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POSITIVE CONES ASSOCIATED WITH A VON NEUMANN ALGEBRA

HIDEKI KOSAKI

Abstract.

Let $\mathscr{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ with a cyclic and separating vector ξ_0. For $\alpha \in [0,\frac{1}{2}]$, a cone P^α in $\mathfrak{H}$ is given by $P^\alpha = (\Delta^\alpha \mathscr{M}_+ \xi_0)^-$, where Δ is the associated modular operator. We characterize the cone in terms of Haagerup' $L^p(\mathscr{M})$-space and prove the followings:

(i) For $\alpha \in [0,\frac{1}{4}]$ ($\alpha \in [0,\frac{1}{2}]$ when $\mathscr{M}$ is finite), the map: $\xi \in P^\alpha \mapsto \omega_\xi \in \mathscr{M}_*^+$ is bijective,

(ii) Any fixed point under $J\Delta^{\frac{1}{2}-2\alpha}$ can be written as a difference of two elements of P^α.

Introduction.

Following the development of the Tomita–Takesaki theory, Araki [2] introduced a one parameter family of pointed cones P^α, $\alpha \in [0,\frac{1}{2}]$, associated with a von Neumann algebra admitting a cyclic and separating vector. (In [2] the cone was denoted by V^α.) Among them, there are three distinguished cones P^0, $P^{\frac{1}{4}}$, and $P^{\frac{1}{2}}$, which are also denoted by $P^\sharp$, $P^\natural$, and $P^\flat$ in the literature. It was shown by Araki [2] and Connes [3] that $P^\natural$ is neutral in many aspects. It seems however that the cones P^α, $\alpha \in [0,\frac{1}{2}]$, deserve further investigation.

In the present paper, by using Haagerup's $L^p(\mathscr{M})$-spaces, [7], we study the above mentioned one parameter family of cones. Especially we obtain a Radon–Nikodym theorem for the cones.

We shall freely use the standard results as well as notations in the Tomita–Takesaki theory, which are found in [12].

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1. Notations and main results.

Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ with a cyclic separating vector ξ_0. Throughout the paper, we shall leave the vector ξ_0 fixed. For the convenience we normalize ξ_0 so that the corresponding functional $\omega_0 = \omega_{\xi_0}$ is a state. Associated with $\{\mathcal{M}, \mathfrak{H}, \xi_0\}$ we have the modular operator Δ and the modular conjugation J. Since ξ_0 is fixed, we shall denote the associated modular automorphism group simply by $\{\sigma_t\}$.

For each $\alpha \in [0, \frac{1}{2}]$, we have $\mathcal{M}_+\xi_0 \subseteqq \mathfrak{D}(\Delta^{\frac{1}{2}}) \subseteqq \mathfrak{D}(\Delta^{\alpha})$ so that the following definition makes sense:

DEFINITION 1.1. ([2]). The cone P^{α}, $\alpha \in [0, \frac{1}{2}]$, is the closure of the convex cone $\Delta^{\alpha}\mathcal{M}_+\xi_0$ in $\mathfrak{H}$.

We then have $P^0 = P^{\sharp}$, $P^{\frac{1}{4}} = P^{\natural}$ and $P^{\frac{1}{2}} = P^{\flat}$. Araki, [2], showed that

$$P^{\frac{1}{2}-\alpha} = \{\eta \in \mathfrak{H} : \langle \eta \mid \xi \rangle \geqq 0 \quad \text{for all } \xi \in P^{\alpha}\} = JP^{\alpha},$$

$$P^{\alpha} \subseteqq \mathfrak{D}(\Delta^{\frac{1}{2}-2\alpha}) \quad \text{and} \quad \Delta^{\frac{1}{2}-2\alpha}\xi = J\xi, \ \xi \in P^{\alpha}.$$

With this set up, we state the main result.

THEOREM 1.2 (Radon–Nikodym theorem). *For each $\alpha \in [0, \frac{1}{4}]$, the map: $\xi \in P^{\alpha} \mapsto \omega_{\xi} \in \mathcal{M}_*^+$ is bijective. If $\mathcal{M}$ is finite, then the correspondence is bijective for each $\alpha \in [0, \frac{1}{2}]$. Here ω_{ξ} means the functional on $\mathcal{M}$ given by $\omega_{\xi}(x) = \langle x\xi \mid \xi \rangle$, $x \in \mathcal{M}$.*

In the course of proving this, we shall characterize the cone P^{α}, $\alpha \in [0, \frac{1}{2}]$, in terms of Haagerup's $L^2(\mathcal{M})$-space (Proposition 2.2) and derive certain properties (Propositions 2.3 and 2.4).

Before proceeding further, we make a few comments. For $\alpha = 0$ the bijectivity of the map was shown by Takesaki, [12]. For $\alpha = \frac{1}{4}$, Araki [2] and Connes [3] proved the result. For this special value of α, they exhibited further interesting properties, such as bicontinuity, concavity and so on. For the detail we refer their original papers. Skau [11] showed that the bijectivity of the map: $\xi \in P^{\frac{1}{2}} \mapsto \omega_{\xi} \in \mathcal{M}_*^+$ is equivalent to the finiteness of $\mathcal{M}$.

