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# SPECTRA OF ACTIONS ON TYPE I C*-ALGEBRAS

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**Abstract.**

Let $(A,G,\alpha)$ be a C*-dynamical system. For $A$ commutative, a characterization of the Arveson and Connes spectra in terms of the isotropy groups is obtained. For $A$ type I and $G$-simple, relationships between the spectra of $(A,G,\alpha)$ and certain subgroups of the isotropy groups are obtained in special cases. However, an example is given, with $A$ homogeneous of degree 2, to show that the natural analogue of the commutative characterization fails in general.

## Introduction.

Let $A=C_0(\hat{A})$, the continuous functions vanishing at infinity on the locally compact Hausdorff space $\hat{A}$. Let $\alpha$ be an action of a locally compact abelian group $G$ on $A$, or equivalently, let $(G,\hat{A})$ be a locally compact topological transformation group. For $\pi \in \hat{A}$, let $G_\pi$ denote the isotropy group at $\pi$ for the action of $G$ on $\hat{A}$. In section 1 we characterize the Arveson spectrum $\mathrm{Sp}(\alpha)$ [1] and the Connes spectrum $\Gamma(\alpha)$ [19] in terms of isotropy subgroups. Specifically, we show $\mathrm{Sp}(\alpha)=\overline{\bigcup_{\pi\in\hat{A}} G_\pi^\perp}$, (1.2), $G_\pi^\perp$ denoting the annihilator of $G_\pi$ in $\hat{G}$, and from this we obtain a characterization of $\Gamma(\alpha)$ as the subgroup of $\hat{G}$ consisting of elements which belong to $G_\pi^\perp$ for all $\pi$ in a certain dense subset of $\hat{A}$ (1.6). This, in turn, leads to an independent proof (1.8), for commutative algebras, of the characterization in [21] of $\Gamma(\alpha)$ in terms of the dual action $\hat{\alpha}$ of $\hat{G}$ on the crossed product algebra $G \times_\alpha A$.

$A$ has no non-trivial closed $G$-invariant ideals if and only if $\hat{A}$ has no non-trivial closed $G$-invariant subsets. In this case we say $A$ is $G$-simple, or $(G,\hat{A})$ is minimal. When $A$ is $G$-simple and commutative, there is one common isotropy group, denoted $G_\pi$. It follows then from the results of section 1 that

$$G_\pi^\perp = \Gamma(\alpha) = \mathrm{Sp}(\alpha)\,.$$

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Furthermore, the crossed product algebra $G \times_\alpha A$ is simple if and only if the above subset of $\hat{G}$ equals $\hat{G}$. (To see this, observe that if $A$ is $G$-simple and $G_\pi = \{0\}$, then all induced representations of $G \times_\alpha A$ are weakly equivalent, while the generalized Effros–Hahn conjecture [14] implies all irreducible representations of $G \times_\alpha A$ are weakly equivalent to induced representations. The converse follows by [21, Proposition 6.3].) Finally, for $\pi \in \hat{A}$, there are natural actions of $G_\pi$ on $A$ and of $G_\pi$ on $\pi(A)$, denoted, respectively, by $\alpha|_{G_\pi}$ and by $\alpha^\pi$. Let $k_\pi$ be the natural map of $\hat{G}$ onto $\hat{G}_\pi = \hat{G}/G_\pi^\perp$ given by restriction to $G_\pi$. It follows trivially from the remarks above that each of the sets $k_\pi(\Gamma(\alpha))$, $\Gamma(\alpha \mid G_\pi)$, $\Gamma(\alpha^\pi)$, $k_\pi(\mathrm{Sp}(\alpha))$, $\mathrm{Sp}(\alpha \mid G_\pi)$, $\mathrm{Sp}(\alpha^\pi)$ just consists of the identity in $\hat{G}_\pi$, and hence one has

$$\overline{k_\pi(\Gamma(\alpha))} = \Gamma(\alpha|_{G_\pi}) = \Gamma(\alpha^\pi)$$

and that

$$\overline{k_\pi(\mathrm{Sp}(\alpha))} = \mathrm{Sp}(\alpha|_{G_\pi}) = \mathrm{Sp}(\alpha^\pi) .$$

In sections 2 and 3, we investigate the extent to which suitable analogues (see below) of the above statements hold for $G$-simple actions on type I, non-commutative C*-algebras $A$. Although some of our results can be extended to non-$G$-simple actions as well, we generally stick to the $G$-simple case for greater coherence. Also, particular references to other work on these questions is given in the body of sections 2–3.

For general type I C*-algebras $A$ it is immediately obvious that the isotropy groups for the action of $G$ on $\hat{A}$ yield insufficient information about either the spectra of the action or the ideal structure of the crossed product algebra. For example, let $A = M_2$, the algebra of all $2 \times 2$ matrices over $C$. Then $\hat{A}$ consists of just one point, the identity representation $\pi$, so every action is $G$-simple, with $G_\pi^\perp = \{0\}$. The spectra and the ideal structure of $G \times_\alpha A$ can be quite different, however. If $\mathbf{Z}_2$ acts by $\alpha = \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $\mathrm{Sp}(\alpha) = \mathbf{Z}_2$, $\Gamma(\alpha) = \{0\}$ and $G \times_\alpha A$ is not simple. If $\mathbf{Z}_2 \times \mathbf{Z}_2$ acts by $\alpha_1 = \alpha$ as above and $\alpha_2 = \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\Gamma(\alpha) = \mathbf{Z}_2 \times \mathbf{Z}_2$, and the crossed product algebra is simple. Rather than $G_\pi$, it is known that a certain subgroup $S_\pi$ (defined below) is more relevant for the spectra and the ideal structure. For $\pi \in \hat{A}$, there is a multiplier representation $U$, with multiplier $w_\pi$ which implements the action of $G_\pi$:

$$U(s)\pi(a)U(s^{-1}) = \pi(\alpha_s(a)), \qquad \forall\, a \in A,\ s \in G_\pi .$$

Then $S_\pi \equiv \{s \in G_\pi : w_\pi(s,t) = w_\pi(t,s)\ \forall\, t \in G_\pi\}$, and $w_\pi$ is called totally skew if $S_\pi$ is trivial. The relevance of $S_\pi$ for ideal structure arises from the fact that if $w_\pi$ is totally skew, all irreducible $w_\pi$-representations are weakly equivalent ([3],

[15]). Furthermore, if $A = C(H)$, the compact operators on $H$, it is known that $\Gamma(\alpha) = S_\pi^\perp$ [15]. Thus, in questions of spectra and ideal structure, $G_\pi$ is replaced in sections 2–3 by $S_\pi$.

In section 2, we obtain some positive results for $G$-simple actions on arbitrary type I C*-algebras, as well as positive results for certain special classes of actions. In section 3, we obtain more positive results for $G$-simple actions on continuous trace algebras, and on continuous trace algebras of finite degree. We also present an example of a $G$-simple action on a homogeneous algebra of degree 2 which provides negative answers to a number of the questions arising from the results of section 1. Some of the questions arising from section 1 are still unsettled.

For general notation on C*-dynamical systems, the reader is referred to [23, Chapters 7–8] or [18]. We mention only that for an action $\alpha$ of $G$ on $A$, we write $\alpha_t(a)$ for the action of a group element $t$ in $G$ on $a$ in $A$, $\alpha_f(a) = \int_G f(t)\alpha_t(a)\,dt$ for the corresponding action of $f$ in $L^1(G)$ on $A$, and $\hat{\alpha}$ for the dual action of $\hat{G}$ on the crossed product algebra $G \times_\alpha A$. Of course $G$ is always assumed to be abelian. We consider only separable C*-dynamical systems $(A, G, \alpha)$, i.e., $A$ is a separable C*-algebra, $G$ is a second countable locally compact group, and all representations are on separable Hilbert spaces. $K(G)$ and $K(G, A)$ denote the continuous, compactly supported functions on $G$ with values in $\mathbf{C}$, respectively, $A$. We sometimes identify a representation $L$ of $G \times_\alpha A$ with the corresponding covariant pair of representations $(\pi, V)$ of $(A, G, \alpha)$, and unless otherwise mentioned, identify representations with their unitary equivalence classes. If $H$ is a closed subgroup of $G$, we use the notation $\mathrm{IND}_{H\uparrow G} L$ to denote either the unitary representation of $G$ induced from a unitary representation $L$ of $H$, or the representation of the crossed product algebra $G \times_\alpha A$ induced from the representation $L$ of $H \times_\alpha A$.

## 1. The commutative case.

Let $(C_0(X), G, \alpha)$ be a separable C*-dynamical system, or equivalently, let $(G, X)$ be a second countable locally compact topological transformation group. For $x$ in $X$, let

$$G_x = \{t \in G \mid tx = x\}$$

denote the isotropy group at $x$, and for any closed subgroup $H$ of $G$ let $H^\perp$ denote the annihilator of $H$ in $\hat{G}$, i.e. $H^\perp = \{\gamma \in \hat{G} \mid \gamma \equiv 1 \text{ on } H\}$.

