$ be the group given by the central extension

$$0 \to \mathbf{Z}/e\mathbf{Z} \to \mathscr{R}_{g,e} \to \mathbf{Z}^{2g} \to 0$$

determined by the "commutator map" (see [11]) $c\colon \mathbf{Z}^{2g} \otimes \mathbf{Z}^{2g} \to \mathbf{Z}/e\mathbf{Z}$ as follows:

$$c(A_i, B_i) = 2 \in \mathbf{Z}/e\mathbf{Z}, \quad 1 \leqq i \leqq g$$

$$c(A_i, B_j) = 0 \quad \text{if } i \neq j$$

$$c(A_i, A_j) = c(B_i, B_j) = 0, \quad 1 \leqq i, j \leqq g\,.$$

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Our result is:

THEOREM 1. *Let $\xi$ be an oriented 3-plane bundle over an oriented surface $\Sigma$ of genus g, with $\xi=\eta\oplus\varepsilon^1$, where the trivial line subbundle is given by a section s. Suppose that $\chi(\eta)=e\sigma$, for $e \in \mathbf{Z}$. Then $\pi_1(\Gamma, s)=\mathscr{R}_{g,e}$.*

REMARK 1. A similar, but somewhat more complicated, result is given in section 1 for the case $\xi$ a non-orientable 3-plane bundle.

REMARK 2. If $\xi$ is the trivial bundle then $s$ is given by a map $f$ from $\Sigma$ to $S^2$, which in turn is classified up to homotopy by an integer $d$, the degree of the map. In this case the exact sequence in Theorem 1 coincides with the sequence given in Theorem 1 of [6]. In that sequence the left-hand group is $\mathbf{Z}/2d\mathbf{Z}$. The "2" arises in the following way, from the point of view of our more general result: In the universal fibration

$$S^2 \xrightarrow{i} B\,SO(2) \to B\,SO(3)\,,$$

the restriction of the universal 2-plane bundle on $B\,SO(2)$ is the tangent bundle, $\tau$, of $S^2$. Since $\chi(\tau)=2\sigma \in H^2(S^2;\,\mathbf{Z})$, we see that if the section $s$ is regarded as a map $i\circ f$, with $f\colon \Sigma \to S^2$, then $\chi(\eta)=2d$, where $d=$degree $f$.

Note that Theorem 1 enables us to determine when $\pi_1(\Gamma, s)$ is abelian (and so answer the question raised in [6]), namely the group is abelian precisely when the commutator $c=0$. Thus we obtain:

COROLLARY 1. *The group $\pi_1(\Gamma, s)$, in Theorem 1, is abelian if and only if, $e=\pm 1$ or $\pm 2$, or $\Sigma=S^2$.*

Suppose now that $\Sigma$ is a non-orientable surface. We then have (again answering a question raised in [6]):

THEOREM 2. *Let $\xi$ be an oriented 3-plane bundle over a non-orientable surface $\Sigma$, and let the oriented 2-plane subbundle $\eta$ be determined by a section s. Then,*

(i) $\pi_1(\Gamma, s)=H^1(\Sigma;\,\mathbf{Z})\oplus\mathbf{Z}/2\mathbf{Z}$, if $\chi(\eta)=0$,
(ii) $\pi_1(\Gamma, s)=H^1(\Sigma;\,\mathbf{Z})$, if $\chi(\eta)\neq 0$.

*In particular, case* (i) *holds if $\xi$ is trivial.*

Instead of considering the sphere bundle associated to a vector bundle $\xi$ one can instead take the associated projective bundle, $P\xi$. A section of this bundle gives a splitting $\xi=\eta\oplus L$, where $L$ is a line bundle—in the next section we show that without loss of generality we may take $L$ to be $\varepsilon^1$. Our result is:

THEOREM 3. *Let $\xi$ be an oriented 3-plane bundle over a surface $\Sigma$ with a splitting $\xi = \eta \oplus \varepsilon^1$, and let s be the section of the projective bundle $P\xi$ determined by that trivial line subbundle. Suppose that $\chi(\eta) \neq 0$. Then $\pi_1(\Gamma, s)$ is given precisely as in Theorem 1, if $\Sigma$ is oriented, and precisely as in Theorem 2, if $\Sigma$ is non-orientable. On the other hand, suppose that $\chi(\eta) = 0$. Then $\pi_1(\Gamma, s) = \mathscr{R}_{g,0} \oplus Z/2Z$ if $\Sigma$ is oriented, while $\pi_1(\Gamma, s) = H^1(\Sigma; Z) \oplus Z/2Z \oplus Z/2Z$ if $\Sigma$ is non-orientable.*

REMARK 3. We have defined the group $\mathscr{R}_{g,e}$ by means of an extension. One can equally give the group by generators and relations, namely:

> $\mathscr{R}_{g,e}$ is the multiplicative group with generator $\alpha_i, \beta_i$, $1 \leqq i \leqq g$, and $\gamma$, subject to the relations:
>
> $$\gamma^e = 1, \quad [\alpha_i, \beta_i] = \gamma^2, \ 1 \leqq i \leqq g,$$
>
> all other pairs of generators commute.

In the above results the group $\pi_1(\Gamma, s)$ in general is non-abelian of infinite order, however, one can give simple examples where the group is finite and non-abelian. Thus, if $n$ is an even integer greater than three, let $X = S^{n-2} \cup_{3l} e^{n-1}$. If $\xi$ is any $(n+1)$-plane bundle over $X$, and $s$ any section of $p: E = P\xi \to X$, then, by theorem 1.7:

$$\pi_1(\Gamma, s) = S_3, \text{ the symmetric group on 3 letters}.$$

Moreover, we show that on the manifold $P^2 \times S^2$ there is a $P^5$-bundle $E \to P^2 \times S^3$, which has sections $s_1$ and $s_2$ such that

$$\pi_1(\Gamma, s_1) = D_8, \text{ the dihedral group of order 8},$$

$$\pi_1(\Gamma, s_2) = Q_8, \text{ the quaternionic group of order 8}.$$

Theorems 1, 2, and 3 follow from more general results stated in the next section. Our method of proof is based on the following observation. As above, let $p: E \to X$ be a fibration with fiber $F$, and with a section $s$. Define $\Omega_X E$ to be the subspace of the space of free loops on $E$ consisting of all fiber-wise loops, i.e.,

