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GEOMETRIC ASPECTS OF THE TOMITA–TAKESAKI THEORY I

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Our goal is to relate the various objects of geometrical nature that is naturally encountered in the Tomita–Takesaki theory of von Neumann algebras with properties of the normal states they correspond to. In section 1 we give a background for this paper by establishing a fixed reference frame on which to base the geometrical objects we introduce. In section 2 we start with a standard form $(M,H,J,P^\natural)$ of a σ-finite von Neumann algebra M and fix a cyclic and separating vector ξ_0 in $P^\natural$. We visualize the real subspaces $K=M_h\xi_0^-$ and $\tilde{K}=M'_h\xi_0^-$ as operator graphs of an essentially unique operator A, $0\leqq A \leqq I$, defined on $H^\natural = P^\natural - P^\natural$. Thus we may introduce a well-defined notion of relative position of K and $\tilde{K}$. In section 3 we study the cones $P^\sharp_{\xi_0}=M_+\xi_0^-$ and $P^\flat_{\xi_0}=M'_+\xi_0^-$ and show that these determine $\varphi=\omega_{\xi_0}$ up to a Jordan isomorphism of M.

In a forthcoming joint paper with U. Haagerup [9] we will explore further how the geometry of the various cones characterize the associated normal states.

1. Background.

Throughout this paper we will assume that M is a σ-finite von Neumann algebra, i.e. any orthogonal family of projections in M is at most countable. It is well known that this is equivalent to the existence of a faithful φ in $(M_*)_+$, where $(M_*)_+$ denotes the set of all normal positive linear functionals on M, M_* being the predual of M. This, in turn, is equivalent to, by the GNS construction, that M can be faithfully represented as acting on a complex Hilbert space with a cyclic and separating vector. So, to fix our ideas, we may assume that M acts on the complex Hilbert space H with $\xi_0 \in H$ a cyclic and separating vector for M. Let $J=J_{\xi_0}$ and $\Delta=\Delta_{\xi_0}$ be the conjugate linear isometric involution and modular operator, respectively, associated with (M,ξ_0) by the Tomita–Takesaki theory. Thus $JMJ=M'$, where M' denotes the commutant of M. The modular operator gives rise to the one-parameter

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modular group $\{\sigma_t\}$ of automorphisms of M defined by $\sigma_t(x)=\Delta^{it}x\Delta^{-it}$, $x \in M$, $t \in \mathbf{R}$.

The fix points of the modular group, i.e. the set of those x in M such that $\sigma_t(x)=x$, $\forall\, t \in \mathbf{R}$, is called the *centralizer* of ω_{ξ_0} and denoted by $C=C_{\xi_0}$. (Here ω_{ξ_0} in $(M_*)_+$ is defined by $\omega_{\xi_0}(x)=\langle x\xi_0,\xi_0\rangle$, $x \in M$). Using the KMS-boundary condition one shows easily that

$$C = \{x \in M \mid \omega_{\xi_0}(xy)=\omega_{\xi_0}(yx),\ \forall\, y \in M\}$$

[18; Lemma 15.8].

We introduce some important geometrical objects associated with (M,ξ_0). Let $P^\sharp=P^\sharp_{\xi_0}$ denote the closure of $M_+\xi_0$, where M_+ denotes the positive part of M, that is, $P^\sharp=M_+\xi_0^-$. $P^\sharp$ is a proper cone, that is, $P^\sharp \cap (-P^\sharp)=(0)$, with the property that to every positive normal functional φ on M (that is, $\varphi \in (M_*)_+$), there exists a unique $\xi \in P^\sharp$ such that $\varphi=\omega_\xi$ [18; Theorem 15.1]. Likewise we define $P^\flat=P^\flat_{\xi_0}$ to be the cone $M'_+\xi_0^-$. Then it is known that $P^\sharp$ and $P^\flat$ are dual cones, i.e. $P^\flat=(P^\sharp)^\circ=\{\xi \in H \mid \langle\xi,\eta\rangle \geqq 0,\ \forall\, \eta \in P^\sharp\}$. The cone $P^\sharp=P^\sharp_{\xi_0}$ is not a geometrical invariant of M but depends upon the particular cyclic and separating vector ξ_0, or more precisely, depends upon the faithful positive normal functional ω_{ξ_0}.

In 1972/73 Connes [4], Araki [1] and Haagerup [7,8] independently introduced the cone $P^\natural$ which really is a geometrical invariant of M. To be specific, let $P^\natural=P^\natural_{\xi_0}=\Delta^{1/4}(P^\sharp)^-$. This is easily seen to be equal to $\Delta^{-1/4}(P^\flat)^-$. $P^\natural$ is self-dual, i.e. $P^\natural=(P^\natural)^\circ=\{\xi \in H \mid \langle\xi,\eta\rangle \geqq 0,\ \forall\, \eta \in P^\natural\}$. More importantly, $P^\natural=P^\natural_{\xi_0}$ is an invariant of M. In fact, we have the following invariance theorem [4; Théorème 2.7]:

Let Φ: $M \to \tilde{M}$ be a $*$-isomorphism, where $\tilde{M}$ is a von Neumann algebra acting on the Hilbert space $\tilde{H}$ with $\eta_0 \in \tilde{H}$ a cyclic and separating vector for $\tilde{M}$. Define $\tilde{P}^\natural=\tilde{P}^\natural_{\eta_0}$ with respect to $(\tilde{M},\eta_0)$ analogously as $P^\natural=P^\natural_{\xi_0}$ was defined with respect to (M,ξ_0). Then there exists a unique unitary operator v: $H \to \tilde{H}$ such that $v(P^\natural)=\tilde{P}^\natural$ and $\Phi(x)=vxv^*$, $x \in M$.

In particular, if $\tilde{M}=M$ and $\Phi=\mathrm{id}$ we get:

Let $\eta_0 \in H$ be a cyclic and separating vector for M. Then there exists a unique unitary operator $v \in M'$ such that $v(P^\natural_{\xi_0})=P^\natural_{\eta_0}$. Furthermore, if $\eta_0 \in P^\natural_{\xi_0}$ then $P^\natural_{\eta_0}=P^\natural_{\xi_0}$ and $J=J_{\xi_0}=J_{\eta_0}$. Also, to every $\varphi \in (M_*)_+$ there exists a unique $\xi \in P^\natural_{\xi_0}$ such that $\varphi=\omega_\xi$.

The invariance theorem leads directly to the following property of $P^\natural$ which we will make extensive use of in proving the main theorem of section 3. It relates the Jordan isomorphisms of M and the unitaries on H leaving $P^\natural$ invariant. (Cf. [4; Section 3] and [7; Corollary 5.12]).

(†) We state this property for M a *factor* since we shall only need it in that case.

Let $\alpha: M \to M$ be an automorphism, that is, $\alpha \in \mathrm{Aut}(M)$. Then there exists a unique unitary operator u on H such that $\alpha(x) = uxu^*$, $x \in M$, and $u(P^\natural) = P^\natural$. In particular, $uMu^* = M$. Correspondingly, let $\beta: M \to M$ be an antiautomorphism. Then there exists a unique unitary operator v on H such that $\beta(x) = Jvx^*v^*J$, $x \in M$, and $v(P^\natural) = P^\natural$. In particular, $vMv^* = M'$.

Conversely, if w is a unitary operator on H such that $w(P^\natural) = P^\natural$, then either $wMw^* = M$ or $wMw^* = M'$. In the first case w implements an automorphism $\alpha: M \to M$ by $\alpha(x) = wxw^*$, $x \in M$. In the second case there is an antiautomorphism $\beta: M \to M$ such that $\beta(x) = Jwx^*w^*J$, $x \in M$.