Let

$$(\Phi_H f)(\dot{t}) = \int_H f(t+s)\,d_H s$$

be the canonical map of $L^1(G)$ onto $L^1(G/H)$. Then one has the following

1.1. LEMMA. *Let* $(C_0(X), G, \alpha)$ *be a separable* C*-*dynamical system. For* $f \in L^1(G)$,

$$\|\alpha_f\| = \sup_{x \in X} \|\Phi_{G_x} f\| .$$

PROOF. For $x$ in $X$, and $h$ in $C_0(X)$, we have

$$\begin{aligned} \alpha_f(h)(x) &= \int_G f(t)\alpha_t(h)(x)\,dt \\ &= \int_G f(t)h((-t)x)\,dt \\ &= \int_{G/G_x}\int_{G_x} f(t+s)h((-s-t)x)\,d_{G_x}s\,d\bar{t} \\ &= \int_{G/G_x} h((-t)x)\left(\int_{G_x} f(t+s)\,d_{G_x}s\right)d\bar{t} \\ &= \int_{G/G_x} h((-t)x)\Phi_{G_x}f(\bar{t})\,d\bar{t} . \end{aligned}$$

Hence $\|\alpha_f\| \leqq \sup_{x \in X} \|\Phi_{G_x} f\|$.

To prove the converse, take $f$ in $K(G)$, with support contained in the compact set $E$. For $x$ in $X$, $\Phi_{G_x} f$ is supported by the compact set $\bar{E}$ in $G/G_x$, and the map $\bar{E} \to X: cG_x \to (-c)x$ is continuous and one-to-one, hence a homeomorphism with its image. Now for any $g$ in $C_0(G/G_x)$, and for $c$ in $E$, $\tilde{g}((-c)x) \equiv g(cG_x)$ is a continuous function on $(-E)x \subseteqq X$, thus extends to $g'$ in $C_0(X)$ with $\|g'\|_\infty = \|\tilde{g}\|_\infty \leqq \|g\|_\infty$. Hence

$$\begin{aligned} \left|\int_{G/G_x} \Phi_{G_x} f(\bar{t})g(\bar{t})\,d\bar{t}\right| &= \left|\int_{\bar{E}} \Phi_{G_x} f(\bar{t})g(\bar{t})\,d\bar{t}\right| \\ &= \left|\int_{G/G_x} \Phi_{G_x} f(\bar{t})g'((-\bar{t})x)\,d\bar{t}\right| \\ &= \left|\int_G f(t)g'((-t)x)\,dt\right| \\ &= \left|\int_G f(t)\alpha_t(g')(x)\,dt\right| \\ &= |\alpha_f(g')(x)| \leqq \|\alpha_f\|\,\|g\|_\infty . \end{aligned}$$

Clearly then, for each $x$ in $X$,

$$\|\Phi_{G_x} f\| \leqq \|\alpha_f\| ,$$

and by the density of $K(G)$ in $L^1(G)$ this holds for all $f$ in $L^1(G)$.

The following theorem was conjectured and partly proved by M. B. Landstad during the second named author's stay in Trondheim in August 1977. Here we give a complete proof.

1.2 THEOREM. *Let $(C_0(X), G, \alpha)$ be a separable C*-dynamical system. Then*

$$\mathrm{Sp}(\alpha) = \overline{\bigcup_{x \in X} G_x^{\perp}} .$$

PROOF. By definition,

$$\mathrm{Sp}(\alpha) = \mathrm{hull}\{f \in L^1(G) \mid \alpha_f \equiv 0\} .$$

Now by [24, 2.7.4 Theorem] one has $\Phi_{G_x} f = 0$ if and only if the Fourier transform $\hat{f}$ of $f$ vanishes on $G_x^{\perp}$. Hence by 1.1 $\alpha_f \equiv 0$ if and only if $\hat{f}$ vanishes on $\bigcup G_x^{\perp}$, and by regularity of $L^1(G)$ we have that if $\gamma \notin \overline{\bigcup G_x^{\perp}}$, there is an $f$ in $L^1(G)$ such that $\hat{f}(\overline{\bigcup G_x^{\perp}}) = 0$, that is, $\Phi_{G_x} f = 0$ for every $x$ but $\hat{f}(\gamma) \neq 0$.

1.3 REMARK. In [19, 5.3 & 5.4] it is shown that for a single automorphism, the Arveson spectrum for $(C_0(X), \mathbf{Z}, \alpha)$ is always a finite union of subgroups of the circle. This is also easily derivable from 1.2, since if $G_x = \{0\}$ for some $x$, then $\mathrm{Sp}(\alpha) = T$ by 1.2, and if $G_x$ is never $\{0\}$, then $G_x = \{n_x \mathbf{Z}\}$, $n_x \neq 0$. Now if $\sup n_x = \infty$, $\mathrm{Sp}(\alpha) = T$, and if $\sup n_x < \infty$, it is a finite union of proper subgroups. Note that $\mathrm{Sp}(\alpha)$ for $(C_0(X), \mathbf{Z}, \alpha)$ coincides with the usual operator spectrum $\sigma(\alpha)$ for $\alpha$, cf. [5, 2.3.8].

Recall that, by definition, the Connes spectrum $\Gamma(\alpha)$ for a commutative algebra is the intersection of $\mathrm{Sp}(\alpha \mid I)$ where $I$ runs through all non-zero $G$-invariant ideals, i.e. by the above theorem

$$(*) \qquad \Gamma(\alpha) = \bigcap_{Y \in \mathcal{O}^G(X)} \left( \overline{\bigcup_{y \in Y} G_y^{\perp}} \right),$$

$\mathcal{O}^G(X)$ denoting the non-empty open $G$-invariant subsets of $X$. The Borchers spectrum is defined to be the larger set

$$(**) \qquad \Gamma_B(\alpha) = \bigcap_{Y \in \mathcal{D}^G(X)} \left( \overline{\bigcup_{y \in Y} G_y^{\perp}} \right),$$

$\mathcal{D}^G(X)$ denoting the dense open $G$-invariant subsets of $X$.

1.4 LEMMA. *Let* $(C_0(X), G, \alpha)$ *be as above. Let* $\gamma \in \hat{G}$. *Then*

(i) $\gamma \in \Gamma(\alpha)$ *if and only if for every neighbourhood* $V$ *of* $\gamma$ *the set*

$$X_\gamma^V = \{x \in X \mid G_x^\perp \cap V \neq \emptyset\}$$

*is dense in* $X$.

(ii) $\gamma \in \Gamma_B(\alpha)$ *if and only if* $X_\gamma^V$, *for every neighbourhood* $V$ *of* $\gamma$, *is not nowhere dense.*

PROOF. (i) Let $\gamma \in \Gamma(\alpha)$ and let $V$ be a neighbourhood of $\gamma$. If $X_\gamma^V$ is not dense, the complement $(X_\gamma^V)^c$ contains a non-empty open set, hence by the $G$-invariance of $X_\gamma^V$ it contains a non-empty open $G$-invariant set, so by (*) there is a $y$ in $(X_\gamma^V)^c$ such that $V \cap G_y^\perp \neq \emptyset$, a contradiction.

Conversely, assume $X_\gamma^V$ to be dense for every $V$. For any non-empty open set $Y \subseteq X$, there is a point $y_V$ in $Y \cap X_\gamma^V$, hence $V \cap G_{y_V}^\perp \neq \emptyset$,

$$V \cap \left( \bigcup_{y \in Y} G_y^\perp \right) \neq \emptyset ,$$

and we conclude that $\gamma \in \overline{\bigcup_{y \in Y} G_y^\perp}$.

(ii) Let $\gamma \in \Gamma_B(\alpha)$. Reasoning as in (i) we see that if $X_\gamma^V$ is nowhere dense, i.e. $(\overline{X_\gamma^V})^c$ is open and dense, there is a $y$ in $(X_\gamma^V)^c$ such that $V \cap G_y^\perp \neq \emptyset$. Conversely, if $X_\gamma^V$ is never nowhere dense, then choosing $Y$ to be dense, open and $G$-invariant there is for every $V$ a $y_V$ in $Y \cap X_\gamma^V$ and hence $\gamma \in \overline{\bigcup_{y \in Y} G_y^\perp}$.

Let $C$ denote the set of points of continuity for the map $x \mapsto G_x$ from $X$ into $\Sigma$, the space of closed subgroups of $G$ endowed with the Fell topology. By (the proof of) [14, Lemma 1.1] the complement of $C$ is meagre in $X$, hence $C$ contains a dense $G_\delta$ subset of $X$. We now strengthen 1.4 (i):

1.5 LEMMA. *Let* $(C_0(X), G, \alpha)$, $\gamma$ *and* $X_\gamma^V$ *be as in* 1.4. *Then* $\gamma \in \Gamma(\alpha)$ *if and only if* $X_\gamma^V \supseteq C$ *for all neighbourhoods* $V$ *of* $\gamma$.