$$\Omega_X E = \bigcup_{x \in X} \Omega p^{-1}(x),$$

where $s(x)$ is the basepoint in $p^{-1}(x)$. Then $\Omega_X E \to X$ is again a fibration, with fiber $\Omega F$ and with a section $\omega$ given by $\omega(x) =$ constant loop at $s(x)$. By using the exponential law for mappings one readily shows that $\Omega\Gamma(E) = \Gamma(\Omega_X E)$, and so by iteration we obtain:

$$\Omega^n \Gamma(E) = \Gamma(\Omega^n_X E), \quad n \geqq 1 .$$

Consequently, $\pi_n \Gamma(E) = \pi_0 \Gamma(\Omega^n_X E)$. But the right-hand group is simply the group of homotopy classes of sections of the fibration $\Omega^n_X E \to X$ and so can be computed by a spectral sequence (see [13]). Note in particular that the Postnikov invariants for the fibration $\Omega^n_X E \to X$ can be obtained from those for the fibration $E \to X$ simply by $n$-fold looping in the category of spaces "over-and-under $X$". In the remainder of the paper we exploit this point of view to obtain our results.

In the final section (5) we comment briefly on prior work of Barratt, Federer, Thom and R. Brown on homotopy groups of function spaces.

## 1. The main results: spherical case.

For theorems 1.1 through 1.3 below, we assume, as before, that $X$ is a finite dimensional C.W. complex, and $p: E \to X$ has fiber $S^n$. We also assume that $E = S\xi$, the total space of the $S^n$ bundle associated with a real vector bundle $\xi = \xi^{n+1}$, and that a section $s: X \to E$ is given. Let $\Gamma = \Gamma(\xi)$ be the space of all such sections.

Now $s$ determines a splitting $\xi = \eta^n \oplus \varepsilon^1$. We set $w_i = w_i(\eta)$, the $i$th Stiefel–Whitney class of $\eta$, and $\chi = \chi(\eta) \in H^n(X; \mathbf{Z}[w_1])$ the Euler class of $\eta$. ($\mathbf{Z}[w_1]$ = integer sheaf over $X$ twisted by $w_1$.)

THEOREM 1.1. *Let $n=2$, $\dim X \leqq 2$. Then we have an exact sequence:*

$$H^0(X; \mathbf{Z}[w_1]) \xrightarrow{\cup \chi} H^2(X; \mathbf{Z}) \xrightarrow{\lambda} \pi_1(\Gamma, s) \xrightarrow{p} H^1(X; \mathbf{Z}[w_1]) \to 0 .$$

*The following information is sufficient to determine $\pi_1(\Gamma, s)$ as an extension:*

(i) *$\lambda H^2(X; \mathbf{Z})$ is central in $\pi_1(\Gamma, s)$.*

(ii) *If $g_1, g_2 \in \pi_1(\Gamma, s)$ and if $pg_1 = x_1$, $pg_2 = x_2$, then $g_1 g_2 g_1^{-1} g_2^{-1} = \lambda(2x_1 \cup x_2)$.*

(iii) *If $w_1 \neq 0$, let $\bar{w}_1 \in H^1(X; \mathbf{Z}[w_1])$ be the (unique) element of that group of order 2. Choose $z \in H^2(X; \mathbf{Z})$ such that $\varrho z = w_2$ (where $\varrho$ = reduction mod 2). Then there exists $\omega \in \pi_1(\Gamma, s)$ such that $p\omega = \bar{w}_1$ and $\omega^2 = \lambda(\bar{w}_1^2 + z)$.*

REMARK. The class $\bar{w}_1$ is the Euler class of the line bundle twisted by $w_1$; also $\bar{w}_1 = \delta^{T_1}$, where $\delta^T$ is the Bokstein homomorphism associated with the coefficient sequence $\mathbf{Z}[w_1] \to \mathbf{Z}[w_1] \to \mathbf{Z}_2$. Note also that $H^0(X; \mathbf{Z}[w_1]) = 0$ if $w_1 \neq 0$.

We recall briefly the definition of twisted (stable) cohomotopy (see, for example, [9]). If $\zeta^k$ is any real $k$-plane bundle over a space $X$, and if $A \subset X$, we define, for any integer $i$:

$$\pi^i(X, A; \zeta) = \lim_{N \to \infty} [X, A; \Omega_X^N \Sigma_X^{N+i-k} S_X(S\zeta)]_X$$

where $\Omega_X$, $\Sigma_X$, $S_X$ denote fiberwise looping, fiberwise based suspension, and fiberwise two-point suspension, respectively. If $\zeta^k$ and $\varkappa^j$ are stably equivalent, then $\pi^i(X, A; \zeta) \cong \pi^i(X, A; \varkappa)$. Now let $\xi = \eta \oplus \varepsilon$, as above. Note that $S_X(S\eta) = E$, and therefore there is a homomorphism

$$\iota\colon \pi_k(\Gamma, s) = [X; \Omega_X^k E]_X \to \pi^{n-k}(X; \eta) \cong \pi^{n-k}(X; \xi)\,.$$

THEOREM 1.2. *Let* $\dim X \leqq 2n-k-2$. *Then* $\iota$ *is an isomorphism.*

We omit the proof. See. for example [7, lemma 2.2].

As a special case, take $k=1$, $n \geqq 3$, $\dim X \leqq n$. Then by 1.2, $\pi_1(\Gamma, s) \cong \pi^{n-1}(X\ ;\ \eta)$, and the latter can be computed by an exact sequence using [7]. Thus we have

COROLLARY 1.3. *Let* $n \geqq 3$, *and* $\dim X \leqq n$. *Then* $\pi_1(\Gamma, s)$ *is commutative, and we have an exact sequence:*

$$H^{n-2}(X; \mathbf{Z}[w_1]) \xrightarrow{\theta} H^n(X; \mathbf{Z}_2) \xrightarrow{\lambda} \pi_1(\Gamma, s) \xrightarrow{p} H^{n-1}(X; \mathbf{Z}[w_1]) \to 0$$

*where* $\theta = (\mathrm{Sq}^2 + w_2 \cup)\varrho$.