Let $H^\natural = P^\natural - P^\natural$. Then it is easily seen that $H^\natural$ is the (closed) *real* eigenspace of J corresponding to the eigenvalue 1, that is, $H^\natural = \{\xi \in H \mid J\xi = \xi\}$. The restriction of the inner product on H to $H^\natural$ is *real* and we have $H = H^\natural + iH^\natural$. Furthermore, we have $H^\natural \amalg iH^\natural$, where we use the symbol $\amalg$ to denote orthogonality with respect to the *real* inner product $\langle \xi, \eta \rangle_r = \mathrm{Re}\langle \xi, \eta \rangle$, $\xi, \eta \in H$. Also $J(\xi + i\eta) = \xi - i\eta$, where $\xi, \eta \in H^\natural$. So J is the (real) symmetry with respect to $H^\natural$. We say that the quadruple $(M, H, J, P^\natural)$ is a *standard form* of the von Neumann algebra M, and by slight abuse of language we say that the cone $P^\natural$ is *the natural cone* of M. Set $K = K_{\xi_0} = M_h\xi_0^-$, where M_h denotes the hermitian elements of M. Then K is a closed *real* subspace of H and $K = P^\# - P^\#$, where $P^\# = P^\#_{\xi_0} = M_+\xi_0^-$. Likewise set $K = K_{\xi_0} = M'_h\xi_0^-$. Then $\tilde{K} = P^\flat - P^\flat$, where $P^\flat = P^\flat_{\xi_0} = M'_+\xi_0^-$. The closed conjugate linear operator $S = S_{\xi_0}$ defined as the closure of $x\xi_0 \to x^*\xi_0$, $x \in M$, has polar decomposition $S = J\Delta^{\frac{1}{2}}$, where as before $J = J_{\xi_0}$, $\Delta = \Delta_{\xi_0}$. Also $\mathscr{D}(\Delta^{\frac{1}{2}}) = \mathscr{D}(S) = K + iK$ and $S(\xi + i\eta) = \xi - i\eta$, $\xi, \eta \in K$. The adjoint operator of S is the conjugate linear operator $F = F_{\xi_0}$ with $\mathscr{D}(F) = \tilde{K} + i\tilde{K}$ and $F(\xi + i\eta) = \xi - i\eta$, $\xi, \eta \in \tilde{K}$, and the polar decomposition of F is $F = J\Delta^{-\frac{1}{2}}$, cf. [16; Appendix]. We have $K^{\amalg} = i\tilde{K}$, which is a special case of the following lemma that we shall need later and whose proof can be found in [15].

LEMMA 1.1. *Let $\xi \in H$ be a cyclic vector for M. Then* $(M_h\xi^-)^{\amalg} = i(M'_h\xi^-)$.

We also write down the following lemma for later reference. The proof can be obtained using Lemma 1.1. We also refer to [14] for proof.

LEMMA 1.2. *Let $\xi_0 \in H$ be a cyclic and separating vector for M. We retain the* *previous notation. (Recall that $C=C_{\xi_0}$ is the centralizer of ω_{ξ_0}). Let ξ be a vector in H. The following is equivalent:*

(i) $\Delta\xi=\xi$
(ii) $\xi \in C\xi_0^-$
(iii) $\xi \in K \cap \tilde{K}+i(K \cap \tilde{K})$.

Furthermore,

$$C_h\xi_0^- = K \cap \tilde{K} = K \cap H^\natural = \tilde{K} \cap H^\natural$$

and

$$C_+\xi_0^- = P^\sharp \cap P^\flat = P^\sharp \cap P^\natural = P^\flat \cap P^\natural .$$

2. Representation of $K=M_h\xi_0^-$ and $\tilde{K}=M_h'\xi_0^-$ as operator graphs.

Let $(M,H,J,P^\natural)$ be a standard form of the von Neumann algebra M and let ξ_0 be a cyclic and separating vector in $P^\natural$. Then we know from section 1 that $P^\natural=P^\natural_{\xi_0}$ and $J=J_{\xi_0}$. It is clear, especially from the Rieffel–van Daele paper [16], that the "relative position", whatever that term means, of the real subspaces $K=M_h\xi_0^-$ and $\tilde{K}=M_h'\xi_0^-$ is closely related to the modular operator $\Delta=\Delta_{\xi_0}$. We want to bring this out explicitly. Now $J(K)=\tilde{K}$, and since J is the (real) symmetry with respect to $H^\natural=P^\natural-P^\natural=\{\xi \in H \mid J\xi=\xi\}$, we have that K and $\tilde{K}$ lie symmetrically with respect to $H^\natural$. The (real) projection onto $H^\natural$ is clearly $\frac{1}{2}(I+J)$. Now $J|_K=\Delta^{\frac{1}{2}}|_K$, $J|_{\tilde{K}}=\Delta^{-\frac{1}{2}}|_{\tilde{K}}$, and so

$$\tfrac{1}{2}(I+J)|_K = \tfrac{1}{2}(I+\Delta^{\frac{1}{2}})|_K \quad \text{and} \quad \tfrac{1}{2}(I+J)|_{\tilde{K}} = \tfrac{1}{2}(I+\Delta^{-\frac{1}{2}})|_{\tilde{K}} .$$

We note that $\Delta^{\frac{1}{2}}|_K$ and $\Delta^{-\frac{1}{2}}|_{\tilde{K}}$ are bounded maps.

LEMMA 2.1. *We retain the notation above. Then we have*

$$K = \{\xi+UA\xi \mid \xi \in H^\natural\} \quad \text{and} \quad \tilde{K} = \{\xi-UA\xi \mid \xi \in H^\natural\} ,$$

where

$$A = \frac{|I-\Delta^{\frac{1}{2}}|}{I+\Delta^{\frac{1}{2}}} \quad \text{and} \quad U = p_1-p_2 .$$

Here p_1 is the spectral projection of Δ corresponding to the open interval $]0,1[$ and p_2 is the spectral projection corresponding to the open interval $]1,\infty[$. In fact, $(I-\Delta^{\frac{1}{2}})/(I+\Delta^{\frac{1}{2}})=UA$ is the polar decomposition of $(I-\Delta^{\frac{1}{2}})/(I+\Delta^{\frac{1}{2}})$ and we have $p_2=Jp_1J$.

PROOF. It is easily seen that the A and U given above are the factors in the polar decomposition of $(I-\Delta^{\frac{1}{2}})/(I+\Delta^{\frac{1}{2}})$. (Note that $|I-\Delta^{\frac{1}{2}}|$ denotes the absolute value of $I-\Delta^{\frac{1}{2}}$ (in the polar decomposition of $I-\Delta^{\frac{1}{2}}$)). Recall that $J\Delta J$

$=\Delta^{-1}$, and so $p_2=Jp_1J$. Set $q=\frac{1}{2}(I+J)$, the (real) projection onto $H^\natural$. We first prove that $q(K)$ $(=q(\tilde{K}))$ is dense in $H^\natural$. Indeed, assume $\xi \in H^\natural$ and $\xi \amalg q(K)$. Let $\eta \in K$. Then $0=\langle \xi, q\eta\rangle_r=\langle q\xi,\eta\rangle_r=\langle \xi,\eta\rangle_r$. Hence $\xi \amalg K$ and so $\xi \in H^\natural \cap K^{\amalg}$. By Lemma 1.1 we have $K^{\amalg}=i\tilde{K}$. Now $F=F_{\xi_0}=J\Delta^{-\frac{1}{2}}$, and so $J\Delta^{-\frac{1}{2}}\xi=-\xi$ since $\xi \in i\tilde{K}$. As $J\xi=\xi$ we get $\Delta^{-\frac{1}{2}}\xi=-\xi$, which implies that $\xi=0$. So $q(K)$ is dense in $H^\natural$. Now $q|_K=\frac{1}{2}(I+\Delta^{\frac{1}{2}})|_K$. Since $\frac{1}{2}(I+\Delta^{\frac{1}{2}})$ has bounded inverse $2/(I+\Delta^{\frac{1}{2}})$ we get that $q(K)$ is closed, hence K (and likewise $\tilde{K}$) projects *onto* $H^\natural$.

Let $\xi \in H^\natural$. Then

$$\xi+UA\xi \quad \left(= \xi+\frac{I-\Delta^{\frac{1}{2}}}{I+\Delta^{\frac{1}{2}}}\xi = \frac{2}{I+\Delta^{\frac{1}{2}}}\xi\right)$$

lies in K because $2/(I+\Delta^{\frac{1}{2}})|_{H^\natural}$ is the inverse of $\frac{1}{2}(I+\Delta^{\frac{1}{2}})|_K$. So we have proved that $K=\{\xi+UA\xi \mid \xi \in H^\natural\}$. Likewise we prove that $\tilde{K}=\{\xi-UA\xi \mid \xi \in H^\natural\}$.