PROOF. If $X_\gamma^V \supseteq C$ for all $V$, then by 1.4 (i) $\gamma \in \Gamma(\alpha)$. Assume $\gamma \in \Gamma(\alpha)$ and let $V$ be a compact neighbourhood of $\gamma$. Take $x_0 \in C$. As $x_0 \in \overline{X_\gamma^V} = X$, we can find a sequence $x_i$ in $X_\gamma^V$ with $x_i \to x_0$. Now for each $x_i$ there is a character $\chi_i$ in $\hat{G} \cap V \cap G_{x_i}^\perp$, and by the compactness of $V$ we may pass to a subsequence and, by re-indexing, assume $\chi_i$ converges to some $\chi_0$ in $V$. As $x_0 \in C$, $x_i \to x_0$

implies $G_{x_i} \to G_{x_0}$ in $\Sigma$, which in turn implies that for each $t \in G_{x_0}$ there is (again, perhaps, after passing to a subsequence and relabelling) a sequence $t_i \in G_{x_i}$ with $t_i \to t$. Hence

$$\chi_0(t) = \lim_i \chi_i(t_i) = 1, \quad \chi_0 \in V \cap G_{x_0}^{\perp} \quad \text{and} \quad X_\gamma^V \supseteqq C .$$

1.6 THEOREM. *Let* $(C_0(X), G, \alpha)$ *be a separable* C*-*dynamical system, and let* $\gamma \in \hat{G}$. *Then the following are equivalent*

(a) $\gamma \in \Gamma(\alpha)$
(b) *For every neighbourhood* $V$ *of* $\gamma$,
$X_\gamma^V = \{x \in X \mid G_x^{\perp} \cap V \neq \emptyset\}$ *is dense in* $X$
(c) *For every neighbourhood* $V$ *of* $\gamma$, $X_\gamma^V \supseteqq C$
(d) $X_\gamma = \{x \in X \mid \gamma \in G_x^{\perp}\}$ *is dense in* $X$
(e) $X_\gamma \supseteqq C$
(f) $\gamma \in \bigcap_{Y \in \mathcal{O}^G(X)} (\bigcup_{y \in Y} G_y^{\perp})$, $\mathcal{O}^G(X)$ *being the non-empty open G-invariant subsets of* $X$.

PROOF. (a) $\Leftrightarrow$ (b) is Lemma 1.4 (i) and (a) $\Leftrightarrow$ (c) is Lemma 1.5. As $G_x^{\perp}$ is closed in $\hat{G}$, $X_\gamma = \bigcap_V X_\gamma^V$. Thus (c) $\Rightarrow$ (e), and (e) $\Rightarrow$ (d) by density of $C$. As density of $X_\gamma$ implies $X_\gamma \cap Y \neq \emptyset$ for every $Y$ in $\mathcal{O}^G(X)$, and thus that $\gamma \in G_y^{\perp}$ for some $y$ in $Y$, we have (d) $\Rightarrow$ (f). Finally, (f) $\Rightarrow$ (a) by the formula (*) for $\Gamma(\alpha)$ which precedes Lemma 1.4.

1.7 REMARK. It is immediate from the equivalence of (a) and (e) in Theorem 1.6 that $\Gamma(\alpha) = \bigcap_{x \in C} G_x^{\perp}$. Thus the characterizations of $\Gamma(\alpha)$ given in Theorem 1.6 yield directly—for commutative C*-algebras—the fact that $\Gamma(\alpha)$ is a subgroup of $\hat{G}$. (For a proof of this fact in general, see [19, Prop. 2.3].) We also give—again, for commutative C*-algebras—an alternate and simpler proof of the characterization [21, 5.4] of $\Gamma(\alpha)$ in terms of the dual action of $\hat{G}$ on the crossed product or transformation group C*-algebra.

1.8 THEOREM. *Let* $(C_0(X), G, \alpha)$ *be a separable* C*-*dynamical system, and let* $\gamma \in \hat{G}$. *Then* $\gamma \in \Gamma(\alpha)$ *if and only if* $\hat{\alpha}_\gamma(I) \cap I$ *is non-zero for every non-zero ideal* $I$ *in* $G \times_\alpha C_0(X)$.

PROOF. Assume $\gamma \notin \Gamma(\alpha) = \bigcap_{Y \in \mathcal{O}^G(X)} (\overline{\bigcup_{y \in Y} G_y^{\perp}})$. Then there is a compact neighbourhood $V$ of $\hat{e}$ in $\hat{G}$, and some $Y \in \mathcal{O}^G(X)$, such that $\gamma V \cap G_y^{\perp} = \emptyset$ for all $y \in Y$. We want to find some ideal $I \neq 0$ in $G \times_\alpha C_0(X)$ so that $\hat{\alpha}_\gamma(I) \cap I = 0$. As $G \times_\alpha C_0(Y)$ is clearly a non-zero ideal in $G \times_\alpha C_0(X)$, it suffices to find $I \subseteqq G \times_\alpha C_0(Y)$.

Choose a compact neighbourhood $K$ of $\hat{e}$ in $\hat{G}$ such that

$$\gamma K \cap K = \emptyset \quad \text{and} \quad \gamma K \cap G_y^\perp = \emptyset \ \forall y \in Y .$$

Let $U$ be a compact neighbourhood of $\hat{e}$ in $\hat{G}$ with $U + U \subseteq K$, $U = -U$. Let $U^0$ be the interior of $U$, $k_y$ the canonical map of $\hat{G}$ onto $\hat{G}_y$, $O_y = k_y(U^0)$, and $\gamma_y$ the restriction of $\gamma$ to $G_y$. Then for every $y$ in $Y$, $O_y \cap \gamma_y O_y = \emptyset$, $O_y \neq \emptyset$ open. Let $F_y$ denote the complement of $O_y$ in $\hat{G}_y$, so that for every $y$ in $Y$, $F_y \cup \gamma_y F_y = \hat{G}_y$, and $F_y$ is a proper closed subset of $\hat{G}_y$.

Let $\delta_y(f) = f(y)$ for $f \in C_0(Y)$, let $\chi_y \in F_y$, and let $\delta_y \times \chi_y$ denote the corresponding representation of $G_y \times C_0(Y)$, cf. [23, 7.6.4]. Let $\text{IND}(\delta_y \times \chi_y)$ be the induced representation $\text{IND}_{G_y \uparrow G}(\delta_y \times \chi_y)$. Let

$$I = \text{kernel} \sum_{y \in Y} \sum_{\chi_y \in F_y} \text{IND}(\delta_y \times \chi_y) .$$

Then $I \neq 0$ since for each $y$, the representation $\text{IND}\, \chi_y$ of $G$ is supported on

$$k_y^{-1}(F_y) = \{\chi \in \hat{G} \mid \chi|_{G_y} \in F_y\} ,$$

and the closure of the union of all these is proper in $\hat{G}$. In particular, $U^0 \cap \bigcup_{y \in Y} k_y^{-1}(F_y) = \emptyset$, so $\hat{e} \notin \overline{\bigcup_{y \in Y} k_y^{-1}(F_y)}$, and the identity representation of $G$ is not weakly contained in $\{\text{IND}\, \chi_y \mid y \in Y, \chi_y \in F_y\}$. Fix $t \in Y$. The identity representation of $G$ is weakly contained in $\text{IND}(\hat{e}_t)$, $\hat{e}_t$ denoting the identity of $\hat{G}_t$, and thus $\text{IND}(\hat{e}_t)$ is not weakly contained in $\{\text{IND}\, \chi_y \mid y \in Y, \chi_y \in F_y\}$, and the kernel of $\text{IND}(\delta_t \times (\hat{e}_t))$ does not contain $I$. On the other hand, $F_y \cup \gamma_y F_y = \hat{G}_y$, so

$$\hat{\alpha}_\gamma(I) \cap I = \text{kernel} \sum_{y \in Y} \sum_{\chi_y \in \hat{G}_y} \text{IND}(\delta_y \times \chi_y) .$$

As the family of irreducible induced representations is dense [13, Theorem 4.3], $\hat{\alpha}_\gamma(I) \cap I = 0$ as claimed.

Conversely, suppose $I$ is a non-zero ideal in $G \times_\alpha C_0(X)$ but $\hat{\alpha}_\gamma(I) \cap I = 0$. Then for every $x$ in $X$ and $\chi$ in $\hat{G}_x$,

$$\text{Ker}\, \text{IND}(\delta_x \times \chi) \supseteq I \quad \text{or} \quad \hat{\alpha}_{-\gamma}(I)$$

so either $\text{IND}(\delta_x \times \chi)$ or $\text{IND}(\delta_x \times k_x(\gamma)\chi)$ has a kernel containing $I$. Now if $\Delta$ is a dense subset of $X$ then

$$\{\text{Ker}\, \text{IND}(\delta_x \times \chi) \mid x \in \Delta, \chi \in \hat{G}_x\}$$

is dense in the primitive ideal space of $G \times_\alpha C_0(X)$, almost exactly as in the proof of Theorem 5.11 of [10]. If $\gamma \in \Gamma(\alpha)$, then

$$X_\gamma = \{x \in X \mid \gamma \in G_x^\perp\}$$

is dense in $X$ by 1.6 (d), and if $\gamma \in G_x^{\perp}$, $k_x(\gamma) \equiv 1$, so

$$\mathrm{IND}\,(\delta_x \times k_x(\gamma)\chi) = \mathrm{IND}\,(\delta_x \times \chi), \qquad \forall\, x \in \Delta,\ \chi \in \hat{G}_x\,.$$

It follows that $I=0$.