Note that elements in $H^n(X; \mathbf{Z}_2)$ all have order two. Thus by 4.1 in [12], the following information suffices to determine $\pi_1(\Gamma, s)$ as an extension:

> If $x \in H^{n-1}(X; \mathbf{Z}[w_1])$ and $2x=0$, choose $z$ such that $\delta^T z = x$. Then there exists $\chi \in \pi_1(\Gamma, s)$ such that $p\chi = x$, and $\chi^2 = \lambda(\mathrm{Sq}^2 + w_1 \cup \mathrm{Sq}^1 + w_2 \cup + w_1^2 \cup)z$.

*The Projective case.* For the rest of this section we assume that $E = P\xi^{n+1}$ and that a section $s\colon X \to E$ is given. We further assume that $s$ is the image (under the identification $\tilde{E} = S\xi \to P\xi$) of a section $\tilde{s}\colon X \to \tilde{E}$. This section $\tilde{s}$ then determines a splitting $\xi = \eta^n \oplus \varepsilon^1$. The assumption that $\tilde{s}$ exists does not result in any loss of generality, for any section $s$ determines a splitting $\xi = \eta \oplus L$, where $L$ is some line bundle. Since $P(\xi \otimes L) = P\xi$, we could have taken $\xi \otimes L$ instead of $\xi$, and $\xi \otimes L = (\eta \otimes L) \oplus (L \otimes L) = (\eta \otimes L) \oplus \varepsilon^1$.

As above, write $w_i = w_i\eta$, and $\chi = \chi(\eta) \in H^n(X; \mathbf{Z}[w_1])$. Let $\tilde{\Gamma}$ be the space of sections of $\tilde{E}$; since $\tilde{E} \to E$ is a covering map, $i\colon \pi_1(\tilde{\Gamma}, \tilde{s}) \to \pi_1(\Gamma, s)$ is injective. Restriction of each loop of sections of $p\colon E \to X$ to the base-point of $X$

determines a homomorphism $q\colon \pi_1(\Gamma, s) \to \pi_1(F) = \mathbf{Z}_2$, and we clearly have an exact sequence:

$$0 \to \pi_1(\tilde{\Gamma}, \tilde{s}) \overset{i}{\subset} \pi_1(\Gamma, s) \xrightarrow{q} Z_2 .$$

We call elements of $\pi_1(\Gamma, s)$ *odd* or *even*, depending on their images under $q$. We write both $\pi_1(\Gamma, s)$ and $\pi_1(\tilde{\Gamma}, \tilde{s})$ multiplicatively.

If $\eta$ has a non-zero section, i.e., $\chi = 0$, we write $\eta = \zeta \oplus \varepsilon^1$, i.e., $\xi = \zeta \oplus \varepsilon^2$. Then $S^1 \cong P^1$ determines a homotopy of sections $X \times S^1 \cong X \times P^1 = P\varepsilon^2 \subset P\xi$. Let $t \in \pi_1(\Gamma, s)$ be the element represented by this homotopy; note that $t$ is odd. We consider $t^2 \in \pi_1(\tilde{\Gamma}, \tilde{s})$.

In either case, we have a diagram with exact row and column:

(1.1)
$$\begin{array}{c}
H^{n-2}(X;\mathbf{Z}[w_1]) \\
\big\downarrow \theta \\
H^n(X;\pi_{n+1}S^n) \\
\big\downarrow \lambda \\
0 \to \pi_1(\tilde{\Gamma}, \tilde{s}) \overset{i}{\subset} \pi_1(\Gamma, s) \xrightarrow{q} \mathbf{Z}_2 \\
\big\downarrow p \\
H^{n-1}(X;\mathbf{Z}[w_1]) \\
\big\downarrow \\
0
\end{array}$$

(Note that the column of diagram (1.1) is the exact sequence in the statement of either theorem 1.1 or 1.3, since $\tilde{\Gamma}$ is the space of sections of a sphere bundle.)

THEOREM 1.4. *Let $n = 2$. Then $\pi_1(\Gamma, s)$ is determined by the following information*:

(i) $\pi_1(\tilde{\Gamma}, \tilde{s}) = \pi_1(\Gamma(\tilde{E}), \tilde{s})$, *which is given in Theorem* 1.1.

(ii) $\chi \neq 0$ *if and only if* $\pi_1(\Gamma, s) = \pi_1(\tilde{\Gamma}, \tilde{s})$.

*The following four conditions apply when* $\chi = 0$.

(iii) $i\lambda H^2(X; \mathbf{Z}) \subset \pi_1(\tilde{\Gamma}, \tilde{s})$ *is central.*

(iv) *For each* $x \in H^1(X; \mathbf{Z}[w_1])$, *there exists* $g \in \pi_1(\tilde{\Gamma}, \tilde{s})$ *such that* $pg = x$ *and* $tgt^{-1}g = \lambda(x^2)$.

(v) *If* $w_1 = 0$, *then* $t^2 = 1$.

(vi) *If* $w_1 \neq 0$, *then* $pt^2 = \bar{w}_1$. *In fact, by choosing* $z \in H^2(X; \mathbf{Z})$ *to be zero (see* 1.1. (iii)), *we may insist that* $t^2 = \omega$.

Consider now the case $n>2$. We state three theorems which suffice to determine $\pi_k(\Gamma, s)$ for $k \geqq 1$, $\dim X \leqq 2n-k-2$.

THEOREM 1.5. *Let* $k \geqq 2$, *and* $\dim X \leqq 2n-k-2$. *Then* $\pi_k(\Gamma, s) = \pi_k(\tilde{\Gamma}, \tilde{s}) \cong \pi^{n-k}(X; \eta)$.