We will henceforth use the term "unitary operator" to mean a *complex* linear isometric mapping of one complex Hilbert space onto another. We use the term "isometry", or "isometric operator", to mean a *real* linear isometric mapping of one real Hilbert space onto another. We will occasionally consider complex Hilbert spaces in their real restrictions, i.e. where the new (real) inner product is given as the real part of the original complex one. Let (L_1,N_1) be a pair of (closed) real subspaces of the Hilbert space H_1, and let (L_2,N_2) be a corresponding pair of the Hilbert space H_2. We say that (L_1,N_1) is (isometrically) equivalent to (L_2,N_2), in symbols, $(L_1,N_1)\cong(L_2,N_2)$, if there exists an isometry $V\colon H_1 \to H_2$ such that $V(L_1)=L_2$ and $V(N_1)=N_2$.

THEOREM 2.2. *There exists an operator* $A\colon H^\natural \to H^\natural$, $0\leqq A\leqq I$, *such that* (graph A, graph $(-A))\cong(K,\tilde{K})$, *i.e. there exists an isometry V from the real Hilbert space $H^\natural \oplus H^\natural$ onto H, in its real restriction, which maps* graph A *onto K and* graph $(-A)$ *onto* $\tilde{K}$.

The operator A is unique up to isometric equivalence, i.e. if B is another operator with the same properties as A then there exists an isometry W: $H^\natural \to H^\natural$ such that $B=WAW^*$.

Specifically, we may choose

$$A = \left.\frac{|I-\Delta^{\frac{1}{2}}|}{I+\Delta^{\frac{1}{2}}}\right|_{H^\natural}.$$

PROOF. Recall that $C=C_{\xi_0}$ is the centralizer of ω_{ξ_0} and let p_0 be the projection onto $C\xi_0^-$. By Lemma 1.2 we have $C\xi_0^-=\{\xi \in H \mid \Delta\xi=\xi\}$. The operator U of Lemma 2.1 maps the orthogonal complement of $C\xi_0^-$

isometrically onto itself. Set $U' = U + ip_0 = p_1 - p_2 + ip_0$, where p_1, p_2 are the projections referred to in Lemma 2.1. Then U' is a unitary operator on H. Define an isometry $V\colon H^\natural \oplus H^\natural \to H = H^\natural + iH^\natural$ from the real Hilbert space $H^\natural \oplus H^\natural$ onto H, in its real restriction, by $V(\xi \oplus \eta) = \xi + U'\eta$. That V is an isometry stems from the claim that U' maps $H^\natural$ onto $iH^\natural$, or, what is equivalent, that $JU' = -U'J$. Indeed, $p_2 = Jp_1J$ and $Jp_0 = p_0J$. (The latter fact follows from Lemma 1.2). From this we get immediately that $JU' = -U'J$.

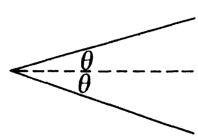
Let $A = |I - \Delta^{\frac{1}{2}}|/(I + \Delta^{\frac{1}{2}})|_{H^\natural}$ and observe that from $J\Delta J = \Delta^{-1}$ we get that $JA = AJ$, so A maps $H^\natural$ into $H^\natural$. Clearly $0 \leqq A \leqq I$, and also $\ker A = \{\xi \in H^\natural \mid \Delta\xi = \xi\} = C_h \xi_0^-$. By Lemma 2.1 we then get:

$$V(\operatorname{graph} A) = V\{(\xi, A\xi) \mid \xi \in H^\natural\} = \{\xi + U'A\xi \mid \xi \in H^\natural\} = \{\xi + UA\xi \mid \xi \in H^\natural\} = K\,.$$

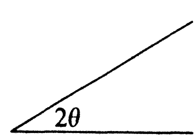
Likewise, $V(\operatorname{graph}(-A)) = \tilde{K}$. (Note that we make the obvious identification of the set of ordered pairs $\{(\xi, \eta) \mid \xi, \eta \in H^\natural\}$ with $H^\natural \oplus H^\natural$).

We prove uniqueness. So assume $B\colon H^\natural \to H^\natural$ such that $0 \leqq B \leqq I$ and $(\operatorname{graph} B, \operatorname{graph}(-B)) \cong (K, \tilde{K})$. To deduce that B is isometrically equivalent to A we proceed along the lines suggested in [10], see Remark below. We observe first that 1 is not an eigenvalue of B. Indeed, if ξ is a non-zero vector in $H^\natural$ such that $B\xi = \xi$, then $(\xi, \xi) \in \operatorname{graph} B$ and $(\xi, -\xi) \in \operatorname{graph}(-B)$. Let $\eta \in H^\natural$. Then the inner product of $(\xi, -\xi)$ and $(\eta, B\eta)$ equals $\langle \xi, \eta \rangle - \langle \xi, B\eta \rangle = \langle \xi, \eta \rangle - \langle B\xi, \eta \rangle = 0$, and so $(\xi, -\xi)$ is orthogonal to $\operatorname{graph} B$. This means that there exists a non-zero vector in $\tilde{K} \cap K^{\amalg}$. Now $K^{\amalg} = i\tilde{K}$ by Lemma 1.1 and so $\tilde{K} \cap i\tilde{K} \neq (0)$. This, however, contradicts that $(\tilde{K} \cap i\tilde{K})^{\amalg} \supseteq \tilde{K}^{\amalg} + i\tilde{K}^{\amalg} = K + iK$ and $K + iK$ is dense in H. So we have proved that $I - B$ has an inverse.

We now exploit trigonometric analogy. The graph of a linear transformation on a Hilbert space is very much like a line in a plane. Since the elements of $\operatorname{graph} B$ are ordered pairs $(\xi, B\xi)$, it follows, purely formally, that the ratio of the second coordinate to the first is always B; so B plays the role of tangent of the inclination θ. Now the geometric fact that the diagram



when rotated upwards through θ becomes



suggests that (graph B, graph $(-B)$) is equivalent to (graph $T, H^\natural \oplus 0$), where $T = 2B/(I - B^2)$. (Use that $\operatorname{tg} 2\theta = 2 \operatorname{tg} \theta / (1 - \operatorname{tg}^2 \theta)$). Observe that T is a well-defined positive self-adjoint operator, possibly unbounded, with $\mathscr{D}(T) = \operatorname{Range}(I - B)$ and $\ker T = \ker B$. The trigonometric analogy suggests further that the equivalence above is implemented by the isometry $U_1 \colon H^\natural \oplus H^\natural \to H^\natural \oplus H^\natural$ defined by

$$U_1((\xi, \eta)) = \left(\frac{I}{(I + B^2)^{\frac{1}{2}}} \xi - \frac{B}{(I + B^2)^{\frac{1}{2}}} \eta, \frac{I}{(I + B^2)^{\frac{1}{2}}} \eta + \frac{B}{(I + B^2)^{\frac{1}{2}}} \xi \right).$$

(Use that rotation counterclockwise through θ has the matrix

$$\left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right).$$

It is readily verified that U_1 is an isometry. By straightforward computation we get

$$U_1(\operatorname{graph}(-B)) = \{ ((I + B^2)^{\frac{1}{2}} \xi, 0) \mid \xi \in H^\natural \} = H^\natural \oplus 0 .$$

Also

$$U_1(\operatorname{graph} B) = \left\{ \left(\frac{I - B^2}{(I + B^2)^{\frac{1}{2}}} \xi, \frac{2B}{(I + B^2)^{\frac{1}{2}}} \xi \right) \middle| \ \xi \in H^\natural \right\}.$$

Now

$$\operatorname{Range}\left(\frac{I - B^2}{(I + B^2)^{\frac{1}{2}}} \right) = \operatorname{Range}(I - B) = \mathscr{D}(T) \quad \text{and} \quad T\left(\frac{I - B^2}{(I + B^2)^{\frac{1}{2}}} \xi \right) = \frac{2B}{(I + B^2)^{\frac{1}{2}}} \xi .$$

Hence graph $T = U_1(\operatorname{graph} B)$.