## 2. $G$-simple actions on type I algebras.

We begin this section by discussing both the structural restrictions on a type I algebra $A$ and the restrictions on the behavior of the isotropy subgroups $G_\pi$ which follow from the assumption that $A$ admits a $G$-simple action. We then consider the relationships between the subgroups $S_\pi$ of $G_\pi$, the spectra of the action, and the ideal structure of the crossed product.

Recall from [11, 3.2] and [22, 3.6] that such an $A$ is necessarily liminal and homogeneous and, if unital, is homogeneous of finite degree and thus continuous trace. Proposition 2.1 shows that $\hat{A}$ is either extremely well-behaved or extremely pathological. We have no examples of the latter, and Proposition 2.2 indicates that perhaps the former always holds.

A point $\pi$ in $\hat{A}$ is called separated if, for every $\tau$ in $\hat{A}$, $\tau \neq \pi$, there exist disjoint open sets containing $\pi$ and $\tau$, respectively. A point which is not separated is called unseparated. Let

$$\mathscr{M}(A)_+ = \{x \in A_+ \mid \pi \to \operatorname{tr}\pi(x) \text{ is finite and continuous on } \hat{A}\}\,.$$

2.1 Proposition. *Let $(A,G,\alpha)$ be a $G$-simple* C*-*dynamical system, with $A$ of type* I. *Then the following are equivalent*:

(a) *$A$ has continuous trace.*
(b) *$\hat{A}$ is Hausdorff.*
(c) $\mathscr{M}(A)_+ \neq \{0\}$.
(d) $\mathscr{U} \equiv \{\pi \in \hat{A} \mid \pi \text{ is unseparated}\}$ *is not dense in $\hat{A}$.*

*Furthermore,* (a)–(d) *automatically hold if $A$ is unital or if $G$ is compact.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) follows from [8, 4.5.3 and 4.7.12]. As $\mathscr{M}(A)_+$ is the positive part of a 2-sided ideal $\mathscr{M}(A)$ in $A$ [8, 4.5.2], (c) and $G$-simplicity implies $\overline{\mathscr{M}(A)}=A$, that is, $A$ has continuous trace. Clearly (b) holds iff the set $\mathscr{U}$ of (d) is empty. As $\mathscr{U}$ is $G$-invariant, it is either empty or dense by $G$-simplicity, and we have (b) iff (d). If $A$ is unital, (a) holds as mentioned above, while if $G$ is compact, then by [15, Lemma 22], $\hat{A}$ is homeomorphic to a quotient of $G$, hence Hausdorff.

Note that if (a)–(d) of Proposition 2.1 do not hold, then $\hat{A}$ has both a dense $G$-invariant set of separated points and a dense $G$-invariant set of unseparated points. Although an example of a type $I$ algebra $A$ with $\hat{A}$ containing dense sets of separated and of unseparated points is known [7], it is not clear how one could make such an example $G$-simple. The usual (trivial) proof that all isotropy subgroups are equal, for a $G$-simple action on $C_0(X)$, is based on

$$(*)\qquad \text{if } x_n \to x \text{ in } X \text{ and } sx_n = x_n \ \forall n, \text{ then } sx = x\,.$$

Of course, (*) also applies to $\hat{A}$ if $\hat{A}$ is Hausdorff, and precisely the same argument shows all isotropy groups are equal in this case. The next proposition shows all isotropy groups are equal in any case, and is followed by two examples indicating that (*) can fail if $A$ is either not type I or not $G$-simple.

2.2 PROPOSITION. *Let $(A, G, \alpha)$ be a $G$-simple $C^*$-dynamical system, with $A$ of type* I. *All the isotropy subgroups for the action of $G$ on $\hat{A}$ are the same.*

PROOF. The set $S$ of separated points in $\hat{A}$ is a dense $G_\delta$ [8, 3.9.4]. Let $\tau \in \hat{A}$, $\pi \in S$. By $G$-simplicity, there is a sequence $\{t_n\}$ in $G$ with $t_n\tau \to \pi$. If $s \in G_\tau$, $t_n\tau = t_n s\tau = st_n\tau \to s\pi$. As $\pi \in S$, we have $G_\tau \subseteq G_\pi$. Thus all points of $S$ have the same isotropy group, which contains all other isotropy groups. $\hat{A}$ also possesses a dense open Hausdorff set $O$ [8, 4.4.5]. Let $s \in G_\pi$. By $G$-simplicity, the non-empty open Hausdorff set $s^{-1}O \cap O$ contains a translate of $\tau$, say $g\tau$, $g \in G$, and we can find a sequence $\{g_n\}$ in $G$ with $g_n\pi \to g\tau$. Then $g_n\pi = sg_n\pi \to sg\tau$. But $g\tau$, $sg\tau$ and hence eventually all $g_n\pi$ lie in the Hausdorff set $O$, so $g\tau = sg\tau = gs\tau$, $\tau = s\tau$ and $G_\pi \subseteq G_\tau$.

2.3 EXAMPLE. If the condition that $A$ is type I is dropped, the conclusion of 2.2, with the action now being that of $G$ on PRIM $A$, the primitive ideal space of $A$, no longer holds. Indeed, let $A = T \times_\alpha \mathscr{F}$, the crossed product of the fermion algebra by the gauge action. Then $A$ is prime, and $\mathbf{Z}$-simple under the dual action $\hat{\alpha}$ of $\mathbf{Z}$. The isotropy group for the $O$-ideal is $\mathbf{Z}$, and is $\{0\}$ for all the non-zero primitive ideals, [20, section 5], see also [4].

2.4 EXAMPLE. If $A$ is type I, but not $G$-simple and $\hat{A}$ not Hausdorff, (*) need not hold. Let $A$ be the algebra of all sequences $x = (x_1, x_2, \ldots)$ of $2 \times 2$ matrices such that $x_n$ converges to a diagonal matrix $\begin{pmatrix} \lambda(x) & 0 \\ 0 & \mu(x) \end{pmatrix}$ as $n \to \infty$. Then $\hat{A}$ is non-Hausdorff (cf. [8, 4.7.19]), and letting $\mathbf{Z}_2$ act by permuting the diagonal elements of $x_n$, for each $n$, we see that the isotropy group for all the separated points in $\hat{A}$ is $\mathbf{Z}_2$ itself, whereas for the two unseparated (limit) points it is $\{0\}$.

2.5 REMARK. Let $G$ act on a type I algebra $A$. In light of 2.2–2.4, the question arises as to whether all points in a given quasi-orbit in $\hat{A}$ have the same isotropy group. We do not know in general, but if the quasi-orbit is locally closed it is homeomorphic to the dual space of a type I C*-algebra which is $G$-simple, (as in [15, pp. 223–224]), and the answer is yes by 2.2.

As discussed in the introduction, it is not $G_\pi$ but rather the subgroup $S_\pi$ of $G_\pi$ which seems to be more relevant for the spectra of the action, and the ideal structure of the crossed-product algebra. For a $G$-simple action on a type I algebra $A$, we do not know if all the $S_\pi$'s must be identical. However, if $A$ satisfies (a)–(d) of Proposition 2.1, they are (3.1). In any case, turning to the relationship of the $S_\pi$'s to the spectrum, we have

2.6 PROPOSITION. *Let $(A, G, \alpha)$ be a C*-dynamical system (not necessarily $G$-simple), with $A$ of type* I. *Then* $\bigcap_{\pi\in\hat{A}} S_\pi^\perp \subseteq \Gamma(\alpha)$.