As an immediate consequence of the theorem, we get the following uniqueness of Araki's one parameter family of Radon–Nikodym theorems [1]:

COROLLARY 1.3 *If $\varphi \leqq l\omega_0$, $\varphi \in \mathcal{M}_*^+$, with some positive number l. then, for each $\alpha \in [0, \frac{1}{4}]$, there corresponds a unique $a_{\alpha} \in \mathcal{M}$ such that*

$$a_{\alpha}\xi_0 \in P^{\alpha} \quad \text{and} \quad \varphi(x) = \omega_0(a_{\alpha}^* x a_{\alpha}), \ x \in \mathcal{M}.$$

PROOF. The existence was proved by Araki [1]. The uniqueness follows from Theorem 1.2.

2. Positive cones in Haagerup's $L^2(\mathcal{M})$-space and the injectivity.

To show the injectivity of Theorem 1.2, we need an apparatus invented by Haagerup [8]. Making use of the crossed product of $\mathcal{M}$ by a modular automorphism group, he constructed $L^p(\mathcal{M})$-spaces, $1 \leqq p \leqq \infty$.

Let $\mathcal{M}_0$ denote the crossed product $\mathcal{M} \times_\sigma \mathbf{R}$ of $\mathcal{M}$ by $\mathbf{R}$ relative to the action $\{\sigma_t\}$, $\{\theta_s : s \in \mathbf{R}\}$ and τ be the dual action of $\hat{\mathbf{R}} = \mathbf{R}$ on $\mathcal{M}_0$ and the faithful semi-finite normal trace on $\mathcal{M}_0$ satisfying:

$$\tau \circ \theta_s = e^{-s}\tau, \ s \in \mathbf{R} \quad \text{(see [13])} .$$

For each normal semi-finite weight φ on $\mathcal{M}$, let $\hat{\varphi}$ be the dual weight on $\mathcal{M}_0$, [4], [6], and $h_\varphi = d\hat{\varphi}/d\tau$ be the Radon–Nikodym derivative of $\hat{\varphi}$ with respect to τ in the sense of Pedersen–Takesaki, [8]. Since the dual weights are precisely the θ_s-invariant weights on $\mathcal{M}_0$, normal semi-finite weights φ on $\mathcal{M}$ are in the bijective correspondence with positive self-adjoint operators h affiliated with $\mathcal{M}_0$ such that

$$\theta_s(h) = e^{-s}h, \quad s \in \mathbf{R} .$$

An interesting fact about this correspondence is that φ is finite, that is, φ belongs to $\mathcal{M}_*^+$, if and only if h_φ is τ-measurable in the sense of Segal, [10]. The space $L^p(\mathcal{M})$, $p \in [1, \infty[$ is defined as the set of all τ-measurable operators k such that

$$\theta_s(k) = e^{-s/p}k, \quad s \in \mathbf{R} .$$

The algebraic structure in $L^p(\mathcal{M})$ is considered on the regular ring of τ-measurable operators.

Imbedding $\mathcal{M}$ into $\mathcal{M}_0$ as the fixed point algebra $\mathcal{M}_0^\theta$ we have the representation (respectively anti-representation) of $\mathcal{M}$ on the Hilbert space $L^2(\mathcal{M})$ defined by

$$\pi_l(x)k = xk \quad \text{(respectively } \pi_r(x)k = kx), \quad x \in \mathcal{M}, \ k \in L^2(\mathcal{M}) .$$

The involution $J: k \in L^2(\mathcal{M}) \mapsto k^* \in L^2(\mathcal{M})$ and $L^2(\mathcal{M})_+$ together with $\pi_l(\mathcal{M})$ form a standard form $\{\pi_l(\mathcal{M}), L^2(\mathcal{M}), J, L^2(\mathcal{M})_+\}$ in the sense of Haagerup, [5]. By the uniqueness of a standard form, $P^\natural = P^{\frac{1}{4}}$ is identified with $L^2(\mathcal{M})_+$. Through this identification, we denote the operator in $L^1(\mathcal{M})_+$ corresponding to ω_0 by h_0, that is, $h_0 = d\hat{\omega}_0/d\tau$. Thus we identify $\{\mathcal{M}, \mathfrak{H}, \xi_0\}$ with $\{\pi_l(\mathcal{M}), L^2(\mathcal{M}), h_0^{\frac{1}{2}}\}$, which will be kept throughout the paper, except for section 4.

LEMMA 2.1. *Let k be a τ-measurable and h be a non-singular τ-measurable self-adjoint operator. If we have either $kh=0$ or $hk=0$, then $k=0$.*

PROOF. By taking the adjoint, we may assume that $kh=0$. Then k vanishes on the intersection of the domain of k and the range of h, which is strongly dense in the sense of Segal, so that k vanishes everywhere, [10, Cor. 5.1].