Recall that if $\alpha^j$ is any vector bundle over a complex $X$, a *Becker–Euler* class $\chi_* \alpha \in \pi^j(X; \alpha)$ may be defined; it is represented by the North polar section $X \to S(\alpha \oplus \varepsilon^1)$. If $\dim X \leqq 2j-3$, then $\chi_* \alpha = 0 \Leftrightarrow \alpha$ has a section.

THEOREM 1.6. *Let* $\dim X \leqq 2n-3$. *If* $\chi_* \eta \neq 0$, *then* $\pi_1(\Gamma, s) = \pi_1(\tilde{\Gamma}, \tilde{s}) \cong \pi^{n-1}(X; \eta)$.

We now consider the case $\chi_* \eta = 0$. Let $\eta = \zeta^{n-1} \oplus \varepsilon^1$ be a splitting. As noted above, this splitting determines an odd element $t \in \pi_1(\Gamma, s)$. Let $T_\zeta$ be the antipodal map on the total space of the sphere bundle of $\zeta$, and let $(T_\zeta)_*$ be the corresponding automorphism of $\pi^*(X; \zeta)$. Recall $\iota: \pi_1(\tilde{\Gamma}, \tilde{s}) \cong \pi^{n-1}(X; \eta) = \pi^{n-1}(X; \zeta)$; and consider $\pi_1(\tilde{\Gamma}, \tilde{s}) \subset \pi_1(\Gamma, s)$.

THEOREM 1.7. *Let* $\dim X \leqq 2n-3$, $\chi_* \eta = 0$. *Then the following information suffices to determine* $\pi_1(\Gamma, s)$ *as an extension of* $\mathbf{Z}_2$ *by* $\pi_1(\tilde{\Gamma}, \tilde{s})$:

(i) $t^2 = \iota^{-1}(\chi_* \zeta)$.

(ii) *For any* $g \in \pi_1(\tilde{\Gamma}, \tilde{s})$, $tgt^{-1} = \iota^{-1}((T_\zeta)_*(\iota g))$.

The proofs of theorems 1.1, 1.4, 1.6 and 1.7 are given in the following sections.

## 2. Proof of Theorem 1.1.

Let $F = S^n$, $n \geqq 2$. Now $E \rightleftarrows X$ is a fibration over-and-under $X$, i.e., an object in the category $\mathscr{X}_X^+$, so $\pi_1(\Gamma, s) = [X; \Omega_X E]$, where $\Omega_X$ is the fiberwise looping. Our technique is to construct a Postnikov tower for $E \rightleftarrows X$ in this category, and then compute $[X; \Omega_X E]$ using the associated spectral sequence.

We consider $n=2$ first. Now $\pi_2 S^2 \approx \pi_3 S^2 \approx \mathbf{Z}$. If a loop in $X$, $f: S^1 \to X$, represents $\alpha \in \pi_1 X$, then $\alpha$ acts on $\pi_2 S^2$ by multiplication by 1 or $-1$, depending on whether $f^* w_1 = 0$ or not. On the other hand, the action on $\pi_3 S^2$ is always trivial. Thus, the first 2 stages of the relative Postnikov tower are as follows:

$$\begin{array}{ccc} K(\mathbf{Z},3)\times X & \xrightarrow{\lambda} & E_2 \\ & & \downarrow p \\ K(\mathbf{Z},2,w_1) & = & E_1 \xrightarrow{\theta} K(\mathbf{Z},4)\times X \end{array}$$

where all spaces are fiber spaces over-and-under $X$, and all maps preserve the over-and-under property [13]. The $k$-invariant $\theta$ can be computed as follows. The fundamental class $\iota_2 \in H^2(E_1, X; \mathbf{Z}[w_1])$ represents the Thom class $U \in H^2(E, X; \mathbf{Z}[w_1])$, and $U^2 = \chi_2 \cup U$, thus $\theta^*\iota_4 = \chi_2 \cup \iota_2 - \iota_2^2$. We have an exact sequence:

$$H^0(X; \mathbf{Z}[w_1]) \xrightarrow{\chi_2 \cup} H^2(X; \mathbf{Z}) \xrightarrow{\lambda} \pi_1(\Gamma, s) \xrightarrow{p} H^1(X; \mathbf{Z}[w_1]) \to 0 .$$

(Note that if $w_1 \neq 0$, $H^0(X; \mathbf{Z}[w_1]) = 0$.)

By the twisted analogue of Theorem 1.1 of [11], $\lambda H^2(X; \mathbf{Z}_2)$ is central. Let

$$c\colon H^1(X; \mathbf{Z}[w_1]) \times H^1(X; \mathbf{Z}[w_1]) \to H^2(X; \mathbf{Z})$$

be the alternating bilinear "commutator", i.e.,

$$c(x_1, x_2) = \lambda^{-1}\mathfrak{x}_1\mathfrak{x}_2\mathfrak{x}_1^{-1}\mathfrak{x}_1^{-1} ,$$

where $p\mathfrak{x}_1 = x_1$ and $p\mathfrak{x}_2 = x_2$. By the twisted analogue of Theorem 2.1 of [11],

$$c(x_1, x_2) = (j^*)^{-1}\theta(i^*)^{-1}(x_1, x_2)$$

in the diagram:

$$\begin{array}{ccc} & H^2(X\times T, X; \mathbf{Z}[w_1]) \xrightarrow{i^*} & H^2(X\times(S_1 \vee S_2), X; \mathbf{Z}[w_1]) \\ & & = H^1(X; \mathbf{Z}[w_1]) \oplus H^1(X; \mathbf{Z}[w_1]) \\ & \downarrow \theta & \\ H^2(X; \mathbf{Z}) & & \\ = H^4(X\times T, X\times(S_1 \vee S_2); \mathbf{Z}) & \xrightarrow{j^*} H^4(X\times T, X; \mathbf{Z}) & \end{array}$$

where $S_1 \cong S_2 \cong S^1$, and $T = S_1 \times S_2$, the torus. Let $\sigma_1, \sigma_2$ be the fundamental classes of $S_1, S_2$. Then

$$\begin{aligned} \theta(x_1\otimes\sigma_1 + x_2\otimes\sigma_2) &= \chi \cup (x_1\otimes\sigma_1 + x_2\otimes\sigma_2) - (x_1\otimes\sigma_1 + x_2\otimes\sigma_2)^2 \\ &= -2x_1 \cup x_2 \otimes \sigma_1 \cup \sigma_2 \end{aligned}$$

since $\dim X \leqq 3$, and so 1.1 (ii) is proved.