PROOF. We shall show that for $\gamma \in \bigcap_{\pi\in\hat{A}} S_\pi^\perp$, $\hat{\alpha}_\gamma(I) = I$ for all ideals $I \subseteq G \times_\alpha A$. The result then follows from the characterization of $\Gamma(\alpha)$ in [21, 5.4]. It suffices to show that $\hat{\alpha}_\gamma(P) = P$ for all $P$ in PRIM $(G \times_\alpha A)$, and by [14] every such $P$ is the kernel of an induced representation $L$ of the following form [26]: there exists $\pi$ in $\hat{A}$, a $w_\pi$-representation $U$ of $G_\pi$ on $H_\pi$ which implements the action of $G_\pi$ on $A$, and an irreducible $\overline{w_\pi}$-representation $W$ of $G_\pi$ such that $L$ is induced from the irreducible covariant pair $(I\otimes\pi, W\otimes U)$ of $(A, G_\pi, \alpha|_{G_\pi})$. Furthermore [3, Theorem 3.1] $w_\pi$ may be chosen so that it is a "lift" of a totally skew multiplier $w'$ on $G_\pi/S_\pi$, and $W$ is then of the form $\chi\tilde{V}$, where $\chi \in \hat{G}_\pi$ and $\tilde{V}$ is the lift to $G_\pi$ of an irreducible $\overline{w'}$-representation $V$ of $G_\pi/S_\pi$. Writing $(G, w)^\wedge$ for the space of unitary equivalence classes of irreducible $w$-representations of a group $G$, we have [3, Theorem 3.1] that the map $\hat{G}_\pi \times (G_\pi/S_\pi, \overline{w'})^\wedge \to (G_\pi, \overline{w_\pi})^\wedge$ sending a pair $(\chi, V)$ into $\chi\tilde{V}$, is continuous. Furthermore, the map $(G_\pi, \overline{w_\pi})^\wedge \to (G_\pi \times_\alpha A)^\wedge$ sending $W$ into $(I\otimes\pi, W\otimes U)$ is easily seen to be continuous, by a routine $w^*$-approximation argument applied to $f \in K(G_\pi, A)$ and an elementary tensor in $H_w \otimes H_\pi$. The continuity of induction from $(G_\pi \times_\alpha A)^\wedge$ to $(G \times_\alpha A)^\wedge$ is routine to check, as is the fact that

$$\hat{\alpha}_\gamma(P) = \ker(\gamma L) = \ker\left(\operatorname*{IND}_{G_\pi\uparrow G}(I\otimes\pi, \chi\gamma|_{G_\pi}\tilde{V}\otimes U)\right).$$

As $\gamma|_{G_\pi} \in S_\pi^\perp$, we can of course consider $\gamma|_{G_\pi}$ as a character $\gamma'$ of $G_\pi/S_\pi$, and the representation $\gamma|_{G_\pi}\tilde{V}$ as the "lift" to $G_\pi$ of the $\overline{w'}$-representation $\gamma' V$ of $G_\pi/S_\pi$. The proposition then follows from the continuity of all the maps defined above, and the fact that all irreducible $\alpha$-representations of an abelian group $G$, where $\alpha$ is a totally skew multiplier, are weakly equivalent [15, Proposition 32].

2.7 PROPOSITION. *Let* $(A, G, \alpha)$ *be a* $G$*-simple* $C^*$*-dynamical system, with* $A$ *of type* I. *Let* $\alpha|_{G_\pi}$ *denote the action of* $G_\pi$ *on* $A$, $\alpha^\pi$ *the canonical action of* $G_\pi$ *on* $\pi(A)$ *and* $k_\pi$ *the canonical map from* $\hat{G}$ *onto* $\hat{G}_\pi$. *Then*

$$\overline{k_\pi(\mathrm{Sp}\,(\alpha))} = \mathrm{Sp}\,(\alpha|_{G_\pi}) = \mathrm{Sp}\,(\alpha^\pi)\,.$$

PROOF. The first equality is a result of Connes [5, 3.4.3] and does not depend on $G$-simplicity. Writing $\alpha_f$ for the natural action of $f \in L^1(G_\pi)$ on $A$, it is clear that $\alpha_f = 0$ implies $(\alpha^\pi)_f = 0$, so that $\mathrm{Sp}\,(\alpha|_{G_\pi}) \supseteq \mathrm{Sp}\,(\alpha^\pi)$ (indeed, it is clear that $\mathrm{Sp}\,(\alpha|_{G_\pi}) = \mathrm{Sp}\,(\alpha^\pi) \cup$ the Arveson spectrum of $\alpha|_{G_\pi}$ on $\ker \pi$). Now if $(\alpha^\pi)_f = 0$ for $f \in L^1(G_\pi)$, then $\pi \circ \alpha_f$: A $\to$ (0) and in fact $(t\pi) \circ \alpha_f = 0$ for all $t \in G$. It follows by $G$-simplicity that $\alpha_f = 0$, and so $\mathrm{Sp}\,(\alpha^\pi) \supseteq \mathrm{Sp}\,(\alpha|_{G_\pi})$.

We shall see in section 3 that Proposition 2.7 does not hold if $\mathrm{Sp}\,(\alpha)$ is replaced by $\Gamma(\alpha)$. Also, regarding Proposition 2.6, we shall see that even for $G$-simple actions with one common $S_\pi$, $S_\pi^\perp$ can be proper subset of both $\Gamma(\alpha)$ and

$$\ker\,(\hat{\alpha}|_{\mathrm{PRIM}\, G \times_\alpha A}) = \{\gamma \in \hat{G} \mid \hat{\alpha}_\gamma(I) = I \ \forall \text{ ideals } I \subseteq G \times_\alpha A\}\,.$$

Furthermore, the relationship between the size of $S_\pi$ and the spectra, on one hand, and the simplicity of the crossed product on the other, is murky-implications hold in one direction (see below), but not necessarily in the opposite direction (see 3.4). For certain classes of actions, though, some positive results along the above lines can be obtained. For example, if $A = C_0(X)$ is $G$-simple, there is one isotropy group $G_\pi$, and, as mentioned in the introduction and from the results of section 1, we have: $G_\pi^\perp = \Gamma(\alpha) = \mathrm{Sp}\,\alpha = \ker\,(\hat{\alpha}|_{\mathrm{PRIM}\, G \times_\alpha A})$, while $G \times_\alpha A$ is simple iff $G_\pi^\perp = \{e\}$. Furthermore, Proposition 2.7 trivially holds for both $\mathrm{Sp}\,(\alpha)$ and $\Gamma(\alpha)$, as $\mathrm{Sp}\,(\alpha^\pi) = \mathrm{Sp}\,(\alpha|_{G_\pi}) = \{e\}$.

2.8 COROLLARY. *Let* $(A, G, \alpha)$ *be a* $G$*-simple* $C^*$*-dynamical system with* $A$ *of type* I. *If* $S_\pi = \{e\}$ *for all* $\pi$ *in* $\hat{A}$, *then*

$$S_\pi^\perp = \ker\,(\hat{\alpha}|_{\mathrm{PRIM}\, G \times_\alpha A}) = \Gamma(\alpha) = \mathrm{Sp}\,(\alpha)\,.$$

*Furthermore,* $G \times_\alpha A$ *is simple, and* $k_\pi(\Gamma(\alpha)) = \Gamma(\alpha^\pi)$.

PROOF. The first statement is trivial since $S_\pi^\perp = \hat{G}$, while the second is [11, Theorem 4.1]. For the third statement, observe that $k_\pi(\Gamma(\alpha)) = k_\pi(\hat{G}) = \hat{G}_\pi$, while $\Gamma(\alpha^\pi)$ equals the annihilator of $S_\pi$ in $\hat{G}_\pi$ by [15, Theorem 18 and Proposition 34].

2.9 PROPOSITION. *Let* $(A, G, \alpha)$ *be a* $G$*-simple* $C^*$*-dynamical system, with* $A$ *of type* I. *If the action of* $G$ *on* $\hat{A}$ *is transitive, then all* $S_\pi$*'s coincide. Denoting this common subgroup by* $S$, *we have*

$$S^{\perp} = \ker(\hat{\alpha}|_{\mathrm{PRIM}\, G \times_{\alpha} A}) = \Gamma(\alpha) .$$

*Furthermore,* $G \times_{\alpha} A$ *is simple iff* $S = \{e\}$, *and also* $k_{\pi}(\Gamma(\alpha)) = \Gamma(\alpha^{\pi})$.

Proof. The $\bar{w}$-representation $U$ of $G_{\pi}$ which implements the action of $G_{\pi}$ on $A$, that is,

$$U(t)\pi(a)U(t)^{-1} = \pi(\alpha_t(a)), \quad \forall t \in G, \ a \in A ,$$

also implements the action of $G_{t\pi} = G_{\pi}$ on $A$, for any $t$ in $G$. Thus $S_{\pi} = S_{t\pi}$ for any $t$ in $G$. Denote this subgroup by $S$. Then

$$S^{\perp} \subseteqq \ker(\hat{\alpha}|_{\mathrm{PRIM}\, G \times_{\alpha} A}) \subseteqq \Gamma(\alpha)$$

by Proposition 2.6 and its proof. Once we show $\Gamma(\alpha) \subseteqq S^{\perp}$, the rest of the proposition follows, since $S = \{e\}$ implies that $G \times_{\alpha} A$ is simple by [11, Theorem 4.1], while $G \times_{\alpha} A$ simple implies $\Gamma(\alpha) = \hat{G}$ by [21, Corollary 5.4], so $S = \Gamma(\alpha)^{\perp} = \{e\}$. As $\Gamma(\alpha) = S^{\perp}$, the last statement follows exactly as in the proof of Corollary 2.8.