Let h be a non-singular τ-measurable self-adjoint operator and k_1, k_2 be τ-measurable operators. If $k_1 h = k_2$ (respectively $hk_1 = k_2$), then by the above lemma k_1 is uniquely determined by h and k_2 so that the notation $k_1 = k_2 h^{-1}$ (respectively $k_1 = h^{-1}k_2$) makes sense. Keeping this fact in mind, we note in our realization of the Hilbert space that

$$\mathfrak{D}(\Delta^\alpha) = \{k \in L^2(\mathcal{M}) : h_0^\alpha k h_0^{-\alpha} \in L^2(\mathcal{M})\},$$

$$\Delta^\alpha k = h_0^\alpha k h_0^{-\alpha}, \qquad k \in \mathfrak{D}(\Delta^\alpha).$$

Now the cone is characterized by the following:

PROPOSITION 2.2. *For each $\alpha \in [0,\frac{1}{4}]$, we have*

$$P^\alpha = \{k \in L^2(\mathcal{M}) : h_0^{\frac{1}{2}-2\alpha} k \geqq 0\}.$$

For each $\alpha \in [\frac{1}{4},\frac{1}{2}]$, we have

$$P^\alpha = \{k \in L^2(\mathcal{M}) : k h_0^{2\alpha-\frac{1}{2}} \geqq 0\}.$$

PROOF. The second assertion follows from the first since $JP^\alpha = P^{\frac{1}{2}-\alpha}$. Thus we may assume $\alpha \in [0,\frac{1}{4}]$. Suppose $k \in P^\alpha$ and choose a sequence $\{x_n\}$ in $\mathcal{M}_+$ such that

$$k = \lim \Delta^\alpha x_n h_0^{\frac{1}{2}} \quad \text{(in norm)}.$$

For each $h \in L^{1/2\alpha}(\mathcal{M})_+$, $hh_0^{\frac{1}{2}-2\alpha}$ belongs to $L^2(\mathcal{M})$ and

$$\begin{aligned}\tau(hh_0^{\frac{1}{2}-2\alpha}k) &= \lim \tau(hh_0^{\frac{1}{2}-2\alpha}\Delta^\alpha x_n h_0^{\frac{1}{2}}) \\ &= \lim \tau(hh_0^{\frac{1}{2}-2\alpha}h_0^\alpha x_n h_0^{\frac{1}{2}-\alpha}) \\ &= \lim \tau(hh_0^{\frac{1}{2}-\alpha}x_n h_0^{\frac{1}{2}-\alpha}) \geqq 0,\end{aligned}$$

since $h_0^{\frac{1}{2}-\alpha}x_n h_0^{\frac{1}{2}-\alpha}$ belongs to $L^{1/1-2\alpha}(\mathcal{M})_+$. Thus $h_0^{\frac{1}{2}-2\alpha}k$ belongs to the dual cone $L^{1/1-2\alpha}(\mathcal{M})_+$ of $L^{1/2\alpha}(\mathcal{M})_+$.

Conversely, if $k \in L^2(\mathcal{M})$ and $h_0^{\frac{1}{2}-2\alpha}k \geqq 0$, then we have, for each $x \in \mathcal{M}_+$,

$$\begin{aligned}\langle k \mid \Delta^{\frac{1}{2}-\alpha}xh_0^{\frac{1}{2}}\rangle &= \langle k \mid h_0^{\frac{1}{2}-\alpha}xh_0^\alpha\rangle \\ &= \tau(h_0^\alpha x h_0^{\frac{1}{2}-\alpha}k) \\ &= \tau(h_0^\alpha x h_0^\alpha h_0^{\frac{1}{2}-2\alpha}k) \geqq 0,\end{aligned}$$

because $h_0^\alpha x h_0^\alpha$ belongs to $L^{1/2\alpha}(\mathscr{M})_+$. Therefore, k belongs to the dual cone P^α of $P^{\frac{1}{2}-\alpha}$.

PROOF OF THE INJECTIVITY (of the map: $\xi \in P^\alpha \mapsto \omega_\xi \in \mathscr{M}_*^+$, $\alpha \in [0,\frac{1}{4}]$). Suppose that

$$\langle \pi_l(x)k_1 \mid k_1\rangle = \langle \pi_l(x)k_2 \mid k_2\rangle; \quad x \in \mathscr{M};\ k_1,k_2 \in P^\alpha .$$

Then there exists a partial isometry u in $\mathscr{M}$ such that $k_1 = \pi_r(u)k_2 = k_2u$. Thus, we have

$$h_0^{\frac{1}{2}-2\alpha}k_1 = h_0^{\frac{1}{2}-2\alpha}k_2u .$$

By Proposition 3.2, both of $h_0^{\frac{1}{2}-2\alpha}k_1$ and $h_0^{\frac{1}{2}-2\alpha}k_2$ are positive self-adjoint. Here the self-adjointness follows from the τ-measurability of respective operators, [1, Theorem 5]. The uniqueness of the polar decomposition implies

$$h_0^{\frac{1}{2}-2\alpha}k_1 = h_0^{\frac{1}{2}-2\alpha}k_2 .$$

Thus, we have $k_1 = k_2$ by Lemma 2.1.

PROPOSITION 2.3. *If ξ in $\mathfrak{D}(\Delta^{\frac{1}{2}-2\alpha})$ satisfies $J\Delta^{\frac{1}{2}-2\alpha}\xi = \xi$, then there exist two vectors ξ_1 and ξ_2 in P^α such that $\xi = \xi_1 - \xi_2$.*

Hence, any element in $\mathfrak{D}(\Delta^{\frac{1}{2}-2\alpha})$ can be written as a linear combination of four elements in P^α.