If $w_1 = 0$, we are done, since $H^1(X; \mathbf{Z})$ is free. If $w_1 \neq 0$, $\bar{w}_1$ is the only torsion element of $H^1(X; \mathbf{Z}[w_1])$, and we recall the definition $\Phi_2\bar{w}_1 = \lambda^{-1}2p^{-1}\bar{w}_1 \in H^2(X; \mathbf{Z})$. In the diagram below (by Theorem 3.2 of [12]), $\Phi_2\bar{w}_1 = (j^*)^{-1}\theta(i^*)^{-1}\bar{w}_1$ (where $S = S^1 \subset P^2$):

$$\begin{array}{ccc}
 & H^2(X\times P^2, X;\ \mathbf{Z}[w_1]) \xrightarrow{\ i^*\ } & H^2(X\times S, X;\ \mathbf{Z}[w_1]) \\
 & & = H^1(X;\ \mathbf{Z}[w_1]) \\
 & \big\downarrow \theta & \\
H^2(X;\ \mathbf{Z}) & & \\
= H^4(X\times P^2, X\times S;\ \mathbf{Z}) & \xrightarrow{\ j^*\ } H^4(X\times P^2, X;\ \mathbf{Z})\,. &
\end{array}$$

Now $\bar{w}_1 = \delta^{T_1}$, thus

$$i^*\delta^T(1\otimes e^1) \;=\; \delta^T(1\otimes\sigma) \;=\; \bar{w}_1\otimes\sigma;$$

$$\theta(\delta^T(1\otimes e^1)) \;=\; (\chi\otimes 1)\cup\delta^T(1\otimes e^1) - (\delta^T(1\otimes e^1))^2\,.$$

Since $\Phi_2\bar{w}_1$ has indeterminacy $2H^2(X;\mathbf{Z})$, it suffices to complete the calculations modulo 2:

$$\begin{aligned}
\varrho\theta\delta^T(1\otimes e^1) &= (w_2\otimes 1)\cup(w_1\otimes e^1 + 1\otimes e^2) - (w_1\otimes e^1 + 1\otimes e^2)^2 \\
&= w_2\otimes e^2 + w_1^2\otimes e^2 \;=\; j^*((w_2+w_1^2)\otimes e^2)\,,
\end{aligned}$$

and hence 1.1 (iii) is proved. This concludes the proof of 1.1, by Theorem 1.3 of [11].

## 3. Proof of Theorems 1.4, 1.5, 1.6 and 1.7.

Recall that $E = P\xi^{n+1} \to X$, $\tilde{E} = S\xi^{n+1} \to X$, $\tilde{s}\colon X \to \tilde{E}$ is a section which determines a splitting $\xi = \eta\oplus\varepsilon^1$, and $s\colon X \to E$ is the composition $X \xrightarrow{\tilde{s}} \tilde{E} \to E$. Let $\Gamma$, $\tilde{\Gamma}$ be the spaces of sections of $E$ and $\tilde{E}$, respectively.

LEMMA 3.1. *Let* $\dim X \leqq 2n-2$. *Then* $\eta$ *has a section* (*which then determines a splitting* $\eta = \zeta^{n-1}\oplus\varepsilon^1$) *if and only if there exists an odd element of* $\pi_1(\tilde{\Gamma},\tilde{s})$.

PROOF. In section 1 we showed how a splitting determines and odd element $t$. Suppose, conversely, that an odd $t$ exists. Now $\tilde{E} = S\xi = S(\eta\oplus\varepsilon^1) = S_X(S\eta)$, which has two canonical sections, $\tilde{s}$ and $\tilde{s}_1$, the section antipodal to $\tilde{s}$. Now a loop in $\Gamma$ based at $s$ corresponds to a path in $\tilde{\Gamma}$ from $\tilde{s}$ to $\tilde{s}_1$, which is precisely a section $X \to P_X S_X(S\eta) = P_X\tilde{E}$, where $P_X$ is the fiber-wise South-to-North path construction (see, for example, [7]). We have an inclusion of fibrations over $X$:

$$\begin{array}{ccc}
S\eta & \overset{i}{\subset} & P_X\tilde{E} \\
 & \searrow \ \ \swarrow\!\nearrow_{t} & \\
 & X\,. &
\end{array}$$

The fiber of $P_X E$ is of the homotopy of $\Omega S^n$, and $S^{n-1}\subset\Omega S^n$ induces isomorphism in homotopy through dimension $2n-1$. Thus obstructions to a section of $S\eta \to X$ must agree with obstructions to a section of $P_X\tilde{E} \to X$ up through dimension $2n-2$, and these latter obstructions are obviously all 0. (Note that we have not established existence of a section $t\colon X \to S\eta$ such that $i \circ t \sim \ell$. This stronger result holds if $\dim X \leqq 2n-3$, as the reader can verify.)

Note that by lemma 3.1, theorem 1.6 is proved, as is part (ii) of theorem 1.4.

For the proof of 1.7 and the remainder of 1.4, assume that we have a splitting $\eta = \zeta^{n-1} \oplus \varepsilon^1$, thus determining, in the manner of section 1, a fixed odd $\ell \in \pi_1(\Gamma, s)$. We thus have an exact sequence, for $n \geqq 2$:

$$(3.1) \qquad 1 \to \pi_1(\tilde{\Gamma}, \tilde{s}) \xrightarrow{i} \pi_1(\Gamma, s) \xrightarrow{q} Z_2 \to 0 .$$

LEMMA 3.2. *The extension* (3.1) *is determined by the two data*:

(i) $\ell^2 \in \pi_1(\tilde{\Gamma}, \tilde{s})$.

(ii) *The automorphism "conjugation by* $\ell$" *on* $\pi_1(\tilde{\Gamma}, \tilde{s})$.