Accordingly, let $\gamma \in \hat{G}$, $\gamma \notin S^{\perp}$. We shall construct an ideal $I \neq (0)$ in $G \times_{\alpha} A$ with $\hat{\alpha}_{\gamma}(I) \cap I = (0)$, so that $\gamma \notin \Gamma(\alpha)$ [21, Corollary 5.4]. Our construction is similar to that of Theorem 1.8. Let $\tau = \gamma|_S$. Then $\tau \neq \hat{e}$ in $\hat{S}$, that is, there is an open set $O$ in $\hat{S}$, $O \neq \varnothing$, $\hat{e} \in O$ such that $O \cap \tau O = \varnothing$. Taking complements, there exists a proper closed subset $C \subseteqq \hat{S}$ such that $\hat{e} \notin C$ and $C \cup \tau C = \hat{S}$. Let $D = \{\chi \in \hat{G}_{\pi} \mid \chi|_S \in C\}$, and let

$$I = \ker\Big(\operatorname*{IND}_{G_{\pi}\uparrow G} \sum_{\chi \in D}\!{}^{\oplus}\ (\pi \otimes I,\ U \otimes \chi W)\Big),$$

where as above, $U$ is the $\bar{w}$-representation of $G_{\pi}$ implementing the action of $G_{\pi}$ on $A$, $w$ is chosen so that it is the "lift" of a multiplier (which we also denote by $w$) of $G_{\pi}/S$, and $W$ is the "lift" to $G_{\pi}$ of some irreducible $w$-representation $W'$ of $G_{\pi}/S$. As $\hat{G}_{\pi} = D \cup (\gamma|_{G_{\pi}})D$, it is clear that

$$I \cap \hat{\alpha}_{\gamma}(I) = \ker\Big(\operatorname*{IND}_{G_{\pi}\uparrow G} \sum_{\chi \in \hat{G}_{\pi}}\!{}^{\oplus}\ (\pi \otimes I,\ U \otimes \chi W)\Big).$$

Since all irreducible $w$-representations of $G_{\pi}/S$ are weakly equivalent [15, Proposition 32], it follows precisely as in the proof of Proposition 2.6 that

$I \cap \hat{\alpha}_{\gamma}(I)$

$= \ker(\sum^{\oplus}$ (all irreducible representations of $G \times_{\alpha} A$ induced from $G_{\pi} \times_{\alpha} A$))

$$= \bigcap_{P \in \mathrm{PRIM}(G \times_{\alpha} A)} P = (0),$$

by transitivity. To show $I \neq (0)$, we shall show $\mathrm{IND}_{G_\pi \uparrow G}(\pi \otimes I, U \otimes W)$ is not weakly contained in the family of representations $\{\mathrm{IND}_{G_\pi \uparrow G}(\pi \otimes I, U \otimes \chi W) \mid \chi \in D\}$. If it were, we could apply the induction-restriction theorem to the restriction to $S \underset{\alpha}{\times} A$ of the induced representations under consideration, and also apply the continuity of the restriction process, to conclude that as representations of $S \underset{\alpha}{\times} A$, $\{(t\pi, U) \mid t \in G\}$ is weakly contained in $\{(t\pi, \chi U) \mid t \in G, \chi \in C\}$. Here by $U$ we mean $U|_S$, which is an ordinary unitary representation of $S$. Thus, $\exists\, \chi_n \in C$, $t_n \in G$ so that

$$(\pi, U) = \lim_n (t_n\pi, \chi_n U) \quad \text{in } (S \underset{\alpha}{\times} A)\hat{}\,.$$

It follows that $t_n\pi \to \pi$ in $\hat{A}$ and thus, by $G$-simplicity and transitivity, that $t_n G_\pi \to G_\pi$ in $G/G_\pi$. By passing to a subsequence, we can thus find a sequence $\{s_n^{-1}\}$ in $G_\pi$ with $t_n s_n^{-1} \to e$ in $G$. Thus

$$(s_n\pi, \chi_n U) = (s_n t_n^{-1})(t_n\pi, \chi_n U) \to (\pi, U)$$

in $(S \underset{\alpha}{\times} A)\hat{}$. But $s_n\pi \cong \pi$, and a unitary implementing the equivalence is $U(s_n)$, so

$$U(s_n)(s_n\pi, \chi_n U)U(s_n)^{-1} = (\pi, \chi_n U) \to (\pi, U)$$

in $(S \underset{\alpha}{\times} A)\hat{}$. Note that for $s_n \in G_\pi$, $U(s_n)$ commutes with all $U(s)$, $s \in S$, precisely because of the definition of $S$ in terms of the multiplier $\bar{w}$.

We shall now show that $(\pi, U)$ is not weakly contained in $\{(\pi, \chi U) \mid \chi \in C\}$, and we shall be done. Pick a neighbourhood $N$ of $\hat{e}$ in $\hat{S}$ such that $N = (-N)$ and $(N+N) \cap C = \varnothing$. Then $N \cap \chi N = \varnothing$ $\forall \chi \in C$. Now observe that there is a certain arbitrariness involved in the choice of $U$, as follows: for any $\tau$ in $\hat{G}_\pi$, $\tau U$ is also a $\bar{w}$-representation of $G_\pi$ which implements the action of $G_\pi$ on $A$; likewise, the restriction of $\tau U$ to $S$ is an ordinary unitary representation. Thus, after replacing $U$ by $\tau U$, for a suitably chosen $\tau$ in $\hat{G}_\pi$, we may, without having changed any of our previous arguments, now suppose that $\hat{e}$ lies in the spectrum of the unitary representation $U|_S$. We can then choose $f = f^*$ in $L^1(S)$ such that $\operatorname{supp} \hat{f} \subseteq N$ and $U(f) \neq 0$. There exists $\xi \in H_\pi$ such that $\|U(f)\xi\| = 1$ and, as $A$ is liminal, $a \in A$ such that $\pi(a)$ is the rank one projection on $U(f)\xi$. Thus $\pi(a)U(f) = (\pi, U)(f \otimes a) \neq 0$, while

$$\begin{aligned}(\pi, \chi U)(f \otimes a)\xi &= \pi(a)U(\chi f)\xi = \langle U(\chi f)\xi, U(f)\xi\rangle U(f)\xi \\ &= \langle U(f^* * \chi f)\xi, \xi\rangle U(f)\xi = 0, \qquad \forall \xi \in H_\pi.\ \chi \in C,\end{aligned}$$

since $\operatorname{supp}(f^* * \chi f)\hat{} \subseteq N \cap \chi N = \varnothing$. So $(\pi, \chi U)(f \otimes a) = 0$ $\forall \chi \in C$, while $(\pi, U)(f \otimes a) \neq 0$, so $(\pi, U)$ cannot be weakly contained in $\{(\pi, \chi U) \mid \chi \in C\}$.

2.10 Remark. Once it is established that all $S_\pi$'s coincide the result that

$\Gamma(\alpha) = S^{\perp}$ could also be deduced from the decomposition of $G \times_{\alpha} A$ for a transitive action as obtained in [16, 2.13] combined with [20, 6.1].

2.11 COROLLARY. *Let* $(A, G, \alpha)$ *be a G-simple* C*-*dynamical system, with A of type* I. *If* $G = \mathbf{R}, \mathbf{Z}$ *or a compact group then*

$$S^{\perp} = \ker(\hat{\alpha}|_{\mathrm{PRIM}\, G \times_{\alpha} A}) = \Gamma(\alpha)\,,$$

*S denoting the common* $S_\pi$, *for all* $\pi \in \hat{A}$. *Furthermore,* $G \times_{\alpha} A$ *is simple if and only if* $S = \{e\}$, *and also* $k_\pi(\Gamma(\alpha)) = \Gamma(\alpha^\pi)$.

PROOF. By (2.2), all the $G_\pi$'s are the same, hence either $G_\pi = \{e\}$ or $G/G_\pi$ is compact for the groups in question. The first case is covered by (2.8), the second by (2.9).

## 3. *G*-simple actions on continuous trace algebras.

As indicated in section 2, there is reason for suspecting that when $(A, G, \alpha)$ is a $G$-simple C*-dynamical system with $A$ of type I, then $A$ must have continuous trace. In this section we restrict our attention to $G$-simple actions on continuous trace algebras. We first show that all the $S_\pi$'s, $\pi \in \hat{A}$, coincide. Then, an example is given with $A$ homogeneous of degree 2 indicating, among other things, that for this common $S_\pi$, $\Gamma(\alpha)$ need not equal $S_\pi^{\perp}$, and that $G \times_{\alpha} A$ can be simple without $S_\pi$ being trivial. Finally, we consider $G$-simple actions on homogeneous algebras of finite degree, for which $\Gamma(\alpha) = \hat{G}$. Although $S_\pi$ need not be trivial, the possibilities for both $G_\pi$ and $S_\pi$ are quite limited. Note, as mentioned in the beginning of section 2, that if $A$ is a $G$-simple type I algebra which is also unital, then it must be homogeneous of finite degree and hence has continuous trace.