PROOF. By considering $J\xi$, if necessary, we may assume $\alpha \in [0,\frac{1}{4}]$. The assumption on $\xi = k$ means that $h_0^{\frac{1}{2}-2\alpha}k$ is a self-adjoint element in $L^{1/1-2\alpha}(\mathscr{M})$. Let $uh = h_0^{\frac{1}{2}-2\alpha}k$ be the polar decomposition. Due to the self-adjointness, we have

$$uh = hu^* = u^*(uhu^*) .$$

Thus, by the uniqueness of the polar decomposition, we have

$$u = u^* ,$$

$$h = uhu^* = uhu .$$

Since $u^2 = uu^* = u^*u$ is a projection in $\mathscr{M}$, the spectrum of u is included in the finite set $\{-1,0,1\}$ so that we can choose projections p_1, p_2 in $\mathscr{M}$ such that

$$u = p_1 - p_2 ,$$

$$p_1p_2 = 0 .$$

Hence we have

$$h = (p_1 - p_2)h(p_1 - p_2) .$$

On the other hand, we have

$$h = (p_1+p_2)h(p_1+p_2)\,,$$

because the range projection of h is given by $u^*u=p_1+p_2$. Thus we have

$$h = p_1hp_1+p_2hp_2\,,$$

$$h_0^{\frac{1}{2}-2\alpha}k = (p_1-p_2)h = p_1hp_1-p_2hp_2\,.$$

Set $\xi_1=kp_1$ and $\xi_2=-kp_2$. Clearly $\xi=\xi_1-\xi_2$ and we compute

$$h_0^{\frac{1}{2}-2\alpha}kp_1 = (p_1hp_1-p_2hp_2)p_1 = p_1hp_1 \geqq 0\,,$$

$$-h_0^{\frac{1}{2}-2\alpha}kp_2 = -(p_1hp_1-p_2hp_2)p_2 = p_2hp_2 \geqq 0\,.$$

Thus, ξ_1 and ξ_2 belong to P^α by Proposition 2.2.

Finally, the second assertion follows simply from the identity

$$\eta = (\eta+J\Delta^{\frac{1}{2}-2\alpha}\eta)/2+i(\eta-J\Delta^{\frac{1}{2}-2\alpha}\eta)/2i, \qquad \eta \in \mathfrak{D}(\Delta^{\frac{1}{2}-2\alpha})\,.$$

PROPOSITION 2.4. *The real subspace* $\mathfrak{R}_\alpha=P^\alpha-P^\alpha$ *in* $\mathfrak{H}$ *is closed and the mapping*

$$S_\alpha\colon \xi+i\eta \in \mathfrak{R}_\alpha+i\mathfrak{R}_\alpha \mapsto \xi-i\eta \in \mathfrak{R}_\alpha+i\mathfrak{R}_\alpha$$

is a (conjugate linear) closed operator. Furthermore, the polar decomposition of S_α *is given by*

$$S_\alpha = J\Delta^{\frac{1}{2}-2\alpha}\,.$$

PROOF. The previous proposition means that $\mathfrak{R}_\alpha$ is nothing but the set of all fixed points under $J\Delta^{\frac{1}{2}-2\alpha}$, which is the involutive isometry of the Hilbert space $\mathfrak{D}(\Delta^{\frac{1}{2}-2\alpha})$ equipped with the graph norm. Thus $\mathfrak{R}_\alpha$ is closed in $\mathfrak{D}(\Delta^{\frac{1}{2}-2\alpha})$. For $\eta \in \mathfrak{R}_\alpha$, we compute

$$2\|\eta\|^2 = \|\eta\|^2+\|\eta\|^2 = \|\eta\|^2+\|J\Delta^{\frac{1}{2}-2\alpha}\eta\|^2$$
$$= \|\eta\|^2+\|\Delta^{\frac{1}{2}-2\alpha}\eta\|^2\,,$$

so that, on $\mathfrak{R}_\alpha$, the topology of $\mathfrak{H}$ coincides with the one of $\mathfrak{D}(\Delta^{\frac{1}{2}-2\alpha})$. Therefore, $\mathfrak{R}_\alpha$ is closed in $\mathfrak{H}$. The other assertions of the proposition are trivial.

Propositions 2.3 and 2.4, which are known for special values $\alpha=0, \frac{1}{2}$, [9], mean that we can completely recover Δ and J from the cone P^α, $\alpha\neq\frac{1}{4}$.

3. The surjectivity.

In this section, we keep the identification established in the previous section and prove the surjectivity of the map: $\xi \in P^\alpha \mapsto \omega_\xi \in \mathcal{M}_*^+$, $\alpha \in [0,\frac{1}{4}]$.

Let $\mathscr{A}$ be the set of all $x \in \mathscr{M}$ such that the Fourier transform, as a distribution, of the $\mathscr{M}$-valued function: $t \in \mathbf{R} \mapsto \sigma_t(x) = h_0^{it} x h_0^{-it} \in \mathscr{M}$ has a compact support. It follows that

i) $\mathscr{A}$ is a σ-weakly dense *-subalgebra of $\mathscr{M}$.
ii) For each $x \in \mathscr{A}$, the function $t \in \mathbf{R} \mapsto \sigma_t(x) \in \mathscr{M}$ is extended to an entire function, whose value at $z \in \mathbf{C}$ will be denoted by $\sigma_z(x)$.

The following formulas can be checked by the uniqueness of analytic extension:

$$\sigma_{z+w}(x) = \sigma_z(\sigma_w(x)), \qquad \sigma_z(xy) = \sigma_z(x)\sigma_z(y)\,,$$
$$\sigma_z(x^*) = \sigma_{\bar{z}}(x)^*; \qquad x, y \in \mathscr{A};\ z, w \in \mathbf{C}\,.$$

At first we prove two technical lemmas. The second one is due to Araki [1], however we give the proof for the sake of completeness.