We omit the proof of 3.2 which is well-known.

The following lemma will be needed to prove 1.7:

LEMMA 3.3. *Let* $n \geqq 2$. *Writing* $\tilde{E} = S(\zeta \oplus \varepsilon^2)$, *let* $T_\zeta\colon \tilde{E} \to \tilde{E}$ *be the function induced by multiplication by* $(-1)$ *on each vector of* $\zeta$ *and the identity on* $\varepsilon^2$. *Let also* $T_\zeta\colon \tilde{\Gamma} \to \tilde{\Gamma}$ *denote the corresponding map on the space of sections. Then the automorphism* $(T_\zeta)_*$ *on* $\pi_1(\tilde{\Gamma}, \tilde{s})$ *is precisely conjugation by* $\ell$.

PROOF. Writing $\xi = \zeta \oplus \varepsilon^2$, we have $\tilde{E} = S\xi = S\xi *_X S\varepsilon^2$ ($*_X$ = fiberwise join) and $T_\xi = T_\zeta *_X T_\varepsilon$. For each $0 \leqq \theta \leqq \pi$, let $R_\theta\colon S\varepsilon^2 \to S\varepsilon^2$ be rotation of each fiber of $S\varepsilon^2 = X \times S^1$ by an angle $\theta$. Then $R_\pi = T_\varepsilon$, $R_0 = 1$, and $\ell$ is classified by the composition

$$X \times [0,1] \xrightarrow{\varrho} S\varepsilon^2 \to P\varepsilon^2 \subset P\xi = E ,$$

where $\varrho(x,t) = R_{\pi t}\tilde{s}x$ for all $x \in X$, $0 \leqq t \leqq 1$. If $g \in \pi_1(\tilde{\Gamma}, \tilde{s})$ is classified by $\alpha$: $X \times [0,1] \to S\zeta *_X S\varepsilon^2$, let $\beta_\theta\colon X \times [0,1] \to S\zeta *_X S\varepsilon^2$, $0 \leqq \theta \leqq \pi$ be given by:

$$\varrho(x,t) = \begin{cases} R_{3\theta t}\tilde{s}x & \text{if } 0 \leqq t \leqq 1/3 \\ (T_\zeta *_X R_\theta)\alpha(x, 3t-1) & \text{if } 1/3 \leqq t \leqq 2/3 \\ R_{(3-3t)\theta}\tilde{s}x & \text{if } 2/3 \leqq t \leqq 1 . \end{cases}$$

Now $\beta_0$ is homotopic to $(T_\zeta *_X 1)\alpha$ and $\beta_\pi$ classifies $\ell g \ell^{-1}$, hence lemma 3.3 is proved.

PROOF OF THEOREM 1.7. Part (ii) now follows, since $\imath\colon \pi_1(\tilde{\Gamma}, \tilde{s}) \to \pi^{n-1}(X; \zeta)$ is clearly consistent with the action $(T_\zeta)_*$:

PROOF OF 1.7 (i): Let $s_0, s_1\colon X \to S_X(S\zeta) = S\eta$ be the South and North polar sections. As elements of $\pi^{n-1}(X;\zeta)$, 0 is classified by $s_0$, and $\chi_*\zeta$ is classified by $s_1$, by definition. The inclusion $i$: $S\eta \subset \Omega_X S_X S\eta = \Omega_X S\xi$ maps each $a \in (S\eta)_X$ to the map $ia$: $[0,1] \to S_X S\eta$ where $iat=[a,2t]$ or $[s_0x, 2-2t]$. Thus $is_1$ is simply $X\times[0,1] \to X\times S^1 = S\varepsilon^2 \subset S\xi$, which classifies $t^2$; this completes the proof.

Our application which yields the dihedral and quaternionic groups will require the following computational interpretation of $(T_\zeta)_*$:

LEMMA 3.4. *Let $\zeta^k$ be any real vector bundle over any* C.W. *complex $X$*, $\dim X \leqq k+1$. *Let $(T_\zeta)_*$: $\pi^k(X;\zeta) \to \pi^k(X;\zeta)$ be the automorphism induced by the antipodal map on each fiber of $S\zeta$. Then, in the exact sequence (where $w_i = w_i\zeta$, and $\theta = (\mathrm{Sq}^2 + w_2 \cup)\varrho$)*

$$H^{k-1}(X;\mathbf{Z}[w_1]) \xrightarrow{\theta} H^{k+1}(X;\mathbf{Z}_2) \xrightarrow{\lambda} \pi^k(X;\zeta) \xrightarrow{p} H^k(X;\mathbf{Z}[w_1]) \to 0$$

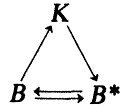
*for any $g \in \pi^k(X;\zeta)$ such that $pg=x$, $(T_\zeta)_* g = (-1)^k g + \lambda(w_1 \cup \varrho x)$.*

Lemma 3.4 is needed only for the specific application, and its proof is fairly lengthy. Hence we omit it. The proof utilizes the theory developed in § 4 of [8], in a fairly straightforward way.

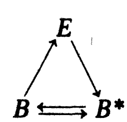
The remaining portion of this section is devoted to completing the proof of Theorem 1.4 in the case $\chi=0$.

Part (iii) of 1.4 follows from the fact that $\lambda H^2(X;\mathbf{Z})$ is central in $\pi_1(\tilde{\Gamma},\tilde{s})$ (see [11]), the fact that conjugation by $t$ is filtration preserving, and the fact that $\pi_1 P^2$ acts trivially on $\pi_3 P^2 \cong \mathbf{Z}$. Part (v) follows from the fact that if $w_1\zeta=0$, $\zeta$ is a trivial line bundle. Then $t^2=1$ because the universal example is $X=\mathrm{pt}$.