3.1 THEOREM. *Let* $(A, G, \alpha)$ *be a G-simple* C*-*dynamical system, with A having continuous trace. Then for all* $\pi$ *in* $\hat{A}$, *the subgroups* $S_\pi$ *coincide.*

For the proof of Theorem 3.1, we need the following two lemmas:

3.2 LEMMA. *Let H be a Hilbert space,* $(P_n)$ *a sequence of rank one projections converging strongly to a rank one projection P, and* $(U_n)$ *a sequence of unitaries such that* $U_n P_n U_n^*$ *converges strongly to P. Then* $U_n P U_n^*$ *converges strongly to P.*

PROOF. By writing rank one projections explicitly as projections on unit vectors, one easily sees that rank one projections converging strongly to a rank one projection in fact converge in norm. Hence $U_nPU_n^* \to P$ in norm.

3.3 LEMMA. *Let $(U_n)$ be a sequence of unitaries such that for all rank one projections $P$, $U_nPU_n^* \to P$ strongly (hence in norm). Let $\Psi$ denote the canonical map of the unitary group of $H$, endowed with the strong topology, onto the projective unitary group, endowed with the quotient topology. Then*

(1) $\exists$ *sequence $(z_n)$ in $\mathbf{C}$ with $|z_n|=1$, $z_nU_n \to I$ strongly.*
(2) $\Psi(U_n) \to \Psi(I)$.

PROOF. That (1) is equivalent to (2) follows from [27, p. 97, 10.1]. As in the proof there, it suffices to show that for any subsequence $U_{n_i}$ which converges weakly to an operator $A$, we have $A=\alpha I$, with $|\alpha|=1$. Accordingly, let $U_{n_i} \to A$ weakly. As

$$\|U_{n_i}PU_{n_i}^*-P\| = \|U_{n_i}P-PU_{n_i}\| \to 0,$$

for all rank one projections $P$, we clearly have $AP=PA$ for all such $P$, so that $A=\lambda I$, $|\lambda|\leqq 1$. Also, $P$ being compact implies $PU_{n_i}^* \to \bar{\lambda}P$ strongly, so $U_{n_i}PU_{n_i}^* \to |\lambda|^2P$ weakly. Thus $|\lambda|=1$.

PROOF OF THEOREM. Let $\pi, \tau \in \hat{A}$. By $G$-simplicity, $t_n\pi \to \tau$ in $\hat{A}$ for some sequence $(t_n)$ in $G$. By (2.2), $G_\pi = G_{t\pi} = G_\tau$, for every $t$ in $G$. For the rest of this proof, we wish to consider only concrete irreducible representations of $A$ of equivalence class $\pi$ and $\tau$, respectively. By abuse of notation, we denote these concrete representations by $\pi$ and $\tau$, and note that by homogeneity of $A$, they can be assumed to act on the same space $H$. It follows from [8, 3.5.8] that, after passing to a subsequence if necessary, there exists a sequence $(V_n)$ of unitaries so that

$$(*) \qquad V_n(t_n\pi)(a)V_n^* \to \tau(a) \quad \text{strongly, } \forall a \in A.$$

Let $U^\pi$ and $U^\tau$ denote the multiplier representations of $G_\pi = G_\tau$ which intertwine correctly with $\pi$ and $\tau$, respectively. $U^\pi$ intertwines correctly with all the representations $t\pi$, $t \in G$, also. To verify that $S_\pi \subseteq S_\tau$, we must show that if

$$(**) \qquad U^\pi(t)U^\pi(s) = U^\pi(s)U^\pi(t) \quad \forall s \in G_\pi,$$

then

$$U^\tau(t)U^\tau(s) = U^\tau(s)U^\tau(t) \quad \forall s \in G_\pi.$$

Accordingly, fix $t$ in $G_\pi$ satisfying (**), and let $s \in G_\pi$. We have

$$(***) \qquad (U^\tau(-s)V_nU^\pi(s)V_n^*)(V_n(t_n\pi)(a)V_n^*)(V_nU^\pi(-s)V_n^*U^\tau(s)) = (U^\tau(-s)V_n)((t_n\pi)(\alpha_s(a)))(V_n^*U^\tau(s))$$

converges strongly by (*) to $U^\tau(-s)\tau(\alpha_s(a))U^\tau(s)=\tau(a)$, $\forall\, a \in A$. Since $A$ has continuous trace, it follows by [8, 4.5.3] and the proof of [8, 4.4.2] that for any rank one projection $P$ on $H$, there exists $a \in A$ such that $\tau(a)=P$ and $\varrho(a)$ is a rank one projection for all irreducible representations $\varrho$ whose equivalence class is in a suitable neighbourhood of the equivalence class of $\tau$ in $\hat{A}$. Thus for $n$ large enough, $V_n(t_n\pi)(a)V_n^*$ will be a rank one projection $P_n$. By applying (*), (**), (3.2) and then (3.3), with the $U_n$ of (3.2) and (3.3) standing for $U^\tau(-s)$ $V_nU^\pi(s)V_n^*$, we have $c_n$ in $\mathbf{C}$, $|c_n|=1$ such that $c_nV_nU^\pi(s)V_n^*$ converges to $U^\tau(s)$ strongly. Similarly, for $t$ in $G_\pi$ we have $d_n$ in $\mathbf{C}$, $|d_n|=1$, with $d_nV_nU^\pi(t)V_n^*$ converging to $U^\tau(t)$ strongly. If $t$ satisfies (**), it follows immediately by strong continuity of multiplication (on bounded sets) that $U^\tau(t)U^\tau(s)=U^\tau(s)U^\tau(t)$, and we have $S_\pi \subseteq S_\tau$. The argument is clearly symmetric in $\pi$ and $\tau$, and we are done.

3.4 Example. Let $A=C(T, M_2)$ be the C*-algebra of continuous functions from the unit circle into the $2\times 2$ matrix algebra. Let

$$(\alpha f)(t) = Wf(e^{-i\theta}t)W^*, \quad f\in A,\ t\in T,$$

where $\theta$ is an irrational multiple of $2\pi$, and $W=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Then $n \to \alpha^n$ defines an action of $\mathbf{Z}$ on $A$, which commutes with the action $\beta$ of $\mathbf{Z}_2$ on $A$ given by

$$(\beta f)(t) = Uf(t)U^*, \quad f\in A,\ t\in T,$$

where $U=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Combining these two actions, we get an action $\alpha\times\beta$ of the discrete abelian group $G=\mathbf{Z}\times\mathbf{Z}_2$ on $A$. The action is clearly $G$-simple, and $G_\pi=\mathbf{Z}_2$ for every $\pi$ in $\hat{A}$, while $S=S_\pi=G_\pi$ since $U^2=I$. We claim that the crossed product algebra is simple and $\Gamma(\alpha\times\beta)=(\mathbf{Z}\times\mathbf{Z}_2)\hat{}$. Given this claim, we will thus have an example of a $G$-simple action $\alpha\times\beta$ on a homogeneous algebra of degree 2 such that: (1) $\Gamma(\alpha\times\beta)\neq S^\perp$; (2) the crossed product is simple but $S$ is non-trivial (so that the converse of [11, Theorem 4.1] does not hold); and (3) $k_\pi(\Gamma(\alpha\times\beta))\neq\Gamma((\alpha\times\beta)^\pi)$. To see (3), note that $k_\pi(\Gamma(\alpha\times\beta)) = k_\pi((\mathbf{Z}\times\mathbf{Z}_2)\hat{}\,)=\hat{\mathbf{Z}}_2$, while $\Gamma((\alpha\times\beta)^\pi)=$ the annihilator of $S_\pi$ in $\hat{G}_\pi$ [15, Prop. 34], hence is $\{0\}$.

To verify the claim, observe that since $\mathbf{Z}\times\mathbf{Z}_2$ is a discrete group, $\Gamma(\alpha\times\beta) =(\mathbf{Z}\times\mathbf{Z}_2)\hat{}$ if and only if the crossed product algebra is simple [21, Theorem 6.5]. We shall actually give 2 independent, quite different proofs of the claim. The first uses a characterization of $\Gamma(\alpha\times\beta)^\perp$ in terms of the canonical action on the multiplier algebra $M(A)$ of $A$; the second involves checking that all irreducible induced representations of the crossed product algebra are weakly equivalent, and then invoking the generalized Effros–Hahn conjecture [14] to prove simplicity of the crossed product.

For the first proof, we need

3.5 LEMMA. *Let* $(A, G, \alpha)$ *be a G-simple* C*-*dynamical system, with A having continuous trace. If* $\mathrm{Sp}(\alpha)/\Gamma(\alpha)$ *is compact (as a subset of* $\hat{G}/\Gamma(\alpha)$*), then*

$$\Gamma(\alpha)^{\perp} = \{t \in G \mid \exists u_t \in Z(M(A)^{\alpha''});\ \alpha_t = \mathrm{Ad}\, u_t\},$$

$Z(M(A)^{\alpha''})$ *denoting the center of the fixed-point algebra for the canonical extension* $\alpha''$ *of* $\alpha$ *to* $M(A)$.