LEMMA 3.1. *Let T be a positive self-adjoint operator on $\mathfrak{H}$ such that $h_0^{\frac{1}{4}}$ belongs to the domain of T and T is affiliated with $\mathscr{M}$, then for $\alpha, \beta \in [0, \frac{1}{2}]$, $\Delta^\alpha T \Delta^\beta$ is a densely defined closable operator and $h_0^{\frac{1}{4}}$ belongs to its domain.*

PROOF. If $x \in \mathscr{A}$, then we have

$$\pi_r(x)h_0^{\frac{1}{4}} = h_0^{\frac{1}{4}}x \in \mathfrak{D}(\Delta^\beta)$$
$$\Delta^\beta \pi_r(x) h_0^{\frac{1}{4}} = h_0^\beta h_0^{\frac{1}{4}} x h_0^{-\beta} = h_0^{\frac{1}{4}} h_0^\beta x h_0^{-\beta}$$
$$= h_0^{\frac{1}{4}} \sigma_{-i\beta}(x) = \pi_r(\sigma_{-i\beta}(x)) h_\sigma^{\frac{1}{4}}\,,$$

so that we have

$$T\Delta^\beta \pi_r(x) h_0^{\frac{1}{4}} = \pi_r(\sigma_{-i\beta}(x)) T h_0^{\frac{1}{4}}\,,$$

because T is affiliated with $\mathscr{M}$. The spectral decomposition for T yields that

$$Th_0^{\frac{1}{4}} \subseteqq P^0 \subseteqq \mathfrak{D}(\Delta^{\frac{1}{2}}) \subseteqq \mathfrak{D}(\Delta^\alpha)\,,$$

so that we have

$$\pi_r(\sigma_{-i\beta}(x)) T h_0^{\frac{1}{4}} \in \mathfrak{D}(\Delta^\alpha)\,,$$

because $\pi_r(\mathscr{A})$ leaves the domain of any power of Δ invariant. We thus have

$$\Delta^\alpha T \Delta^\beta \pi_r(x) h_0^{\frac{1}{4}} = \Delta^\alpha \pi_r(\sigma_{-i\beta}(x)) T h_0^{\frac{1}{4}}\,,$$
$$= \pi_r(\sigma_{-(\alpha+\beta)}(x)) \Delta^\alpha T h_0^{\frac{1}{4}}\,.$$

Hence $\mathfrak{D}(\Delta^\alpha T \Delta^\beta)$ contains $\pi_r(\mathscr{A})h_0^{\frac{1}{4}}$, which is dense in $\mathfrak{H}$. Since we have

$$(\Delta^\alpha T\Delta^\beta)^* \supseteq \Delta^\beta T\Delta^\alpha ,$$

and $\mathfrak{D}(\Delta^\beta T\Delta^\alpha)$ is dense as seen above, $\Delta^\alpha T\Delta^\beta$ is closable.

LEMMA 3.2. *Let T be a positive self-adjoint operator. If*

1) $$\langle \pi_r(x)k \mid Th\rangle = \langle \pi_r(\sigma_{-i\alpha}(x))Tk \mid h\rangle; \quad k,h \in \mathfrak{D}(T);\ x \in \mathscr{A} ,$$

then we have

2) $$\langle \pi_r(x)k \mid T^{\frac{1}{2}}h\rangle = \langle \pi_r(\sigma_{-i\alpha/2}(x))T^{\frac{1}{2}}k \mid h\rangle; \quad k,h \in \mathfrak{D}(T^{\frac{1}{2}});\ x \in \mathscr{A} .$$

PROOF. We have only to show 2) for k and h in a core of $T^{\frac{1}{2}}$. Thus we may assume that both of k and h belong to a bounded spectral subspace of T. Set

$$f_1(z) = \langle \pi_r(x)k \mid T^{\bar{z}}h\rangle \quad \text{and} \quad f_2(z) = \langle \pi_r(\sigma_{-iz\alpha}(x))T^z k \mid h\rangle .$$

Then both of f_1 and f_2 are entire functions of exponential type and

$$\limsup_{r\to\infty} \frac{1}{r}\log|f_j(re^{\varepsilon\pi i/2})| = 0; \quad j=1,2;\ \varepsilon=\pm 1 .$$

Hence Carleson's theorem (for example, Boas: *Entire Functions*, Academic Press, New York, 1954, Chap. 9.2) yields $f_1(z)=f_2(z)$, $z \in \mathbf{C}$, since $f_1(n)=f_2(n)$, $n=0,1,2,\ldots$, by repeated use of 1). In particular, we have $f_1(\frac{1}{2})=f_2(\frac{1}{2})$.

We take an arbitrary φ in $\mathscr{M}_*^+$ and fix it throughout the section. By a result of Takesaki, [12], there exists a positive self-adjoint operator A affiliated with $\mathscr{M}$ such that

$$k_\varphi = Ah_0^{\frac{1}{2}} \in P^0, \quad \varphi = \omega_{k_\varphi} .$$

We will construct a representative vector $k_\alpha \in P^\alpha$ for φ by making use of the operator A.

By Lemma 3.1, $\Delta^{\frac{1}{2}}A\Delta^{\frac{1}{2}-2\alpha}$ is a densely defined closable operator on $\mathfrak{H}$ and $h_0^{\frac{1}{2}}$ belongs to its domain if $0\leqq\alpha\leqq\frac{1}{4}$. We denote the closure of $\Delta^{\frac{1}{2}}A\Delta^{\frac{1}{2}-2\alpha}$ by T_α. Since A is affiliated with $\mathscr{M}$, we expect a certain commutation relation between T_α and $\pi_r(x)$, $x \in \mathscr{A}$.

