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A NEW PROOF OF LÖWNER'S THEOREM ON MONOTONE MATRIX FUNCTIONS

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By a definition due to Löwner [4], a function h on $(0,\infty)$ is a monotone (non-decreasing) matrix function of order n if, for positive $n\times n$ matrices A and B,

$$B \leqq A \text{ implies } h(B) \leqq h(A)\,. \tag{1}$$

(Here $B\leqq A$ means that $A-B$ is positive semidefinite and $h(A)$ is defined by

$$h(A) = Q\operatorname{diag}(h(\alpha_1),\ldots,h(\alpha_n))Q^{-1} \quad \text{if } A = Q\operatorname{diag}(\alpha_1,\ldots,\alpha_n)Q^{-1})\,.$$

Below the scalars are supposed to be real. In the complex case some notational changes are needed.

Let P_n denote the class of monotone matrix functions of order n. Then, apparently, P_1 consists of all non-decreasing functions on $(0,\infty)$ and $P_1\supset P_2 \supset P_3\ldots.$

The functions that are monotonic of every order are characterized by

LÖWNER'S THEOREM. *$h\in \bigcap_1^\infty P_n$ if and only if*

$$h(\lambda) = \int_0^\infty \frac{t\lambda-1}{t+\lambda}\,d\mu(t)+a\lambda+b, \qquad \lambda > 0\,, \tag{2}$$

where $d\mu$ is a positive finite measure on $[0,\infty)$, $a\geqq 0$, $b\in \mathrm{R}$.

(It is well known (cf. Donoghue [3, ch. II]) that, permitting λ to be complex, the functions (2) are exactly those Pick-functions in the upper halfplane that can be continued analytically across the positive real axis to the lower halfplane. This fact will however not be used in the sequel).

In his book [3] Donoghue presents the three different proofs of this theorem, existing in literature: Löwner 1934, Bendat–Sherman 1955, Koranyi–Nagy 1956. They are all based on a precise description (of interest in itself) of the classes P_n by means of certain difference quotients.

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In our approach we make use of another function class: $h \in M_n$ if and only if, for $a_j \in \mathbb{R}$, $\lambda_j > 0$, $j = 1, \ldots, 2n$,

$$(3) \quad \sum_1^{2n} a_j \frac{t\lambda_j - 1}{t + \lambda_j} \geqq 0 \quad \text{for} \quad t > 0, \ \sum_1^{2n} a_j = 0 \ \text{implies} \ \sum_1^{2n} a_j h(\lambda_j) \geqq 0\,.$$

M_n is closely related to P_n. In fact:

LEMMA 1. $P_{n+1} \subset M_n \subset P_n$, $n = 1, 2, \ldots$.

Knowing this, in proving Löwner's theorem one may consider $\bigcap M_n$ instead of $\bigcap P_n$. Thereby, in the validity of (3) for all n one recognizes a condition for the solvability of a moment problem, where the function value $h(\lambda)$ plays the role of the moment with respect to the function $t \mapsto (t\lambda - 1)/(t + \lambda)$, $t > 0$. More precisely we use a variant of M. Riesz' method for the extension of positive linear functionals (cf. [1, sect. 2.6]):

LEMMA 2. *Let $G \subset F$ be two linear spaces, n a seminorm on F, K a cone in F, l a linear functional on G. If*

$$(4) \qquad l(g) \leqq n(g + k), \quad g \in G, \ k \in K$$

then there exists a linear extension L of l to F such that

$$(5) \qquad L(k) \geqq 0 \ \text{for} \ k \in K, \quad L(f) \leqq n(f) \ \text{for} \ f \in F\,.$$

PROOF. By (4) holds

$$l(g) \leqq \inf_{k \in K} n(g + k), \quad g \in G\,,$$

where the last expression is a seminorm. But then the Hahn–Banach theorem ensures the existence of an extension L such that

$$L(f) \leqq \inf_{k \in K} n(f + k), \quad f \in F\,.$$

In particular $L(f) \leqq n(f)$ and, taking $f = -k$, $L(-k) \leqq 0$, that is, $L(k) \geqq 0$.