By analogy we get that (graph A, graph $(-A)$) is equivalent to (graph T_1, $H^\natural \oplus 0$), where $T_1 = 2A/(I - A^2)$. Here T_1 is a positive self-adjoint operator, possibly unbounded, with $\mathscr{D}(T_1) = \operatorname{Range}(I - A)$ and $\ker(T_1) = \ker(A)$. Since $\cong$ clearly is an equivalence relation we get that (graph T_1, $H^\natural \oplus 0$) $\cong$ (graph T, $H^\natural \oplus 0$). Let $U_2 \colon H^\natural \oplus H^\natural \to H^\natural \oplus H^\natural$ implement the equivalence. Since U_2 maps $H^\natural \oplus 0$ onto $H^\natural \oplus 0$ we get that U_2 is a direct sum $W_1 \oplus W$, where W and W_1 are isometries on $H^\natural$. The assumption that $W_1 \oplus W$ maps graph T_1 onto graph T implies that if $\xi \in \mathscr{D}(T_1)$, then $W_1 \xi \in \mathscr{D}(T)$ and $T W_1 \xi = W T_1 \xi$; in other words, $T W_1 = W T_1$. Taking adjoints we get $W_1^* T = T_1 W^*$ and by multiplication we get $T^2 = W T_1^2 W^*$. Hence $T = W T_1 W^*$ because both T and T_1 are positive self-adjoint operators and the positive square root is unique. In particular, W maps $\ker T_1 = \ker A$ onto $\ker T = \ker B$ and $L_1 = H^\natural \ominus \ker T_1$ onto

$L = H^\natural \ominus \ker T$. (We use the symbol $\ominus$ to denote orthogonal complement). The restriction of T_1 to $\mathscr{D}(T_1) \cap L_1$ is self-adjoint and non-singular in L_1 and the same is true for the restriction of T to $\mathscr{D}(T) \cap L$ in L. Now

$$T_1 = \frac{2A}{I-A^2} \quad \text{and} \quad T = \frac{2B}{I-B^2},$$

respectively, and restricting to L_1 and L, respectively, we can solve in terms of A and B, respectively, to get $A = f(T_1)$ and $B = f(T)$, respectively, where

$$f(\lambda) = \frac{-1+\sqrt{1+\lambda^2}}{\lambda}, \quad \lambda > 0 .$$

All this adds up to the conclusion that $B = WAW^*$, and the proof is complete.

REMARK. In [10] pairs of subspaces K_1 and K_2 are studied which are in generic position, i.e. the pairwise intersections of K_1 and K_2 and their orthogonal complements are all (0). In our situation the pair $(K, \tilde{K})$ is not in generic position, since $K \cap \tilde{K} = C_h\xi_0^-$ and $K^{\perp\!\!\perp} \cap \tilde{K}^{\perp\!\!\perp} = i(K \cap \tilde{K}) = i(C_h\xi_0)^-$ by Lemma 1.1 and Lemma 1.2. The above theorem is in its purely geometric aspect closely linked to the main theorem of [10; Theorem 3], though the unicity of the representation is not explicitly proved there. (See also [16; Theorem 2.4]).

Owing to Theorem 2.2 we may now be precise and define what we mean by the vaguely intuitive term "relative position" of $K = M_h\xi_0^-$ and $\tilde{K} = M'_h\xi_0^-$, where $\xi_0 \in P^\natural$ is a cyclic and separating vector for M.

DEFINITION 2.3. By the expression "relative position of $(K, \tilde{K})$" we mean the isometric equivalence class of the operator $A = |I - \Delta^{\frac{1}{2}}|/(I + \Delta^{\frac{1}{2}})$ (restricted to $H^\natural$), where $\Delta = \Delta_{\xi_0}$ is the modular operator associated with (M, ξ_0).

This is equivalent to say the *unitary* equivalence class of the operator $|I - \Delta^{\frac{1}{2}}|/(I + \Delta^{\frac{1}{2}})$ on H, with the proviso that the unitaries in question commute with $J = J_{\xi_0}$. (In fact, an isometry on $H^\natural$ has a unique extension to a unitary operator on H that commutes with J).

In view of Definition 2.3 a natural question to ask is the following: Given another cyclic and separating vector $\eta_0 \in P^\natural$ with associated (real) subspaces $K_1 = M_h\eta_0^-$, $\tilde{K}_1 = M'_h\eta_0^-$, and (essentially) unique operator $A_1 = |I - \Delta_1^{\frac{1}{2}}|/(I + \Delta_1^{\frac{1}{2}})$ on $H^\natural$, where $\Delta_1 = \Delta_{\eta_0}$; if the relative position of $(K, \tilde{K})$ is the same as the relative position of $(K_1, \tilde{K}_1)$, what is the relation between the faithful positive normal linear functionals $\varphi = \omega_{\xi_0}$ and $\psi = \omega_{\eta_0}$? To be specific, assume there exists an isometry $V\colon H^\natural \to H^\natural$ such that $A_1 = VAV^*$. Let the unique unitary extension of V to H again be denoted by V. We then have

$$(*) \qquad \frac{|I-\Delta_1^{\frac{1}{2}}|}{I+\Delta_1^{\frac{1}{2}}} = V\frac{|I-\Delta^{\frac{1}{2}}|}{I+\Delta^{\frac{1}{2}}}V^* .$$

If $\Delta_1 = V\Delta V^*$ (or $\Delta_1^{-1} = V\Delta V^*$) it follows from spectral theory that equation $(*)$ holds. Conversely, assuming M is a factor, it seems to be a reasonable conjecture that if $(*)$ holds, then either $\Delta_1 = V\Delta V^*$ (or $\Delta_1^{-1} = V\Delta V^*$). (Note that if the absolute value is removed from the two numerators in $(*)$, then it follows from spectral theory that $\Delta_1 = V\Delta V^*$). Now $S_{\xi_0} = J\Delta^{\frac{1}{2}}$, $F_{\xi_0} = J\Delta^{-\frac{1}{2}}$ and $S_{\eta_0} = J\Delta_1^{\frac{1}{2}}$, $F_{\eta_0} = J\Delta_1^{-\frac{1}{2}}$. Using that the polar decomposition is unique we conclude (cf. discussion preceding Lemma 1.1): If there exists a *unitary* operator V on H such that $V(K) = K_1$ (or $V(K) = \tilde{K}_1$) then the relative position of $(K, \tilde{K})$ is the same as the relative position of $(K_1, \tilde{K}_1)$. Conversely, assuming M is a factor, we conjecture that if the relative position of $(K, \tilde{K})$ is the same as the relative position of $(K_1, \tilde{K}_1)$, then there exists a *unitary* operator V on H such that $V(K) = K_1$ (or $V(K) = \tilde{K}_1$).

Returning to our original question we assume that $\varphi = \omega_{\xi_0}$ and $\psi = \omega_{\eta_0}$ are normal *states*, i.e. $\|\xi_0\| = \|\eta_0\| = 1$. If $\psi = \varphi \circ \alpha$, where α is a Jordan isomorphism of M, then one can show that the relative position of $(K, \tilde{K})$ is the same as the relative position of $(K_1, \tilde{K}_1)$, cf. section 3. The converse, however, is not true as the following example shows. (The example was shown us by U. Haagerup. Actually, as explained in the next section, the case when M is finite requires some elaboration, but the example below avoids this problem).

EXAMPLE. We construct an infinite factor M (of type I) on standard form $(M, H, J, P^\natural)$ with the following property: there exist two cyclic and separating unit vectors ξ_0 and η_0 in $P^\natural$ and a unitary operator V on H such that $V(K) = K_1$, but $\omega_{\eta_0} \neq \omega_{\xi_0} \circ \alpha$ for every Jordan isomorphism α of M. (Recall that $K = M_h\xi_0^-$, $K_1 = M_h\eta_0^-$).