PROOF. This is a verbatim repetition of the proof in [19, 4.2] for the case where $A$ is simple, using that, as shown in [23, 8.6.11], every derivation of a C*-algebra $A$ with continuous trace is inner in $M(A)$.

As $\mathbf{Z} \times \mathbf{Z}_2$ is discrete, (3.5) yields the desired result once we verify that the action is never inner in the fixed-point algebra. But $A = M(A)$, and $A^{\alpha \times \beta} = \{\lambda I \mid \lambda \in \mathbf{C}\}$, where $I$ is the identity element of $A$, as follows: if $f \in A$ and $\beta f = f$, then $f(t) = \begin{pmatrix} a(t) & 0 \\ 0 & d(t) \end{pmatrix}$, while

$$f(t) = (\alpha^2 f)(t) = \begin{pmatrix} a(e^{-2i\theta}t) & 0 \\ 0 & d(e^{-2i\theta}t) \end{pmatrix}$$

implies $a$ and $d$ are constant functions, which must then be equal since

$$f(t) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = (\alpha f)(t) = \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}.$$

Hence no non-trivial automorphism of $A$ is inner in the fixed-point algebra, and we are done.

For the second proof, note that $\hat{A}$ is homeomorphic to $T$, with $\pi_t(f) = f(t)$ being the representation determined by $t$ in $T$. The unitary representation of $\mathbf{Z}_2 = \{0, 1\}$ sending $1 \to U$ (which we denote by $U$) intertwines with $\pi_t$ correctly, and thus for each $t$ in $T$ we have two irreducible representations of $G_\pi \times_\beta A = \mathbf{Z}_2 \times_\beta A$: $\langle \pi_t, U \rangle$ and $\langle \pi_t, \chi U \rangle$, $\chi$ being the character of $\mathbf{Z}_2$ with $\chi(1) = -1$. Now

$$(\alpha\pi_t)(f) = (\alpha^{-1} f)(t) = W^* f(e^{i\theta}t) W,$$

and $WUW^{-1} = \chi U$, so

$$W\langle \alpha\pi_t, U \rangle W^{-1} = \langle \pi_{e^{i\theta}t}, \chi U \rangle$$

and thus

$$\langle \alpha\pi_t, U\rangle \cong \langle \pi_{e^{i\theta}t}, \chi U\rangle .$$

As $\chi^2 \equiv 1$, it is clear that for any $s, t \in T$ we can find a sequence $(n_i)$ of *even* integers with $\langle \alpha^{n_i}\pi_t, U\rangle \to \langle \pi_s, U\rangle$ in $(\mathbf{Z}_2 \times_\beta A)\hat{}$, and similarly $\langle \alpha^{n_i}\pi_t, \chi U\rangle \to \langle \pi_s, \chi U\rangle$. Likewise, for some sequence $(k_i)$ of *odd* integers,

$$\langle \alpha^{k_i}\pi_t, U\rangle \to \langle \pi_s, \chi U\rangle \quad \text{and} \quad \langle \alpha^{k_i}\pi_t, \chi U\rangle \to \langle \pi_s, U\rangle$$

in $(\mathbf{Z}_2 \times_\beta A)\hat{}$. It follows as in Proposition 2.1 of [11] that all irreducible induced representations of $(\mathbf{Z} \times \mathbf{Z}_2) \times_{\alpha \times \beta} A$ are weakly equivalent, and thus by [14, Corollary 3.2] that $(\mathbf{Z} \times \mathbf{Z}_2) \times_{\alpha \times \beta} A$ is simple.

We now consider $G$-simple actions on homogeneous algebras of finite degree.

3.6 LEMMA. *Let $(A, G, \alpha)$ be a $G$-simple C*-dynamical system, with $A$ homogeneous of degree $n < \infty$. If $\mathrm{Sp}(\alpha) = \hat{G}$, then the (common) isotropy group $G_\pi$ is a finite group.*

PROOF. By (2.7), $\mathrm{Sp}(\alpha^\pi) = \hat{G}_\pi$. Now $\alpha^\pi$ denotes the action of $G_\pi$ on $\pi(A) = M_n$, and hence $\mathrm{Sp}(\alpha^\pi)$ contains at most $n^2$ elements. Indeed, if $\mathrm{Sp}(\alpha^\pi)$ contained $n^2+1$ distinct elements $\gamma_k$, $1 \leqq k \leqq n^2+1$, we could choose non-empty, mutually disjoint closed neighbourhoods $V_k$ of $\gamma_k$. The corresponding spectral subspaces $M^{\alpha^\pi}(V_k)$ [18, 2.1.1] are non-zero and linearly independent, a contradiction.

3.7 REMARK. The above lemma improves [22, 3.9] where it was shown, under the same hypotheses, that the common isotropy group must be discrete.

3.8 LEMMA. *Let $(A, G, \alpha)$ be a $G$-simple C*-dynamical system, with $A$ homogeneous of degree $n < \infty$, and $\Gamma(\alpha) = \hat{G}$. Let $S$ denote the common $S_\pi$ $(\pi \in \hat{A})$, $|S|$ the order of $S$, and $U^\pi$ the multiplier representation of $G_\pi$ which implements the action of $G_\pi$ on $\pi(A)$. Then $|G_\pi/S|$ is a perfect square, $|S|\sqrt{|G_\pi/S|}$ divides $n$, and $U^\pi | S$, the restriction of $U^\pi$ to $S$, is a multiple of the left regular representation of $S$.*

Before proving the lemma, we make several comments.

3.9 REMARK. Although we know (3.4) that under the hypotheses of the above lemma, $S$ need not be trivial, the lemma proves that there are severe restrictions on the size of $S$. For the case $n=2$, Example (3.4) exactly exhibits the only alternative to $S = \{e\}$, namely both that $|S| = 2$ so $S = \mathbf{Z}_2$, and that $U^\pi$ is the 2-dimensional regular representation of $\mathbf{Z}_2$.

We also note that if $G$ has no non-trivial finite subgroups, the hypotheses of (3.8) imply, by (3.6), free action of $G$ on $\hat{A}$.

3.10 COROLLARY. *Let* $(A, G, \alpha)$ *be as in Lemma* 3.8, *with n prime. Either* $S = \{e\}$, *in which case* $G_\pi = \{e\}$ *or a group of order* $n^2$, *or* $S = G_\pi = \mathbf{Z}_n$, *the cyclic group of order n.*

PROOF OF LEMMA 3.8. Let $w$ denote the multiplier on $G_\pi$ determined by $U^\pi$. We may without loss of generality [3, Theorem 3.1] assume $w$ is the "lift" to $G_\pi$ of a totally skew multiplier (also denoted by $w$) on $G_\pi/S$. As $U_\pi$ is an $n$-dimensional representation of the finite group $G_\pi$ (3.6), it can be decomposed as a direct sum of irreducible $w$-representations of $G_\pi$, each of which is of the form $\chi_i W$, for $\chi_i \in \hat{G}_\pi$ and $W$ the unique irreducible $w$-representation of $G_\pi/S$ [3, Theorem 3.1 and Lemma 3.1]. Thus $U^\pi = \sum \oplus \chi_i W = W \otimes (\sum \oplus \chi_i)$, the direct sum being taken over some subset of $\hat{G}_\pi$, with multiplicity. By Lemma 3.1 of [3], $|G_\pi/S| = (\dim W)^2$, so $n = \sqrt{|G_\pi/S|} \cdot \dim(\sum \oplus \chi_i)$. As $U|_S = I \otimes (\sum \oplus \chi_i|_S)$, we shall be done once we verify that $U|_S$ is a direct sum of all the characters in $\hat{S}$, each occurring with the same multiplicity. For this, we need only check that for any $\chi$ in $\hat{S}$, $\chi \cdot U|_S$ is unitarily equivalent to $U|_S$. Accordingly, let $\chi \in \hat{S}$ and write $U$ for $U|_S$. Then $\langle \pi, U \rangle$ and $\langle \pi, \chi U \rangle$ are both irreducible representations of $S \times_\alpha A$. As $G \times_\alpha A$ is simple [22, 3.10], it follows by continuity of induction-restriction (cf. [11, Prop. 2.1]) that for some sequence $(t_n)$ in $G$, $L_n \equiv \langle t_n \pi, \chi U \rangle$ converges to $L \equiv \langle \pi, U \rangle$ in $(S \times_\alpha A)\hat{}$. After passing to a subsequence, if necessary, we can find by [8, 3.5.8] a sequence of unitaries $(V_n)$ such that $V_n L_n(u) V_n^*$ converges to $L(u)$ strongly for every $u$ in $S \times_\alpha A$. As we are on a finite-dimensional space, however, some subsequence $(V_{n_i})$ of $(V_n)$ converges in norm to a unitary $V$, which clearly implements the unitary equivalence between $\chi U$ and $U$.

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NOTE ADDED IN PROOF. For a $G$-simple action on a type I algebra we now know that all the $S_\pi$'s must be identical. Details of this strengthening of theorem 3.1 will appear elsewhere.

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