PROOF OF LÖWNER'S THEOREM. One readily verifies that every function (2) obeys (3) (integrate with respect to $d\mu$). Conversely, let h satisfy (3) for every n. Referring to Lemma 2, let

$$G = \left\{ g(t) = \sum_{\text{finite}} a_j \psi_{\lambda_j}(t),\ t \geqq 0, \text{ with } \sum a_j = 0 \right\}$$

$$\text{where } \psi_\lambda(t) = \frac{t\lambda - 1}{t + \lambda}\,,$$

$$F = C[0,\infty] \text{ (continuous functions on } [0,\infty]\text{, in particular does } f(\infty) = \lim_{t\to\infty} f(t) \text{ exist)},$$

$$n(f) = \sup_{t \geqq 0} |f(t)| ,$$

$$K = \text{the non-negative functions in } F ,$$

$$l\colon G \to \mathbf{R}\colon \sum a_j \psi_{\lambda_j} \mapsto \sum a_j h(\lambda_j) .$$

The important observation is now that there exist functions g_0 in G such that $g_0(t) \geqq 1$, $t \geqq 0$. For instance, this is the case with

$$g_0 = c(\psi_{\lambda'} - \psi_{\lambda''})$$

for λ' (large), λ'' (small) and $c > 0$ suitably chosen. Apart from the trivial case when h is constant, we may arrange that $l(g_0) > 0$. Multiplying h by a convenient factor we may even assume that $l(g_0) = 1$. Now by means of (3), the positivity of l on G, (4) is readily verified:

If $g_{\max} < 0$, then

$$l(g) < 0 \leqq n(g+k), \qquad k \in K;$$

if $g_{\max} \geqq 0$, then $g \leqq g_{\max} g_0$ so that

$$l(g) \leqq g_{\max} l(g_0) = g_{\max} \leqq (g+k)_{\max} \leqq n(g+k), \qquad k \in K .$$

Then Lemma 2 yields the existence of an extension L. As a bounded positive functional on $C[0,\infty]$ it can be written

$$L(f) = \int_0^\infty f(t)\, d\mu(t) + af(\infty) ,$$

where $d\mu$ is a bounded positive measure on $[0,\infty)$, $a \geqq 0$. Applying $l = L$ to $g = \psi_\lambda - \psi_1 \in G$ we get

$$h(\lambda) - h(1) = l(g) = L(g) = L(\psi_\lambda) - L(\psi_1) .$$

Hence, since $\psi_\lambda(\infty) = \lambda$ and by putting $b = h(1) - L(\psi_1)$,

$$h(\lambda) = \int_0^\infty \psi_\lambda(t)\, d\mu(t) + a\lambda + b .$$

This proves the theorem.

(From the theory of Pick functions (cf. [3, ch II] one knows that the representation (2) is unique. The moment problem we considered in the proof thus is determinate).

Before proving Lemma 1 we reformulate the definitions of M_n and P_n.

For M_n we note that in (3)

$$\sum_1^{2n} a_j \frac{t\lambda_j - 1}{t+\lambda_j} = \sum_1^{2n} a_j\left(\left(t+\frac{1}{t}\right)\frac{\lambda_j}{t+\lambda_j} - \frac{1}{t}\right) = \left(t+\frac{1}{t}\right)\sum_1^{2n} a_j \frac{\lambda_j}{t+\lambda_j},$$

since $\sum a_j = 0$. Hence $h \in M_n$ if and only if, for $a_j \in \mathsf{R}$, $\lambda_j > 0$, $j = 1,\ldots,2n$,

$$(3') \qquad \sum_1^{2n} a_j \frac{\lambda_j}{t+\lambda_j} \geqq 0 \text{ for } t > 0,\ \sum_1^{2n} a_j = 0 \ \text{ implies } \sum_1^{2n} a_j h(\lambda_j) \geqq 0\,.$$