Let M_1 be the 3×3 matrices acting standardly as a von Neumann algebra on $H_1 = M_1$ (in Hilbert–Schmidt norm) by left multiplication. The natural cone $P_1^\natural$ for M_1 is the set of positive matrices. Let $\lambda_1, \lambda_2, \lambda_3$ be (non-zero) positive real numbers and set

$$h = c_1 \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}, \quad k = c_2 \begin{bmatrix} \lambda_1^{-1} & & 0 \\ & \lambda_2^{-1} & \\ 0 & & \lambda_3^{-1} \end{bmatrix}$$

where c_1 and c_2 are chosen so that $\mathrm{Tr}_1(h) = \mathrm{Tr}_1(k) = 1$, Tr_1 being the canonical trace on M_1. Choose $\lambda_1, \lambda_2, \lambda_3$ so that h and k do not have the same eigenvalues, e.g. $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$. Set $\xi = h^{\frac{1}{2}}$, $\eta = k^{\frac{1}{2}}$. Then ξ and η are cyclic and separating unit vectors in $P_1^\natural$. Let $(\mu_{ij}) \in H_1$. Then $\Delta_\xi(\mu_{ij}) = (\lambda_i \lambda_j^{-1} \mu_{ij})$, $\Delta_\eta(\mu_{ij}) = (\lambda_i^{-1} \lambda_j \mu_{ij})$. Also $J_\xi = J_\eta = J_1$, where $J_1(\mu_{ij}) = (\overline{\mu_{ji}})$. Let U be the unitary operator on H_1 defined by $U(\mu_{ij}) = (\mu_{ji})$, i.e. U is the transpose map. We easily verify that $UJ_1U^* = J_1$ and $U\Delta_\xi U^* = \Delta_\eta$. Now let M_2 be the unique type I_∞ factor acting standardly on the Hilbert space H_2 of Hilbert–Schmidt operators in M_2 by left multiplication. The natural cone $P_2^\natural$ for M_2 is the set of positive Hilbert–Schmidt operators. Let $a \in \,]0,1[$ and set

$$h_a = \frac{1-a}{a} \begin{bmatrix} a & & & 0 \\ & a^2 & & \\ & & a^3 & \\ 0 & & & \ddots \end{bmatrix} .$$

Then h_a is a trace-class operator in M_2 and $h_a^{\frac{1}{2}}$ is a unit vector in H_2. Notice that $h \otimes h_a$ and $k \otimes h_a$ do not have the same eigenvalues. (Compare the two largest eigenvalues). Thus $h \otimes h_a$ and $k \otimes h_a$ are not unitarily equivalent. We set $M = M_1 \otimes M_2$, acting on $H = H_1 \otimes H_2$ ($\cong$ Hilbert–Schmidt operators in M). M acts standardly with natural cone $P^\natural$ equal to the positive Hilbert–Schmidt operators in M. Set

$$\xi_0 = \xi \otimes h_a^{\frac{1}{2}} = h^{\frac{1}{2}} \otimes h_a^{\frac{1}{2}} \quad \text{and} \quad \eta_0 = \eta \otimes h_a^{\frac{1}{2}} = k^{\frac{1}{2}} \otimes h_a^{\frac{1}{2}} .$$

Then ξ_0 and η_0 are cyclic and separating vectors for M in $P^\natural$. We have that $J_{\xi_0} = J_{\eta_0} = J$, where J is equal to the map $x \to x^*$, $x \in H$. Let $U\colon H_1 \to H_1$ be as above and let $\mathrm{id}\colon H_2 \to H_2$ be the identity operator. Set $V = U \otimes \mathrm{id}$. Then V is a unitary operator on H and it is easily verified that $VJV^* = J$ and $V\Delta_{\xi_0}V^* = \Delta_{\eta_0}$. So V maps $M_h\xi_0^-$ onto $M_h\eta_0^-$. However,

$$\omega_{\xi_0}(\cdot) = \operatorname{Tr}((h \otimes h_a)\cdot) \quad \text{and} \quad \omega_{\eta_0}(\cdot) = \operatorname{Tr}((k \otimes h_a)\cdot)$$

are not related by a Jordan isomorphism α, since this would entail that $h \otimes h_a$ and $k \otimes h_a$ were unitarily equivalent. (Indeed, this is a consequence of the fact that the canonical trace Tr on M is invariant under Jordan isomorphisms and that automorphisms of M are inner).

REMARK. We think it is appropriate at this point to mention that if we had chosen a more restrictive notion of relative position, taking the cone $P^\natural$ into account, then it would be true that normal states with the same relative position are equal modulo a Jordan isomorphism. Specifically, if we require the unitary operator V that implements the equivalence in (*) above to map $P^\natural$ onto $P^\natural$, then $\psi = \varphi \circ \alpha$ for some Jordan isomorphism α of M. (Here we assume M is a factor). Indeed, if $V\xi_0 = \gamma_0$ then it follows easily from property (†) of section 1 that $\theta = \varphi \circ \alpha$ for some Jordan isomorphism α of M, where $\theta = \omega_{\gamma_0}$.

Besides it can be verified that either $V\Delta V^* = \Delta_2$ or $V\Delta V^* = \Delta_2^{-1}$, where $\Delta_2 = \Delta_{\gamma_0}$. (Cf. proof of Theorem 3.1). So by (*) we get

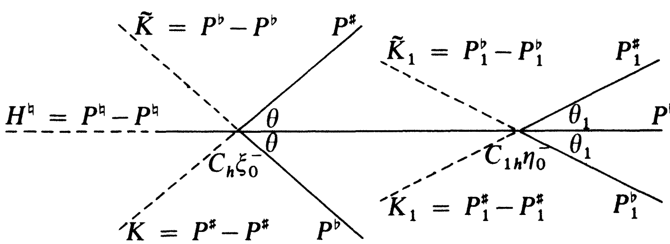
$$\frac{|I-\Delta_1^{\frac12}|}{I+\Delta_1^{\frac12}} = \frac{|I-\Delta_2^{\frac12}|}{I+\Delta_2^{\frac12}}\,.$$

This equality implies by elementary spectral theory that $f(\log\Delta_1) = f(\log\Delta_2)$ for every *even* Borel function f on R, i.e. $f(\lambda) = f(-\lambda)$, $\lambda \in \mathbf{R}$. In particular, $\Delta_1^{it} + \Delta_1^{-it} = \Delta_2^{it} + \Delta_2^{-it}$, $\forall t \in \mathbf{R}$. By the results in [9] this yields $\psi = \theta$, thus proving $\psi = \varphi\circ\alpha$. (For M finite there might be a second alternative, analogous to the situation described in Lemma 3.3, cf. [9]).

We conclude this section by presenting a suggestive, if rather simplified, figure. Retaining the previous notation let $\xi_0 \in P^\natural$ be a cyclic and separating unit vector for M with $\Delta = \Delta_{\xi_0}$, $P^\sharp = M_+\xi_0^-$, $P^\flat = M'_+\xi_0^-$, $K = M_h\xi_0^- = P^\natural - P^\sharp$, $\tilde K = M'_h\xi_0 = P^\flat - P^\flat$. By Lemma 1.2 we have

$$K \cap \tilde K = K \cap H^\natural = \tilde K \cap H^\natural = C_h\xi_0^- = \{\xi \in H \mid J\xi = \xi,\ \Delta\xi = \xi\}\,,$$

where C is the centralizer of ω_{ξ_0}. Likewise let $\eta_0 \in P^\natural$ be another cyclic and separating unit vector and let Δ_1, $P_1^\sharp$, $P_1^\flat$, K_1, $\tilde K_1$ and C_1 be the corresponding objects associated with η_0.



We may think of $H^\natural$ as the x-axis and imagine the orthogonal y-axis as a replica of $H^\natural$. By Theorem 2.2 the "tangents" of the "angles" θ and θ_1, respectively, are the (equivalence classes of the) operators

$$A = \frac{|I-\Delta^{\frac12}|}{I+\Delta^{\frac12}} \quad \text{and} \quad A_1 = \frac{|I-\Delta_1^{\frac12}|}{I+\Delta_1^{\frac12}}, \quad \text{respectively}\,.$$

The example we gave above shows that $\theta = \theta_1$ does not necessarily imply that $\omega_{\eta_0} = \omega_{\xi_0}\circ\alpha$ for some Jordan isomorphism α of M. As the remark above indicates one has to bring the various cones into consideration in order to get this conclusion. This will be clarified in the next section.

The figure also suggests that one should study the projected images of the various $P^\sharp$-cones into $H^\natural$. This is done in [9].

3. The cones $P^{\sharp}=M_{+}\xi_0^{-}$ and $P^{\flat}=M'_{+}\xi_0^{-}$.

The main theorem of this section is stated for factors only. The reason for this is that our results are more awkward to state for general von Neumann algebras, while on the other hand the factor case will elucidate all the significant ideas involved. However, the subsequent results, appropriately modified, are true for general von Neumann algebras. Roughly speaking, in the general case one has to "split" the various objects under consideration by appropriate projections in the center of the von Neumann algebra M. All this stems from Kadison's [11] characterization of Jordan isomorphisms $\alpha: M \to M$, to wit, α can be decomposed as a direct sum of a *-isomorphism and a *-antiisomorphism by splitting M in direct sums by appropriate projections in the center of M. If M is a factor this becomes simply that a Jordan isomorphism is either an automorphism or an antiautomorphism.