To show part (iv) of theorem 1.4, we need lemma 3.5 below, a slight generalization of theorem 2.5 of [10]. Suppose that $B^* \rightleftarrows B$ is a fixed fibration over-and-under $B$, and let




be a fibration over $B^*$ and under $B$. (That is, $K \to B^* \to B$ are fibrations, and $B \to K$ and $B \to B^*$ are inclusions.) Now let $L \rightleftarrows B^*$ be a fibration over-and-under $B^*$ (hence, automatically, over $B^*$ and under $B$), and let $\psi$: $K \to L$ be a map over $B^*$ and under $B$. Let



be the fiber of $\psi$ in the category of fibrations over $B^*$ and under $B$, and then $\Omega_{B^*}L \rightleftarrows B^*$ is the fiber of $B \to K$ in the category of fibrations over $B^*$.

Let $(X, X')$ and $(Y, Y')$ be C.W. pairs over $B^*$, and $f$: $(X, X') \to (Y, Y')$ a map over $B^*$. Let $(M_f, M'_f)$ be the mapping cylinder of $f$, also a pair over $B^*$. Now consider the diagram: (note that each object in the diagram has a distinguished element, which we refer to as 0)

(3)

$$\begin{array}{cccccc}
 & & & & & [X,X';\,\Omega_B,\,L]_{B^*} \\
 & & & & & \downarrow{\scriptstyle\lambda} \\
 & & & [Y,Y';\,E]_B & \xrightarrow{f^*} & [X,X';\,E]_B \\
 & & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
 & & [M_f,M'_f;\,K]_B & \xrightarrow{i^*}\ [Y,Y';\,K]_B & \xrightarrow{f^*} & [X,X';\,K]_B \\
 & & \downarrow{\scriptstyle\psi} & \downarrow{\scriptstyle\psi} & & \\
[M_f,M'_f\cup X\cup Y;\,L]_{B^*} & \xrightarrow{j^*} & [M_f,M'_f;\,L]_{B^*} & \xrightarrow{i^*}\ [Y,Y';\,L]_{B^*} & & \\
\cong| & & & & & \\
[X\times I,X'\times I\cup X\times\{0,1\};\,L]_{B^*} & & & & &
\end{array}$$

LEMMA 3.5. (I) *diagram* (3) *is commutative, and all rows and columns are exact.*
(II) *If* $x \in [Y, Y'; K]_B$ *and* $f^*x=0$, $\psi x=0$; *then*

$$\lambda^{-1}f^*\pi^{-1}x = (j^*)^{-1}\theta(i^*)^{-1}x$$

*under the obvious identification* $[X, X'; \Omega_{B^*}L]_{B^*} \cong [X\times I, X'\times I \cup X \times \{0,1\}; L]_{B^*}$.

We omit the proof, which is a straightforward generalization of that of theorem 2.5 of [10].

Part (vi) of theorem 1.4 follows from lemma 3.5, where we let $B=X$, $B^*=X\times P^\infty$, $K=X\times P^\infty$ (over $B^*$ by the identity and under $B$ by inclusion), $L=K(\mathbf{Z},3,w_1\otimes 1+1\otimes u)$, the twisted Eilenberg–MacLane space over-and-under $X\times P^\infty$, and $\psi$: $K\to L$ is a map with

$$\psi^*\iota_3 = \delta^T(1\otimes u^2) \in H^3(X\times P^\infty;\ \mathbf{Z}[w_1\otimes 1+1\otimes u]);$$

and where $f$: $(X\times S^1, X) \to (X\times S^1, X)$ is the identity on $X$ and the double wrap-around on $S^1$. (We consider the target copy of $X\times S^1$ to be over $X\times P^\infty$ by inclusion, and under $X$ by injection.) Diagram (3) becomes:

$$
\begin{array}{ccccc}
 & & & & H^1(X;\,\mathbf{Z}[w_1]) \xrightarrow[\cong]{s} H^2(X\times S^1, X;\,\mathbf{Z}[w_1\otimes 1+1\otimes\varrho\sigma]) \\
 & & & & \downarrow \\
 & & \mathscr{I} & \xrightarrow[\text{squaring}]{j^*=} & \mathscr{I} \\
 & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\
 & H^1(X\times P^2, X;\,\mathbf{Z}_2) \xrightarrow{i^*} & H^1(X\times S^1, X;\,\mathbf{Z}_2) & \xrightarrow[\substack{\times 2\\ =0}]{j^*=} & H^1(X\times S^1, X;\,\mathbf{Z}_2) \\
 & \downarrow{\scriptstyle \psi^*} & \downarrow{\scriptstyle \psi^*} & & \\
H^3(X\times P^2, X\times S^1;\,\mathbf{Z}[w_1\otimes 1+1\otimes u]) \xrightarrow{j^*} & H^3(X\times P^2, X;\,\mathbf{Z}[w_1\otimes 1+1\otimes u]) \xrightarrow{i^*} & H^3(X\times S^1, X;\,\mathbf{Z}[w_1\otimes 1+1\otimes\varrho\sigma]) & & \\
\uparrow{\scriptstyle s}\,\cong & & & & \\
H^1(X;\,\mathbf{Z}[w_1]) & & & &
\end{array}
$$

where $\mathscr{I}$ is the pushout in the diagram:

$$
\begin{array}{ccccccc}
\pi_1(\tilde{\Gamma},\tilde{s}) & \subset & \pi_1(\Gamma, s) & \xrightarrow{q} & \mathbf{Z}_2 & & \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \| & & \\
H^1(X;\,\mathbf{Z}[w_1]) & \subset & \mathscr{I} & \xrightarrow{q} & \mathbf{Z}_2 & \xrightarrow[\cong]{s} & H^1(X\times S^1, X;\,\mathbf{Z}_2)\,.
\end{array}
$$

Now $qpt=1\otimes\varrho\sigma\in H^1(X\times S^1, X;\,\mathbf{Z}_2)$, $1\otimes\varrho\sigma=i^*(1\otimes u)$, and

$$\psi^*(1\otimes u)\;=\;\delta^T(1\otimes u^2)\;=\;j^*\delta^T(1\otimes u^2)\;=\;j^*s\delta^{T_1}\,,$$

and $\delta^{T_1}=\bar{w}_1$. Hence $pt^2=\bar{w}_1$, by lemma 3.5. The parenthetical remark that we can insist that $t^2=\omega$ is based on the fact that if $\chi_2=0$, $BO(1)=P^\infty$ is a universal example for $X$, and $2H^2(P^\infty;\,\mathbf{Z})=0$.