Concerning P_n, let in (1) A and B have the (positive) eigenvalues $\alpha_1,\ldots,\alpha_n$ and $\beta_1,\ldots,\beta_n$ respectively. Denote diagonalmatrices by

$$D_\alpha = \operatorname{diag}(\alpha_1,\ldots,\alpha_n), \qquad D_{h(\alpha)} = \operatorname{diag}(h(\alpha_1),\ldots,h(\alpha_n))\,.$$

We may suppose the bases so chosen that $A = D_\alpha$. Then $B = Q^t D_\beta Q$ with Q orthogonal. Hence

$$B \leqq A \Leftrightarrow x^t B x \leqq x^t A x,\ x \in \mathsf{R}^n \Leftrightarrow (Qx)^t D_\beta (Qx) \leqq x^t D_\alpha x,\ x \in \mathsf{R}^n$$

$$\Leftrightarrow \sum_1^n (Qx)_i^2 \beta_i \leqq \sum_1^n x_i^2 \alpha_i,\ x \in \mathsf{R}^n\,.$$

In the same way

$$h(B) \leqq h(A) \Leftrightarrow \sum_1^n (Qx)_i^2 h(\beta_i) \leqq \sum_1^n x_i^2 h(\alpha_i),\ x \in \mathsf{R}^n\,.$$

It is convenient here to use the notation of weighted l_2-norms $\|x\|_\alpha = (\sum x_i^2 \alpha_i)^{1/2}$, writing $\|x\|_1 = (\sum x_i^2)^{1/2}$. For matrices we use the norm

$$\|Q\|_{\alpha,\beta} = \sup_{x \neq 0} \|Qx\|_\beta / \|x\|_\alpha\,.$$

What we have shown is that $h \in P_n$ if and only if

$$(1') \qquad Q \text{ orthogonal of type } n \times n,\ \|Q\|_{\alpha,\beta} \leqq 1 \text{ implies}$$
$$\sum_1^n (Qx)_i^2 h(\beta_i) \leqq \sum_1^n x_i^2 h(\alpha_i),\ x \in \mathsf{R}^n\,.$$

REMARK. Since h is not supposed to be positive, we have avoided to write $\|\cdot\|_{h(\alpha)}$, $\|\cdot\|_{h(\beta)}$. Claiming positivity one gets close to the concept of interpolation functions, introduced by Foias and Lions:

$$\|Q\|_{1,1} \leqq 1,\ \|Q\|_{\alpha,\beta} \leqq 1 \text{ implies } \|Q\|_{h(\alpha),h(\beta)} \leqq 1\,.$$

In a natural way, using weighted l_p-norms, the interpolation functions of type p are defined. Foias–Lions gave for $p \geqq 1$ sufficient conditions and for $p = 2$, by means of Löwner's theorem, a complete characterization of the interpolation-functions. In the remaining cases, except for $p \leqq 1$, $p=\infty$, such a characterization is not known. (Cf. [2, sect. 5.4] and the references given there).

PROOF OF LEMMA 1. Starting with the inclusion $M_n \subset P_n$, we use the fact that the expression $\lambda/(t+\lambda)$ in (3′) appears in the solution of an extremal problem:

$$\inf_{x=x_0+x_1} (x_0^2+t^{-1}\lambda x_1^2) = x^2 \frac{\lambda}{t+\lambda}, \quad x \in \mathrm{R} \tag{6}$$

(Schwarz' inequality). For $x \in \mathrm{R}^n$ define

$$K_\lambda(t,x) = \inf_{x=x_0+x_1} \|x_0\|_1^2+t^{-1}\|x_1\|_\lambda^2, \quad t>0$$

(Peetre's K-functional, known from interpolation theory). Then by (6)

$$K_\lambda(t,x) = \inf_{x=x_0+x_1} \sum_1^n (x_{0i}^2+t^{-1}\lambda_i x_{1i}^2) = \sum_1^n \inf_{x_i=x_{0i}+x_{1i}} (x_{0i}^2+t^{-1}\lambda_i x_{1i}^2) \tag{7}$$
$$= \sum_1^n x_i^2 \frac{\lambda_i}{t+\lambda_i}.$$