The following theorem clarifies the significance of the $P^{\sharp}$- and $P^{\flat}$-cones with respect to perturbations of normal states by Jordan isomorphisms.

THEOREM 3.1. *Let $(M,H,J,P^{\natural})$ be a standard form of the factor M. Let φ and ψ be two faithful normal states on M and let ξ_0 and η_0 be the two unique cyclic and separating unit vectors in $P^{\natural}$ such that $\varphi=\omega_{\xi_0}$, $\psi=\omega_{\eta_0}$. Set $P^{\natural}=M_{+}\xi_0^{-}$, $P^{\flat}=M'_{+}\xi_0^{-}$ and $P_1^{\sharp}=M_{+}\eta_0^{-}$, $P_1^{\flat}=M'_{+}\eta_0^{-}$. Then the following statements hold:*

(i) *$\psi=\varphi\circ\alpha$ for some automorphism $\alpha: M \to M$ if and only if there exists a unitary operator u on H such that $u(P^{\sharp})=P_1^{\sharp}$. (If M is finite we have to assume in addition that $u\xi_0=\eta_0$).*

(ii) *$\psi=\varphi\circ\beta$ for some antiautomorphism $\beta: M \to M$ if and only if there exists a unitary operator v on H such that $v(P^{\sharp})=P_1^{\flat}$. (If M is finite we have to assume in addition that $v\xi_0=\eta_0$ and also that ξ_0 (or η_0) is not a trace vector for M).*

COROLLARY 3.2. *There is a $1-1$ correspondence between the automorphisms α of M which leave $\varphi=\omega_{\xi_0}$ invariant (i.e. $\varphi\circ\alpha=\varphi$) and the set of unitary operators u on H which map $P^{\sharp}=M_{+}\xi_0^{-}$ onto itself. (If M is finite we have to assume in addition that the u's fix ξ_0 and also that ξ_0 is not a trace vector for M). There is a $1-1$ correspondence between the anti-automorphisms β of M which leave φ invariant and the set of unitary operators v on H which map $P^{\sharp}=M_{+}\xi_0^{-}$ onto $P^{\flat}=M'_{+}\xi_0^{-}$. (If M is finite we have to assume in addition that the v's fix ξ_0 and also that ξ_0 is not a trace vector for M).*

REMARK. If M is a type I factor then $\psi=\varphi\circ\alpha$ for some automorphism α if and only if $\psi=\varphi\circ\beta$ for some antiautomorphism β. Hence we conclude from the above theorem that in this case $P^{\sharp}$ is unitarily equivalent to $P_1^{\sharp}$ if and only if $P^{\sharp}$ is unitarily equivalent to $P_1^{\flat}$. This is not true in general. In fact, Connes [5] has shown the existence of factors of all types (except type I) with separable predual which are not antiautomorphic. By the above theorem it follows that there exist (non-finite) factors such that no $P^\#$-cone can be mapped onto any $P^\flat$-cone by a *unitary* operator on H. On the other hand we know that $J(P^\#_{\xi_0}) = P^\flat_{\xi_0}$ for every cyclic and separating vector ξ_0 in $P^\natural$; J is an isometry but *conjugate* linear.

We might ask a related question: If $v(P^\#) = P^\flat_1$ for some unitary operator v on H, does it then exist a unitary operator u on H such that $u(P^\#) = P^\#_1$? We give an example to show that this is not the case in general. Indeed, by the above theorem it is sufficient to exhibit a finite factor M with two faithful normal states $\varphi = \omega_{\xi_0}$ and $\psi = \omega_{\eta_0}$ such that $\psi = \varphi\circ\beta$ for some anti-automorphism β of M, while $\psi \neq \varphi\circ\alpha$ for every automorphism α of M. (Here $\xi_0, \eta_0 \in L^2(M,\tau)_+$, where τ is the normalized trace on M. We also make sure that $\eta_0^{-1} \notin L^2(M,\tau)_+$, cf. Lemma 3.3 for a discussion of this point). Let N be a type II_1 factor (with separable predual) not antiisomorphic to itself. Set $M = N\otimes N^{\mathrm{op}}$, where N^{op} is the *opposite* algebra of N, that is, N^{op} is identical with N as a vector space while the multiplication $\circ$ in N^{op} is given by $x\circ y = yx\,(x, y \in N)$. Let β: $N\otimes N^{\mathrm{op}} \to N\otimes N^{\mathrm{op}}$ be the antiautomorphism such that $\beta(x\otimes y) = y\otimes x$ for $x, y \in N$. Set $\varphi = \tau_1\otimes\theta$, where τ_1 is the normalized trace on N and θ is a faithful normal state on N such that the centralizer of θ is a maximal abelian algebra A of N. (If A is maximal abelian in N we first find $h \in N_+$ such that h generates A [6; Ch. I, §7, Exercice 3]. By appropriate scaling we may assume that $h^{-\frac{1}{2}} \notin L^2(N,\tau_1)_+$. Set $\theta = \tau_1(h\cdot) = \omega_{h^{\frac{1}{2}}}$. Then centralizer of θ is $\{h\}' \cap N = A$). We set $\psi = \varphi\circ\beta$. Then clearly $\psi = \theta\otimes\tau_1$, and so $\psi = \omega_{\eta_0}$, where $\eta_0 = h^{\frac{1}{2}}\otimes I \in L^2(M,\tau)_+$. Since $h^{-\frac{1}{2}} \notin L^2(N,\tau_1)_+$ we have that $\eta_0^{-1} \notin L^2(M,\tau)_+$.

We claim that $\psi \neq \varphi\circ\alpha$ for every automorphism α of M. Assume to the contrary that $\psi = \varphi\circ\alpha$ for some automorphism α. We have $\sigma^\psi_t = \alpha^{-1}\circ\sigma^\varphi_t\circ\alpha$, $\sigma^\varphi_t = \mathrm{id}\otimes\sigma^\theta_t$ and $\sigma^\psi_t = \sigma^\theta_t\otimes\mathrm{id}$, $\forall t \in \mathbf{R}$. So $\sigma^\theta_t\otimes\mathrm{id} = \alpha^{-1}\circ(\mathrm{id}\otimes\sigma^\theta_t)\circ\alpha$, $\forall t \in \mathbf{R}$. Considering centralizers we get $A\otimes N^{\mathrm{op}} = \alpha^{-1}(N\otimes A)$. In the central decomposition of $A\otimes N^{\mathrm{op}}$ and $N\otimes A$, respectively, the fibers are almost all equal to N^{op} and N, respectively. By decomposition theory [6; Ch. II, §6] we get that N and N^{op} are *-isomorphic, a contradiction. So $\psi \neq \varphi\circ\alpha$ for every automorphism α of M.

We now turn to the proof of Theorem 3.1. The reason for the special assumption when M is finite is due to the fact that in only this case may a $P^\#$-cone coincide with a $P^\flat$-cone. This is clarified in the following lemma.

LEMMA 3.3. *As above let $(M, H, J, P^\natural)$ be a standard form of the factor M and let ξ_0 and η_0 be two cyclic and separating vectors in $P^\natural$.*

If $\Delta_{\xi_0}=\Delta_{\eta_0}^{-1}$, *then* (i) *M is finite. So we may assume that M acts by left multiplication on* $H=L^2(M,\tau)$, *where* τ *is the unique normalized trace on M, and* $P^\natural=L^2(M,\tau)_+$. ($L^2(M,\tau)$ is the set of all closed operators affiliated with M that are $L^2(\tau)$-summable, cf. [12, 17]).

Then (ii) ξ_0^{-1} *is again in* $L^2(M,\tau)$ *and* $\eta_0=\lambda\xi_0^{-1}$ *for some* $\lambda>0$. *In particular, if* $P^\sharp_{\xi_0}=P^\flat_{\eta_0}$ *then* (i) *and* (ii) *hold.*

Conversely, if (i) *and* (ii) *hold then* $P^\sharp_{\xi_0}=P^\flat_{\eta_0}$.