LEMMA 3.3. *Each $\pi_r(x)$, $x \in \mathscr{A}$, leaves $\mathfrak{D}(T_\alpha)$ invariant and*

1) $$T_\alpha\pi_r(x)k = \pi_r(\sigma_{i(2\alpha-1)}(x))T_\alpha k, \quad k \in \mathfrak{D}(T_\alpha) .$$

Furthermore, $\pi_r(x)$ leaves $\mathfrak{D}(T_\alpha^*)$ invariant and

2) $$T_\alpha^*\pi_r(x)k = \pi_r(\sigma_{i(2\alpha-1)}(x))T_\alpha^* k, \quad k \in \mathfrak{D}(T_\alpha^*) .$$

PROOF. It is straightforward to see that $\pi_r(x)$, $x \in \mathscr{A}$, leaves $\mathfrak{D}(\Delta^{\frac{1}{2}}A\Delta^{\frac{1}{2}-2\alpha})$ invariant and

$$\Delta^{\frac{1}{2}}A\Delta^{\frac{1}{2}-2\alpha}\pi_r(x)k = \pi_r(\sigma_{i(2\alpha-1)}(x))\Delta^{\frac{1}{2}}A\Delta^{\frac{1}{2}-2\alpha}k, \quad k \in \mathfrak{D}(\Delta^{\frac{1}{2}}A\Delta^{\frac{1}{2}-2\alpha}) .$$

For each $k \in \mathfrak{D}(T_\alpha)$, we can choose a sequence $\{k_n\}$ in $\mathfrak{D}(\Delta^{\frac{1}{2}}A\Delta^{\frac{1}{2}-2\alpha})$ such that

$$k = \lim k_n, \quad T_\alpha k = \lim T_\alpha k_n ,$$

because $\mathfrak{D}(\Delta^{\frac{1}{2}}A\Delta^{\frac{1}{2}-2\alpha})$ is a core of T_α. We then get, for each $x \in \mathscr{A}$,

$$\pi_r(x)k = \lim \pi_r(x)k_n ,$$

$$\lim T_\alpha\pi_r(x)k_n = \lim \pi_r(\sigma_{i(2\alpha-1)}(x))T_\alpha k_n$$

$$= \pi_r(\sigma_{i(2\alpha-1)}(x))T_\alpha k ,$$

so that $\pi_r(x)k$ belongs to $\mathfrak{D}(T_\alpha)$ and (1) is valid, or equivalently, we have

$$T_\alpha\pi_r(x) \supseteqq \pi_r(\sigma_{i(2\alpha-1)}(x))T_\alpha .$$

By taking the adjoint, we have

$$T_\alpha^*\pi_r(\sigma_{i(2\alpha-1)}(x)^*) = T_\alpha^*\pi_r(\sigma_{i(1-2\alpha)}(x^*)) \supseteqq \pi_r(x^*)T_\alpha^* .$$

By replacing x by $\sigma_{i(2\alpha-1)}(x)^*$, we have

$$T_\alpha^*\pi_r(x) \supseteqq \pi_r(\sigma_{i(2\alpha-1)}(x))T_\alpha^* .$$

Thus we have the second assertion.

LEMMA 3.4. *If $T_\alpha = u_\alpha H_\alpha$ is the polar decomposition, then we have, for any $x \in \mathscr{A}$ and $k, h \in \mathfrak{D}(H_\alpha)$,*

$$\langle \pi_r(x)k \mid H_\alpha h\rangle = \langle \pi_r(\sigma_{i(2\alpha-1)}(x))H_\alpha k \mid h\rangle .$$

Furthermore, the phase u_α belongs to $\mathscr{M}$.

PROOF. For any $x \in \mathscr{A}$ and $k, h \in \mathfrak{D}(T_\alpha^* T_\alpha)$, we compute

$$\langle \pi_r(x)k \mid T_\alpha^* T_\alpha h\rangle = \langle T_\alpha\pi_r(x)k \mid T_\alpha h\rangle ,$$

$$= \langle \pi_r(\sigma_{i(2\alpha-1)}(x))T_\alpha k \mid T_\alpha h\rangle \quad \text{by Lemma 3.3 (1)}$$

$$= \langle T_\alpha^*\pi_r(\sigma_{i(2\alpha-1)}(x))T_\alpha k \mid h\rangle$$

$$= \langle \pi_r(\sigma_{i(2\alpha-2)}(x))T_\alpha^* T_\alpha k \mid h\rangle \quad \text{by Lemma 3.3}$$

(2) ,

so that we have

$$\langle \pi_r(x)k \mid H_\alpha h\rangle = \langle \pi_r(\sigma_{i(2\alpha-1)}(x))H_\alpha k \mid h\rangle; \qquad k,h \in \mathfrak{D}(H_\alpha);\ x \in \mathscr{A},$$

due to Lemma 3.2.