Now let $h \in M_n$ and, in verifying (1′), let Q be a matrix obeying $\|Qx\|_1 = \|x\|_1$, $\|Qx\|_\beta \leqq \|x\|_\alpha$, $x \in \mathrm{R}^n$. Then for every partition $x=x_0+x_1$

$$\|Qx_0\|_1+t^{-1}\|Qx_1\|_\beta \leqq \|x_0\|_1+t^{-1}\|x_1\|_\alpha, \quad t>0.$$

Taking infimum we obtain

$$K_\beta(t,Qx) \leqq K_\alpha(t,x), \quad t>0.$$

Hence, by (7),

$$\sum_1^n (Qx)_i^2 \frac{\beta_i}{t+\beta_i} \leqq \sum_1^n x_i^2 \frac{\alpha_i}{t+\alpha_i}, \quad t>0.$$

Since the sums of the coefficients are equal, $\sum (Qx)_i^2 = \sum x_i^2$, (3′) applies and yields

$$\sum_1^n (Qx)_i^2 h(\beta_i) \leqq \sum_i^n x_i^2 h(\alpha_i), \quad x \in \mathrm{R}^n.$$

By this (1′) is verified, that is, $h \in P_n$.

In verifying $P_{n+1} \subset M_n$ we consider the converse problem: Interpreting $\sum a_i\lambda_i/(t+\lambda_i) \geqq 0$, $t>0$ as $K_\beta(t,y) \leqq K_\alpha(t,x)$, $t>0$ (for suitable α, β, x, y), does this last inequality guarantee the existence of a matrix Q with $y=Qx$ such that (1′) applies (for some n). If so, the problem would be solved. In a particular case such a Q exists, and this turns out to be enough to settle also the general case.

Put

$$\pi(t) = \prod_{1}^{2n} (t+\lambda_i)$$

with distinct $\lambda_1,\ldots,\lambda_{2n}>0$, arbitrarily chosen, fixed from now on. Let Pol (n) denote the linear space of polynomials of degree at most n. By

$$\frac{p(t)}{\pi(t)} = \sum_{1}^{2n} a_i \frac{\lambda_i}{t+\lambda_i}$$

is defined a linear bijection

$$\text{(8)} \qquad \mathrm{Pol}\,(2n-1) \to \mathbf{R}^{2n}\colon\ p \mapsto a = a(p)$$

where thus $a_i = a_i(p) = p(-\lambda_i)/\pi'(-\lambda_i)\lambda_i$. The relation $\sum a_i = 0$ corresponds to $p(0)=0$. It follows that (3') is equivalent to

$$\text{(3'')} \qquad p \in \mathrm{Pol}\,(2n-1),\ p(t) \geqq 0 \ \text{ for } t > 0,\ p(0) = 0 \text{ implies} \quad \sum_{1}^{2n} a_i(p)h(\lambda_i) \geqq 0\,.$$

The polynomials in question here can be written (cf. [1, p. 77])

$$p(t) = tq_1^2(t)+q_2^2(t), \qquad q_1, q_2 \in \mathrm{Pol}\,(n-1),\ q_2(0) = 0\,.$$

Hence, because of the linearity of (8), in verifying (3'') it suffices to consider the two cases (i) $p(t)=tq^2(t)$, $q \in \mathrm{Pol}\,(n-1)$ and (ii) $p(t)=q^2(t)$, $q \in \mathrm{Pol}\,(n-1)$, $q(0)=0$. Thereby it will be convenient to rewrite $\lambda_1,\ldots,\lambda_{2n}$ as $\beta_1<\alpha_1<\ldots<\beta_n<\alpha_n$. One observes that then

$$\pi'(-\beta_i) > 0, \qquad \pi'(-\alpha_i) < 0, \qquad i=1,\ldots,n\,.$$

(i) Let $p(t)=tq^2(t)$, $q \in \mathrm{Pol}\,(n-1)$. Then

$$\text{(9)} \qquad \frac{tq^2(t)}{\pi(t)} = -\sum_{1}^{n} y_i^2 \frac{\beta_i}{t+\beta_i} + \sum_{1}^{n} x_i^2 \frac{\alpha_i}{t+\alpha_i}$$

with

$$\text{(10)} \qquad y_i = \frac{q(-\beta_i)}{(\pi'(-\beta_i))^{1/2}}, \qquad x_i = \frac{q(-\alpha_i)}{(-\pi'(-\alpha_i))^{1/2}}, \qquad i=1,\ldots,n\,.$$