PROOF. Since $S_{\xi_0}=J\Delta_{\xi_0}^{\frac{1}{2}}$ and $F_{\eta_0}=J\Delta_{\eta_0}^{-\frac{1}{2}}$ we get from the assumption $\Delta_{\xi_0}=\Delta_{\eta_0}^{-1}$ that $S_{\xi_0}=F_{\eta_0}$, and so $M_h\xi_0^-=M'_h\eta_0^-$ (cf. section 1). Define $\theta(x)=\langle x\xi_0,\eta_0\rangle$, $x\in M$. Let x and y be hermitian elements of M. Then $\theta(xy)=\langle xy\xi_0,\eta_0\rangle=\langle y\xi_0,x\eta_0\rangle\in \mathbf{R}$, since by Lemma 1.1 we get $iM_h\eta_0^{\perp\perp}=M'_h\eta_0^-$ $(=M_h\xi_0^-)$. In particular, if $y=I$ we get that $\theta(x)$ is real for x hermitian. Hence

$$\theta(xy) = \overline{\theta((xy)^*)} = \overline{\theta(yx)} = \theta(yx); \quad x,y\in M_h .$$

By linearity we get that $\theta(xy)=\theta(yx)$ for every x and y in M. We want to show that $\theta\geqq 0$. For that purpose it is enough to show that $\eta_0\in P^\flat_{\xi_0}=(P^\sharp_{\xi_0})^\circ$. By the hypothesis it follows that Δ_{ξ_0} fixes η_0 and so $\eta_0=\Delta_{\xi_0}^{\frac{1}{2}}\eta_0\in P^\flat_{\xi_0}$, since we have that $P^\flat_{\xi_0}=\Delta_{\xi_0}^{\frac{1}{2}}(P^\natural\cap\mathscr{D}(\Delta_{\xi_0}^{\frac{1}{2}}))$. (The latter fact is easily verified, cf. [2; Theorem 2.8]). Now $\theta\neq 0$, since ξ_0 is cyclic for M, and θ has a central support by its tracial property. Since M is a factor we conclude that θ is a positive multiple of a unique faithful normalized trace τ on M, say $\theta=c\tau$, $c>0$. In particular, M is finite. With $H=L^2(M,\tau)$, $P^\natural=L^2(M,\tau)_+$, and inner product on H defined by $\langle\xi,\eta\rangle=\tau(\eta^*\xi)$, $\xi,\eta\in L^2(M,\tau)$, we get

$$c\tau(x) = \theta(x) = \langle x\xi_0,\eta_0\rangle = \tau(\eta_0x\xi_0) = \tau(\xi_0\eta_0x)$$

for $x\in M$. (Recall that $\xi_0,\eta_0\in L_2(M,\tau)_+$). Hence we must have $\xi_0\eta_0=cI$, and so $\xi_0^{-1}=c\eta_0\in L_2(M,\tau)$ and $\eta_0=\lambda\xi_0^{-1}$, where $\lambda=1/c$. If $P^\sharp_{\xi_0}=P^\flat_{\eta_0}$ then clearly $M_h\xi_0^-=M'_h\eta_0^-$, and so $S_{\xi_0}=F_{\eta_0}$. This implies $\Delta_{\xi_0}=\Delta_{\eta_0}^{-1}$ and so, as we have just seen, (i) and (ii) hold.

The converse follows by noting that M' acts on $H=L^2(M,\tau)$ by right multiplication. In fact, approximating by cutting down with appropriate spectral projections e of ξ_0 to make the operators bounded, we get

$$\begin{aligned} P^\flat_{\eta_0} &= \{\eta_0x \mid x\in M_+\}^- = \{\xi_0^{-1}x \mid x\in M_+\}^- \\ &= \{\xi_0^{-1}(\xi_0exe\xi_0) \mid x\in M_+ \text{ and } e \text{ “appropriate” spectral projection of } \xi_0\}^- \\ &= \{exe\xi_0 \mid x\in M_+,\ e \text{ as above}\}^- = P^\sharp_{\xi_0} . \end{aligned}$$

This completes the proof.

REMARK. The above proof is direct and in the spirit of this paper. However, if we rephrase the above lemma in terms of the modular groups rather than the modular operators another proof comes more natural. In fact, a more general result including weights as well as elements in $(M_*)_+$ can be proved using the alternative approach. We think both the rephrasing as well as the proof is interesting so we give a brief sketch below. (Note that we do not have to assume that M is a factor).

LEMMA 3.3′. *Assume $\sigma_t^\varphi = \sigma_{-t}^\psi$, $\forall t \in \mathrm{R}$, where φ and ψ are faithful, normal, semifinite weights on M. Then M is semifinite.*
If φ and ψ are in $(M_*)_+$, then M is finite.

SKETCH OF PROOF. The condition $\sigma_t^\varphi = \sigma_{-t}^\psi$ implies that $\psi = \psi \circ \sigma_t^\varphi$, $\forall t \in \mathrm{R}$. Then $\psi = \varphi(h\,\cdot)$ for a (non-singular) positive self-adjoint operator affiliated with $C^\varphi = \{x \in M \mid \sigma_t^\varphi(x) = x,\ \forall t \in \mathrm{R}\}$ and $\sigma_t^\psi(x) = h^{it}\sigma_t^\varphi(x)h^{-it}$, $x \in M$, $t \in \mathrm{R}$. (Cf. [13; Theorems 4.6 & 5.12]). Now

$$\sigma_{-2t}^\varphi(x) = \sigma_{-t}^\varphi(\sigma_{-t}^\varphi(x)) = \sigma_{-t}^\varphi(\sigma_t^\psi(x)) = \sigma_{-t}^\varphi(h^{it}\sigma_t^\varphi(x)h^{-it}) = h^{it}xh^{-it},$$

and so $\sigma_t^\varphi(x) = h^{-it/2}xh^{it/2}$, $\forall t \in \mathrm{R}$. Hence M is semifinite [18; Theorem 14.1]. Set $\tau = \varphi(h^{\frac{1}{2}}\cdot)$. Then

$$\sigma_t^\tau(x) = h^{it/2}(h^{-it/2}xh^{it/2})h^{-it/2} = x$$

for all x in M. So τ is a faithful trace on M. If φ and ψ are in $(M_*)_+$, then $\tau = \varphi(h^{\frac{1}{2}}\cdot)$ is a bounded trace on M since $h^{\frac{1}{2}} \leqq \frac{1}{2}(I+h)$, and so $\tau \leqq \frac{1}{2}(\varphi + \varphi(\mathrm{h}\,\cdot)) = \frac{1}{2}(\varphi+\psi)$, cf. [13, Section 4].

We shall also need the following lemma.

LEMMA 3.4. *With the same setting as in the previous lemma, assume that $\Delta_{\eta_0} = \Delta_{\xi_0}$. (In particular, this is the case if $P_{\xi_0}^\sharp = P_{\eta_0}^\sharp$). Then $\eta_0 = \lambda\xi_0$ for some $\lambda > 0$.*

PROOF. We get that $\sigma_t = \alpha_t = \Delta^{it}\cdot\Delta^{-it}$, $t \in \mathrm{R}$, where $\Delta = \Delta_{\xi_0} = \Delta_{\eta_0}$ and $\{\sigma_t\}$ and $\{\alpha_t\}$ are the modular groups of ω_{ξ_0} and ω_{η_0}, respectively. By [18; Theorem 15.4] we immediately get that $\eta_0 = \lambda\xi_0$ for some $\lambda > 0$. Another proof, more direct and elementary, was shown us by van Daele (private communication) and we think it is worthwhile to sketch it. First we observe that the hypothesis implies that $M_h\xi_0^- = M_h\eta_0^-$. In fact, this is an immediate consequence of $S_{\xi_0} = J\Delta_{\xi_0}^{\frac{1}{2}}$ and $S_{\eta_0} = J\Delta_{\eta_0}^{\frac{1}{2}}$. Now let $x \in M_h$, $x' \in M_h'$. Then $\langle x\xi_0, x'\eta_0\rangle$ is real because of Lemma 1.1. Hence $\langle xx'\xi_0, \eta_0\rangle$ is real for every $x \in M_h$, $x' \in M_h'$. The von Neumann algebra $\{M \cup M'\}''$ is generated by the *-algebra

$$A = \left\{ \sum_{i=1}^{n} x_i x_i' \mid x_i \in M,\ x_i \in M' \right\} ,$$

and any hermitian element in A is a sum of elements of the form

$$\tfrac{1}{2}(x+x^*)(x'+x'^*)-\tfrac{1}{2}i(x-x^*)i(x'-x'^*) \quad (=xx'+x^*x'^*) ,$$

$$\text{where } x \in M,\ x' \in M' .$$

Hence $\langle y\xi_0, \eta_0 \rangle$ is real for every $y \in \{M \cup M'\}_h''$. As ξ_0 is cyclic for $\{M \cup M'\}''$ we get by Lemma 1.1 that $\eta_0 \in (M \cap M')_h \xi_0$. (Observe that the commutant of $\{M \cup M'\}''$ is the center $M \cap M'$ of M). Since M is a factor we get that $\eta_0 = \lambda\xi_0$ for some $\lambda \in \mathbf{R}$. Now ξ_0 and η_0 are in $P^\natural$, and so we must have $\lambda > 0$ since $P^\natural$ is a proper cone.