For each $k \in \mathfrak{D}(T_\alpha^*)$, $h \in \mathfrak{D}(T_\alpha)=\mathfrak{D}(H_\alpha)$, we compute

$$\begin{aligned}\langle \pi_r(x)k \mid u_\alpha H_\alpha h\rangle &= \langle \pi_r(x)k \mid T_\alpha h\rangle \\ &= \langle T_\alpha^* \pi_r(x)k \mid h\rangle \\ &= \langle \pi_r(\sigma_{i(2\alpha-1)}(x))T_\alpha^* k \mid h\rangle, \quad \text{by Lemma 3.3 (2)}, \\ &= \langle \pi_r(\sigma_{i(2\alpha-1)}(x))H_\alpha u_\alpha^* k \mid h\rangle, \\ &= \langle \pi_r(x)u_\alpha^* k \mid H_\alpha h\rangle,\end{aligned}$$

by the first half of the proof. Hence we have $\pi_r(x^*)u_\alpha H_\alpha h = u_\alpha \pi_r(x^*)H_\alpha h$. Set $p = 1-u_\alpha^* u_\alpha$, the projection onto the null space of H_α. For any $k \in \mathfrak{H}$ and $h \in \mathfrak{D}(H_\alpha)$, we have

$$\langle \pi_r(x)pk \mid H_\alpha h\rangle = \langle \pi_r(\sigma_{i(2\alpha-1)}(x))H_\alpha pk \mid h\rangle = 0,$$

so that $(1-p)\pi_r(x)p=0$ for any $x \in \mathscr{A}$. Thus we have $p \in \pi_r(\mathscr{A})' = \pi_l(\mathscr{M}) = \mathscr{M}$, that is, $u_\alpha^* u_\alpha$ belongs to $\mathscr{M}$. Since $\pi_r(x)u_\alpha(1-p) = u_\alpha \pi_r(x)(1-p)$, $x \in \mathscr{A}$, as seen above, we conclude that u_α belongs to $\mathscr{M}$.

LEMMA 3.5. *Setting $k_\alpha = JH_\alpha h_0^{\frac{1}{2}}$, we have*

i) *$\varphi = \omega_{k_\alpha}$, and.*
ii) *the vector k_α belongs to P^α.*

PROOF. (i) We simply compute, for $x = \pi_l(x) \in \mathscr{M}$,

$$\begin{aligned}\omega_{k_\alpha}(x) &= \langle xk_\alpha \mid k_\alpha\rangle = \langle xJH_0 h_0^{\frac{1}{2}} \mid JH_\alpha h_0^{\frac{1}{2}}\rangle \\ &= \langle H_\alpha h_0^{\frac{1}{2}} \mid JxJH_\alpha h_0^{\frac{1}{2}}\rangle \\ &= \langle u_\alpha^* u_\alpha H_\alpha h_0^{\frac{1}{2}} \mid JxJH_\alpha h_0^{\frac{1}{2}}\rangle \\ &= \langle u_\alpha H_\alpha h_0^{\frac{1}{2}} \mid JxJu_\alpha H_\alpha h_0^{\frac{1}{2}}\rangle \quad \text{by Lemma 3.4}, \\ &= \langle \Delta^{\frac{1}{2}} A \Delta^{\frac{1}{2}-2\alpha} h_0^{\frac{1}{2}} \mid JxJ\Delta^{\frac{1}{2}} A \Delta^{\frac{1}{2}-2\alpha} h_0^{\frac{1}{2}}\rangle \\ &= \langle \Delta^{\frac{1}{2}} A h_0^{\frac{1}{2}} \mid JxJ\Delta^{\frac{1}{2}} A h_0^{\frac{1}{2}}\rangle \\ &= \langle xJ\Delta^{\frac{1}{2}} A h_0^{\frac{1}{2}} \mid J\Delta^{\frac{1}{2}} A h_0^{\frac{1}{2}}\rangle \\ &= \langle xJ\Delta^{\frac{1}{2}} k_\varphi \mid J\Delta^{\frac{1}{2}} k_\varphi\rangle \\ &= \langle xk_\varphi \mid k_\varphi\rangle \quad \text{since } k_\varphi \in P^0 \text{ is invariant under } J\Delta^{\frac{1}{2}} \\ &= \varphi(x).\end{aligned}$$

ii) Since $JP^{\frac{1}{2}-\alpha}=P^{\alpha}$, it suffices to show that $H_{\alpha}h_0^{\frac{1}{4}} \in P^{\frac{1}{2}-\alpha}$. For each $x \in \mathscr{A}$, we have

$$\begin{aligned}
&\langle \pi_r(x)\pi_r(\sigma_{i(1-2\alpha)}(x^*))h_0^{\frac{1}{4}} \mid H_{\alpha}h_0^{\frac{1}{4}}\rangle \\
&= \langle \pi_r(\sigma_{i(2\alpha-1)}(x))H_{\alpha}\pi_r(\sigma_{i(1-2\alpha)}(x^*))h_0^{\frac{1}{4}} \mid h_0^{\frac{1}{4}}\rangle \\
&= \langle H_{\alpha}\pi_r(\sigma_{i(1-2\alpha)}(x^*))h_0^{\frac{1}{4}} \mid \pi_r(\sigma_{i(1-2\alpha)}(x^*))h_0^{\frac{1}{4}}\rangle \geqq 0
\end{aligned}$$

by the positivity of H_{α}.