Verifying (3'') we thus have to show that

$$\text{(11)} \qquad -\sum_{1}^{n} y_i^2 h(\beta_i) + \sum_{1}^{n} x_i^2 h(\alpha_i) \geqq 0\,.$$

To do this, we note that by (10) are defined two linear bijections

$$\mathsf{R}^n \to \mathrm{Pol}\,(n-1)\colon\; x \mapsto q$$

$$\mathrm{Pol}\,(n-1) \to \mathsf{R}^n\colon\; q \mapsto y\,.$$

Their composite thus is a linear bijection

$$Q\colon\; \mathsf{R}^n \to \mathsf{R}^n\colon\; x \mapsto y\,.$$

Taking $t=0$ in (9) we obtain

$$-\sum_1^n y_i^2+\sum_1^n x_i^2 \;=\; 0, \qquad \text{that is, } Q \text{ is orthogonal}\,.$$

Multiplying (9) by t and letting $t \to \infty$ we obtain

$$-\sum_1^n y_i^2\beta_i+\sum_1^n x_i^2\alpha_i \;\geqq\; 0, \qquad \text{that is, } \|Q\|_{\alpha,\beta} \leqq 1\,.$$

Hence, if h obeys (1′),

$$\sum_1^n (Qx)_i^2 h(\beta_i) \;\leqq\; \sum_1^n x_i^2 h(\alpha_i)\,.$$

By this (11) is verified. (Observe that in this part we actually only used that $h \in P_n$. This fact will be of importance in parts (ii) and (iii)).

(ii) Let $p(t)=q^2(t)$, $q \in \mathrm{Pol}\,(n-1)$, $q(0)=0$. Then

$$\frac{q^2(t)}{\pi(t)} \;=\; \sum_1^n y_i^2\,\frac{\beta_i}{t+\beta_i}-\sum_1^n x_i^2\,\frac{\alpha_i}{t+\alpha_i}\,, \tag{12}$$

with

$$y_i \;=\; \frac{q(-\beta_i)}{(\beta_i\pi'(-\beta_i))^{1/2}}, \qquad x_i \;=\; \frac{q(-\alpha_i)}{(-\alpha_i\pi'(-\alpha_i))^{1/2}}, \qquad i=1,\ldots,n\,.$$

Verifying (3″) we have to show that

$$\sum_1^n y_i^2 h(\beta_i)-\sum_1^n x_i^2 h(\alpha_i) \;\geqq\; 0\,. \tag{13}$$

Arguing as in (i) one would need an orthogonal Q with $\|Q\|_{\beta,\alpha}\leqq 1$. This is however impossible, since $\beta_i<\alpha_i$, $i=1,\ldots,n$. Instead we note that $q^2(t)/\pi(t)$ is the limit function as $\delta \to 0$, $\omega \to \infty$ of

$$\frac{t}{t+\delta}\,\frac{q^2(t)}{\pi(t)}\,\frac{\omega}{t+\omega}, \qquad \delta<\beta_1<\ldots<\alpha_n<\omega\,.$$

To the latter function the $(n+1)$-dimensional version of (i) applies. Expanding into partial fractions we get

$$(14)\quad \frac{t}{t+\delta}\frac{q^2(t)}{\pi(t)}\frac{\omega}{t+\omega} = -\bar{x}_0^2\frac{\delta}{t+\delta}-\sum_1^n \bar{x}_i^2\frac{\alpha_i}{t+\alpha_i}+\sum_1^n \bar{y}_i^2\frac{\beta_i}{t+\beta_i}+\bar{y}_{n+1}^2\frac{\omega}{t+\omega},$$

with coefficients given by formulas analogous to (10). If $h \in P_{n+1}$, then by (i)

$$(15)\qquad -\bar{x}_0^2h(\delta)-\sum_1^n \bar{x}_i^2h(\alpha_i)+\sum_1^n \bar{y}_i^2h(\beta_i)+\bar{y}_{n+1}^2h(\omega) \geqq 0 .$$