Proof of Theorem 3.1. (i) Assume first that $\psi = \varphi\circ\alpha$ for some $\alpha \in \mathrm{Aut}\,(M)$. By property (†) of section 1 there exists a unitary operator u_1 on H such that $u_1(P^\natural) = P^\natural$ and $\alpha(x) = u_1 x u_1^*$, $x \in M$. Then we have for $x \in M$

$$\langle x\eta_0, \eta_0 \rangle = \psi(x) = \varphi\circ\alpha(x) = \langle u_1 x u_1^* \xi_0, \xi_0 \rangle = \langle x u_1^* \xi_0, u_1^* \xi_0 \rangle .$$

Since $u^*\xi_0 \in P^\natural$, we get by the unique representation of $(M_*)_+$ by vectors in $P^\natural$ (cf. section 1) that $\eta_0 = u_1^*\xi_0$, hence $\xi_0 = u_1\eta_0$. Set $u = u_1^*$. Then

$$u(P^\#) = u_1^*(P^\#) = u_1^*(M_+\xi_0)^- = (u_1^* M_+ u_1 u_1^* \xi_0)^- = M_+\eta_0^- = P_1^\# .$$

On the other hand, assume there exists a unitary operator u on H such that $u(P^\#) = P_1^\#$. Then $u(M_h\xi_0^-) = M_h\eta_0^-$ and so $uS_{\xi_0}u^* = S_{\eta_0}$. By uniqueness of the polar decomposition we get that $u\Delta_{\xi_0}u^* = \Delta_{\eta_0}$. Hence

$$u(P^\natural) = u(\Delta_{\xi_0}^{\frac{1}{4}} P^\#)^- = u\Delta_{\xi_0}^{\frac{1}{4}} u^* u(P^\#)^- = \Delta^{\frac{1}{4}}_{\eta_0}(P_1^\#)^- = P^\natural .$$

By property (†) of section 1 u implements an automorphism α_1 or an antiautomorphism β. Assume the second alternative. Then $uMu^* = M'$ and so

$$u(P^\#) = (uM_+u^*u\xi_0)^- = (M'_+u\xi_0)^- = P_{u\xi_0}^\flat .$$

Since $P_1^\# = P_{\eta_0}^\# = u(P^\#)$ we get that $P_{\eta_0}^\# = P_{u\xi_0}^\flat$. According to Lemma 3.3 this can only occur for M finite. In that case we have the additional assumption that $\eta_0 = u\xi_0$. Then $P_{\eta_0}^\# = P_{\eta_0}^\flat = P^\natural$, and this is easily seen to be equivalent to $\psi = \omega_{\eta_0}$ being a trace on M. Now $u(P^\#) = P_1^\#$ implies that $u(P^\flat) = P_1^\flat$, since $P^\flat = (P^\#)^\circ$, $P_1^\flat = (P_1^\#)^\circ$. So $P_{\xi_0}^\# = P_{\xi_0}^\flat = P^\natural$ and hence $\varphi = \omega_{\xi_0}$ is also a trace on M. As the trace is unique we get that $\psi = \varphi = \psi\circ\mathrm{id}$.

The second alternative we have to consider is that u implements an automorphism $\alpha_1 = u\cdot u^*$. Then $uMu^* = M$ and we get

$$u(P^\#) = (uM_+u^*u\xi_0)^- = (M_+u\xi_0)^- = P_{u\xi_0}^\# .$$

So we have $P_1^\# = P_{\eta_0}^\# = P_{u\xi_0}^\#$. By Lemma 3.4 this implies that $\eta_0 = u\xi_0$. Hence we get for $x \in M$

$$\psi(x) = \langle x\eta_0, \eta_0\rangle = \langle xu\xi_0, u\xi_0\rangle = \langle u^*xu\xi_0, \xi_0\rangle = \langle \alpha(x)\xi_0, \xi_0\rangle = \varphi\circ\alpha(x),$$

where $\alpha = \alpha_1^{-1}$.

(ii) Assume first that $\psi = \varphi\circ\beta$ for some antiautomorphism β and let v_1 be the corresponding unitary operator on H according to property (†) of section 1. Then $\beta(x) = Jv_1x^*v_1^*J$, $x \in M$, and so

$$\langle x\eta_0, \eta_0\rangle = \psi(x) = \varphi\circ\beta(x) = \langle Jv_1x^*v_1^*J\xi_0, \xi_0\rangle = \langle xv_1^*\xi_0, v_1^*\xi_0\rangle .$$

As above this implies that $\eta_0 = v_1^*\xi_0$. Set $v = v_1^*$. Then

$$v(P^\#) = v_1^*(P^\#) = (v_1^*M_+v_1v_1^*\xi_0)^- = M'_+\eta_0^- = P_1^\flat .$$

On the other hand, assume there exists a unitary operator v on H such that $v(P^\#) = P_1^\flat$. Then $v(M_h\xi_0^-) = M'_h\eta_0^-$ and so $vS_{\xi_0}v^* = F_{\eta_0}$. By uniqueness of the polar decomposition we get that $v\Delta_{\xi_0}v^* = \Delta_{\eta_0}^{-1}$. Hence

$$v(P^\natural) = v(\Delta_{\xi_0}^{\frac{1}{4}}P^\#)^- = v\Delta_{\xi_0}^{\frac{1}{4}}v^*v(P^\#)^- = \Delta_{\eta_0}^{-\frac{1}{4}}(P_1^\flat) = P^\natural .$$

So v implements an automorphism α or an antiautomorphism β_1 of M according to property (†) of section 1. Let us assume the first alternative. Then $vMv^* = M$ and so

$$v(P^\#) = (vM_+v^*v\xi_0)^- = (M_+v\xi_0)^- = P_{v\xi_0}^\# .$$

Hence $P_1^\flat = P_{\eta_0}^\flat = v(P^\#) = P_{v\xi_0}^\#$. According to Lemma 3.3 this can only occur for M finite. In that case we have assumed $v\xi_0 = \eta_0$ and this implies as above that $\eta_0 = \xi_0$ is a trace vector for M, which we have ruled out.

Thus the only alternative is that v implements an antiautomorphism β_1. Hence we have $vMv^* = M'$ and so

$$v(P^\#) = (vM_+v^*v\xi_0)^- = (M'_+v\xi_0)^- = P_{v\xi_0}^\flat .$$

Thus $P_1^\flat = P_{\eta_0}^\flat = v(P^\#) = P_{v\xi_0}^\flat$. By Lemma 3.4 this implies that $\eta_0 = v\xi_0$. Now $\beta_1(x) = Jvx^*v^*J$, $x \in M$, and so

$$\psi(x) = \langle x\eta_0, \eta_0\rangle = \langle xv\xi_0, v\xi_0\rangle = \langle Jv^*x^*vJ\xi_0, \xi_0\rangle = \varphi\circ\beta(x) ,$$

where $\beta(x) = Jv^*x^*vJ = \beta_1^{-1}(x)$.

This completes the proof of the theorem.

PROOF OF COROLLARY 3.2. The proof follows by combining the proof of the theorem with the uniqueness part of property (†) of section 1.

REMARK 1. Note that from the proof of the above theorem it follows that if M is a finite factor and u is a unitary operator mapping $P_{\xi_0}^\#$ onto $P_{\eta_0}^\#$ (or $P_{\eta_0}^\flat$), where ξ_0 and η_0 are cyclic and separating unit vectors in $L^2(M,\tau)_+$ (cf. Lemma 3.3), then either $u\xi_0=\eta_0$ or $u\xi_0=\lambda\eta_0^{-1}$ for some $\lambda>0$.

REMARK 2. Theorem 3.1, appropriately modified, can be generalized to (non-finite) weights by working with the cones $P^{\#}$ and $P^{\flat}$ introduced in [14] and applying the results of [7, 8]. Lemma 3.3′ will replace Lemma 3.3 in the proof.

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