Finally, part (iv) of theorem 1.4 follows directly from an unstable version of theorem 4.1 of [8], in a manner similar to the proof of lemma 3.5. But since this unstable version is not given in that reference, we give a direct proof. We have a commutative diagram with exact rows, where $(T_\zeta)_*$ acts on all groups consistent with all maps:

$$
\begin{array}{ccccccccc}
H^0(X;\,\mathbf{Z}[w_1]) & \xrightarrow{\chi\cup} & H^2(X;\,Z) & \xrightarrow{\lambda} & \pi_1(\tilde{\Gamma},\tilde{s}) & \xrightarrow{p} & H^1(X;\,\mathbf{Z}[w_1]) & \to & 0 \\
\| & & \downarrow{\scriptstyle \varrho} & & \downarrow{\scriptstyle \iota} & & \| & & \\
H^0(X;\,Z[w_1]) & \xrightarrow{\theta} & H^2(X;\,Z_2) & \xrightarrow{\lambda} & \pi^1(X;\,\zeta) & \xrightarrow{p} & H^1(X;\mathbf{Z}[w_1]) & \to & 0
\end{array}
$$

where $\theta=(\mathrm{Sq}^2+w_2\cup)\varrho$. Now if $x\in H^1(X;\,\mathbf{Z}[w_1])$, choose $g\in\pi_1(\tilde{\Gamma},\tilde{s})$ such that $pg=x$. Then $tgt^{-1}g=\lambda\varphi x$ for some $\varphi x$. By 3.5, $\varrho\varphi x=w_1\cup\varrho x=\mathrm{Sq}^1\varrho x=(\varrho x)^2=\varrho x^2$. But $\operatorname{Ker}\varrho=2H^2(X;\,\mathbf{Z})=\operatorname{Ind}\varphi x$; thus 1.4 (iv) follows.

## 4. An example involving non-commutative groups of order 8.

We now take $X = P^2 \times S^3$, and denote by $H$ the non-trivial line bundle over $X$. Define $\xi = H \oplus \varepsilon^5$, and let $\Gamma$ be the space of all sections of $E = P\xi \to X$. Denote by $\alpha^5$ the unique non-trivial 5-plane bundle over $S^5$; note that $\alpha^5 \oplus \varepsilon^1 = \varepsilon^6$, $\alpha^5 = \alpha^4 \oplus \varepsilon^1$, since $\pi_5 B\mathrm{O}(4) \to \pi_5 B\mathrm{O}(5)$ is onto. Moreover, $\alpha^4$ does not admit a trivial section. We set $\eta_1 = H \oplus \varepsilon^4$ over $X$ and $\eta_2 = (H \oplus \varepsilon^4) \# \alpha^5$, i.e., a bundle over $X$ obtained by altering $\eta_1$ on the top cell of $X$, using $\alpha^5$. Then $\xi = \eta_1 \oplus \varepsilon^1 = \eta_2 \oplus \varepsilon^1$, while 4-plane bundles $\zeta_1 = H \oplus \varepsilon^3$ and $\zeta_2 = (H \oplus \varepsilon^3) \# \alpha^4$ exist such that $\eta_1 = \zeta_1 \oplus \varepsilon^1$ and $\eta_2 = \zeta_2 \oplus \varepsilon^1$. Let $s_1, s_2$ be the two sections of $E \to X$ determined by the two given splittings of $\xi$. Then:

THEOREM 4.1. (I) $\pi_1(\Gamma, s_1) = \mathrm{D}_8$, *the dihedral group of order* 8.
(II) $\pi_1(\Gamma, s_2) = \mathrm{Q}_8$, *the quaternionic group of order* 8.

PROOF. By 1.3, $\pi_1(\tilde{\Gamma}, \tilde{s}_1) = \pi_1(\tilde{\Gamma}, \tilde{s}_2) = \pi^4(X; H) = \mathbf{Z}_4$, and fits into the exact sequence:

$$0 \to \mathbf{Z}_4 \xrightarrow{i} \pi_1(\Gamma, s_i) \xrightarrow{q} \mathbf{Z}_2 \to 0\,.$$

By lemma 3.4, conjugation by $t$ on $\mathbf{Z}_4$ is the non-trivial automorphism. Thus, if we let $g = i\ (1 \bmod 4) \in \pi_1(\Gamma, s_i)$, we have $tg = g^3 t$.

In case (I), $t^2 = 1$ by 1.6, since $\zeta_1$ has a section.

In case (II), $t^2 = i\chi_*$, again by 1.6, where $\chi_* \in \pi^4(X; H)$ is the Becker–Euler class of $\zeta_2$. But since $\chi(\zeta_2) = 0$, $\chi_*$ must be the image, under $\lambda$, of the secondary obstruction to section of $\zeta_2$. This secondary obstruction is clearly 0 on $(H \oplus \varepsilon^3)$ and non-zero on $\alpha^4$, hence is non-zero on $\zeta_2$. It follows that $t^2 = g^2$.

In either case, a simple algebraic argument finishes the proof.

## 5. Historical remarks.

Following early work of G. Whitehead, Hu, Borsuk, and J. C. Moore on function spaces, Barratt [1], Federer [5], and Thom [14] gave fairly definitive treatments of the properties of homotopy groups of such spaces (1955–56). Barratt and Federer approached the problem via filtration of the domain space by skeleta (Federer used an exact couple), while Thom utilized the Postnikov decomposition of the range. Barratt gave a complete calculation of $\pi_*(Y^X, x_0)$, where $X$ and $Y$ are CW complexes, $X$ is $(n-1)$-connected and $(n+1)$-dimensional, and $x_0$ is the constant map. In particular he computed the group extension involved in an exact sequence giving the above group. R. Brown [3], [4], in the early 1960's, then studied function spaces from a semi-simplicial point of view, incorporating and extending the work of Barratt and Thom.

We note here that in our earlier paper [10], Proposition 3.3 is given by Thom in [14], while Brown [3], [4] has results similar to Theorem 3.6.

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