On the other hand, we have

$$\begin{aligned}
&\pi_r(x)\pi_r(\sigma_{i(1-2\alpha)}(x^*))h_0^{\frac{1}{4}} \\
&= \pi_r(x)\pi_r(h_0^{2\alpha-1}x^*h_0^{1-2\alpha})h_0^{\frac{1}{4}} \\
&= h_0^{\frac{1}{4}}h_0^{2\alpha-1}x^*h_0^{1-2\alpha}x = h_0^{2\alpha-\frac{3}{4}}x^*h_0^{1-2\alpha}x \in P^{\alpha}
\end{aligned}$$

by Proposition 2.2. Furthermore, the above elements, $x \in \mathscr{A}$, form a dense subset of P^{α} (For the detail, see [1, Lemma 5]). Thus we conclude that $H_{\alpha}h_0^{\frac{1}{4}}$ belongs to the dual cone $P^{\frac{1}{2}-\alpha}$ of P^{α}.

Therefore, we have proved the surjectivity of the map: $\xi \in P^{\alpha} \mapsto \omega_{\xi} \in \mathscr{M}_*^+$, $\alpha \in [0,\frac{1}{4}]$.

4. The bijectivity for a finite von Neumann algebra.

To complete the proof of Theorem 1.2, we prove in this section that the map: $\xi \in P^{\alpha} \mapsto \omega_{\xi} \in \mathscr{M}_*^+$ is bijective for each $\alpha \in [0,\frac{1}{2}]$ when $\mathscr{M}$ is finite.

Throughout the section, τ is a fixed tracial state on $\mathscr{M}$. We note at first that any densely defined closed operator affiliated with $\mathscr{M}$ is τ-measurable, [10, Cor. 4.1], due to finiteness. Thus the Hilbert space $L^2(\mathscr{M};\tau)$ associated with τ, [10], consists of all densely defined closed operators k affiliated with $\mathscr{M}$ such that $\tau(|k|^2)$ is finite, and $\mathscr{M}$ acts on $L^2(\mathscr{M};\tau)$ as the left multiplications, that is,

$$\pi_l(a)k = ak, \quad a \in \mathscr{M},\ k \in L^2(\mathscr{M};\tau)\,.$$

As before, $\{\pi_l(\mathscr{M}), L^2(\mathscr{M};\tau), J, L^2(\mathscr{M};\tau)_+\}$ is a standard form, where the involution J is given by $Jk=k^*$, $k \in L^2(\mathscr{M};\tau)$. Set $h_0=dw_0/d\tau$, the Radon–Nikodym derivative of w_0 with respect to τ. As in section 2, we identify $\{\mathscr{M},\mathfrak{H},\xi_0\}$ with $\{\pi_l(\mathscr{M}), L^2(\mathscr{M};\tau), h_0^{\frac{1}{2}}\}$ throughout the section so that the modular operator Δ is given by

$$\mathfrak{D}(\Delta^{\alpha}) = \{k \in L^2(\mathscr{M};\tau) : h_0^{\alpha}kh_0^{-\alpha} \in L^2(\mathscr{M};\tau)\}\,,$$

$$\Delta^{\alpha}k = h_0^{\alpha}kh_0^{-\alpha}, \quad k \in \mathfrak{D}(\Delta^{\alpha})\,.$$

Thus we have Proposition 2.2 again in this version. However, by the finiteness of $\mathcal{M}$, $h_0^{\frac{1}{2}-2\alpha}$ makes sense as a τ-measurable operator for $\alpha \in [0, \frac{1}{2}]$ so that the proposition 2.2 can be reformulated in the following fashion:

$$P^\alpha = \{k \in L^2(\mathcal{M}; \tau) : h_0^{\frac{1}{2}-2\alpha} k \geqq 0\} \qquad \text{for each } \alpha \in [0, \tfrac{1}{2}] .$$

Thus, by the argument given in section 2, we have the injectivity for each $\alpha \in [0, \frac{1}{2}]$.

Finally, we prove the surjectivity. Take $\varphi \in \mathcal{M}_*^+$ and let h_φ be the Radon–Nikodym derivative $d\varphi/d\tau$. For each $\alpha \in [0, \frac{1}{2}]$, set

$$k_\alpha = h_0^{2\alpha-\frac{1}{2}} (h_0^{\frac{1}{2}-2\alpha} h_\varphi h_0^{\frac{1}{2}-2\alpha})^{\frac{1}{2}} .$$

We have

$$h_0^{\frac{1}{2}-2\alpha} k_\alpha = (h_0^{\frac{1}{2}-2\alpha} h_\varphi h_0^{\frac{1}{2}-2\alpha})^{\frac{1}{2}} \geqq 0 ,$$

$$k_\alpha k_\alpha^* = h_0^{2\alpha-\frac{1}{2}} (h_0^{\frac{1}{2}-2\alpha} h_\varphi h_0^{\frac{1}{2}-2\alpha}) h_0^{2\alpha-\frac{1}{2}} = h_\varphi ,$$

so that k_α belongs to P^α. For each $x \in \mathcal{M}$, we compute

$$\begin{aligned} \langle \pi_l(x) k_\alpha \mid k_\alpha \rangle &= \tau(k_\alpha^* x k_\alpha) \\ &= \tau(k_\alpha k_\alpha^* x) \\ &= \tau(h_\varphi x) \\ &= \varphi(x) . \end{aligned}$$

Thus k_α is a representative vector for φ in P^α.

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CENTRE DE PHYSIQUE THÉORIQUE II
CNRS, MARSEILLE, FRANCE

Postal Address:
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KANSAS
LAWRENCE, KANSAS 66045
U.S.A.