Comparing now the rational functions in (12) and (14) it follows that, for $i = 1,\ldots,n$,

$$\bar{x}_i \to x_i, \quad \bar{y}_i \to y_i \quad \text{as } \delta \to 0,\ \omega \to \infty .$$

Moreover

$$\bar{x}_0^2 = \frac{q^2(-\delta)}{\pi(-\delta)}\frac{\omega}{\omega-\delta} = O(\delta^2), \quad \delta \to 0 \text{ since } q(0)=0 ,$$

$$\bar{y}_{n+1}^2 = \frac{\omega}{\omega-\delta}\frac{q^2(-\omega)}{\pi(-\omega)} = O(1/\omega^2), \quad \omega \to \infty \text{ since degree } q \leqq n-1 .$$

Hence, proving that

$$(16)\qquad \lim_{\delta\to 0}\delta^2h(\delta) = \lim_{\omega\to\infty}\frac{h(\omega)}{\omega^2} = 0 \quad \text{if } h \in P_{n+1},\ n=1,2,\ldots$$

we obtain (13) as a limiting case of (15).

(iii) Verifying (16), since $P_{n+1} \subset P_n$, $n=1,2,\ldots$, it suffices to consider $h \in P_2$. By part (i), formula (11) applies with $n=2$, $p(t)=t(t-c)^2$. It gives

$$(17)\quad -\frac{(c-\beta_1)^2}{(\beta_2-\beta_1)(\alpha_1-\beta_1)(\alpha_2-\beta_1)}h(\beta_1)-\frac{(c-\beta_2)^2}{(\beta_2-\beta_1)(\beta_2-\alpha_1)(\alpha_2-\beta_2)}h(\beta_2)+$$
$$+\frac{(c-\alpha_1)^2}{(\alpha_1-\beta_1)(\beta_2-\alpha_1)(\alpha_2-\alpha_1)}h(\alpha_1)+\frac{(c-\alpha_2)^2}{(\alpha_2-\beta_1)(\alpha_2-\beta_2)(\alpha_2-\alpha_1)}h(\alpha_2) \geqq 0 .$$

Choosing $c=\beta_1=\delta/2$, $\alpha_1=\delta$, $\beta_2=1/2$, $\alpha_2=1$ we get

$$\delta h(\delta) \geqq 2(1-\delta)^2h(1/2)+(2-\delta)(1-2\delta)h(1) .$$

It follows that

$$\underline{\lim}_{\delta\to 0}\delta^2h(\delta) \geqq 0 .$$

On the other hand, since h is non-decreasing,

$$\overline{\lim}_{\delta\to 0}\delta^2h(\delta) \leqq 0 .$$

This settles the case $\delta \to 0$ in (16). In the same way, choosing in (17) $\beta_1 = 1$, $\alpha_1 = 2$, $\beta_2 = \omega$, $\alpha_2 = c = 2\omega$, the case $\omega \to \infty$ is treated.

REMARK. In lemma 1 the spaces P_n and M_n do not generally coincide. For $n=2$ this is seen by the example

$$h(t) = \int_1^t 1/\min(1, s^2)\, ds = \begin{cases} 1 - 1/t & \text{for } 0 < t \leqq 1 \\ t - 1 & \text{for } t \geqq 1 \,. \end{cases}$$

Being an indefinite integral of a function of the form $1/c^2(s)$, where c is positive and concave, by [3, p. 81, th. III] h belongs to P_2. But looking at part (ii) of the proof, taking $n=2$, $p(t)=t^2$, β_1 small, α_2 large, α_1 and β_2 close to 1 one easily construct an example where (3′) is violated, that is $h \notin M_2$.

REMARK. A closer analysis shows that the matrix Q in part (i) of the proof is the same as the one used by Löwner [4, § 1], cf. also Donoghue [3, ch. VI].

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