MATH. SCAND. 48 (1981), 41–55

EIGENSPACE REPRESENTATIONS OF NILPOTENT LIE GROUPS

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1. Introduction.

At the International Congress of Mathematicians in Nice 1970, S. Helgason [3] posed the following problems concerning a homogeneous manifold G/H, H being a closed subgroup of a Lie group G.

(A) Determine the algebra $\boldsymbol{D}(G/H)$ of all differential operators on G/H which are invariant under G.
(B) Determine the functions on G/H which are eigenfunctions of each $D \in \boldsymbol{D}(G/H)$.
(C) For each joint eigenspace for the operators in $\boldsymbol{D}(G/H)$, study the natural representation of G on this eigenspace; in particular when it is irreducible, and what representations are so obtained.

These problems initiated several investigations, in particular of semisimple Lie groups. See sections 4 and 5 of the survey paper [4].

Nilpotent groups were in the special case of the $(2n+1)$-dimensional Heisenberg group treated by A. Hole [5]. He showed that if H is a connected, non-normal subgroup, maximal with respect to these properties, then the eigenspace representations from (C) are topologically irreducible (except for the eigenvalue 0).

We present a generalization of Hole's results to any simply connected, connected, nilpotent Lie group G.

Let $\mathfrak{g}$ be the Lie algebra of G, let $\mathfrak{k}$ be a real polarisation at $\alpha \in \mathfrak{g}^*$ and let H be the analytic subgroup of G corresponding to $\mathfrak{h} := \mathfrak{k} \cap \ker \alpha$. Then (Theorem 6.1) $\boldsymbol{D}(G/H)$ is generated by the G-invariant vector field on G/H determined by any $Z \in \mathfrak{k} \setminus \mathfrak{h}$ and the natural representation of G on the joint eigenspace

$$\{u \in \mathscr{E}(G/H) \mid Du = \chi(D)u, \ \forall D \in \boldsymbol{D}(G/H)\}$$

from (C) is equivalent to the restriction $\lambda_{a\alpha,\mathfrak{k}}$ of the left regular representation of G to

Received June 19, 1979; in revised form April 9, 1980.

$$\mathscr{E}_{a\alpha,\mathfrak{k}} := \{u \in \mathscr{E}(G) \mid Xu = -ia\alpha(X)u, \ \forall X \in \mathfrak{k}\},$$

where $a = a(\chi)$ is a constant. If $\chi \neq 0$ then $a \neq 0$.

In terms of $\lambda_{a\alpha,\mathfrak{k}}$ we get the following answers to the problems (B) and (C), when $a \in \mathbf{C} \setminus \{0\}$:

(i) There exist topological isomorphisms of $\mathscr{E}_{a\alpha,\mathfrak{k}}$ onto $\mathscr{E}(\mathbf{R}^n)$ where $n = \dim \mathfrak{g}/\mathfrak{k}$ (Theorem 3.1) such that
(ii) the derived algebra $d\lambda_{a\alpha,\mathfrak{k}}(\boldsymbol{D}(G))$ viewed as operators on $\mathscr{E}(\mathbf{R}^n)$ equals the algebra of all differential operators on $\mathbf{R}^n$ with polynomial coefficients (Theorem 4.1).
(iii) $\lambda_{a\alpha,\mathfrak{k}}$ is both operator and topologically irreducible (Theorem 5.1).
(iv) If $\lambda_{a\alpha,\mathfrak{k}_1}$ is equivalent to $\lambda_{a\alpha,\mathfrak{k}_2}$, then $\mathfrak{k}_1 = \mathfrak{k}_2$ (Proposition 7.4).

The result above on $\boldsymbol{D}(G/H)$ is a corollary of (iii). Another and earlier proof is due to A. Hole (Theorem 3.7 of [6]).

Hole's result for the Heisenberg groups is generalized by (iii).

It can be mentioned that our results are still true if the function space $\mathscr{E}$ is replaced by the distribution space $\mathscr{D}'$ in the statements.

Our key result (ii) is closely related to a theorem of Kirillov ([7, Theorem 7.1, p. 82]), the difference being that we consider

1) representations on spaces of C^∞-functions and distributions,
2) functionals on $\mathfrak{g}$ which may be complex-valued,
3) and that we make the representations explicit in terms of coexponential bases.

In the unitary theory two polarisations at the same point in $\mathfrak{g}^*$ give rise to equivalent representations. (iv) shows this is not the case for representations on spaces of C^∞-functions or distributions.

2. Notation.

We conform to the standard terminology for function and distribution spaces. For example, if M is a manifold, then $\mathscr{D}'(M)$ denotes the strong dual of $\mathscr{D}(M) = C_c^\infty(M)$.

Throughout this paper G denotes a real Lie group with left Haar measure dg, Lie algebra $\mathfrak{g}$ and exponential map exp. By means of dg function spaces like $\mathscr{E}(G)$ and $\mathscr{D}(G)$ are identified with subspaces of $\mathscr{D}'(G)$ and differential operators $D: \mathscr{E}(G) \to \mathscr{E}(G)$ with their unique continuous extensions $D: \mathscr{D}'(G) \to \mathscr{D}'(G)$. The complexified universal enveloping algebra of $\mathfrak{g}$ will be identified with $\boldsymbol{D}(G)$, the complex algebra of all the left invariant differential operators on G.

We denote by $\mathfrak{g}^*$ the dual of $\mathfrak{g}$, by $(\mathfrak{g}^*)^{\mathbf{C}}$ the set of complex valued, real linear functionals on $\mathfrak{g}$, and by $\mathbf{C}\mathfrak{g}^*$ the set of all complex multiples of elements from

$\mathfrak{g}^*$. If $\alpha \in (\mathfrak{g}^*)^{\mathbf{C}}$, we write $S(\alpha; \mathfrak{g})$ for the set whose elements are the subalgebras $\mathfrak{k}$ of $\mathfrak{g}$ subordinate to α, i.e. with $\alpha([\mathfrak{k},\mathfrak{k}])=\{0\}$. The subset of $S(\alpha; \mathfrak{g})$ consisting of subalgebras of maximal dimension subordinate to α is denoted by $M(\alpha; \mathfrak{g})$.

If $\alpha \in (\mathfrak{g}^*)^{\mathbf{C}}$ and $\mathfrak{k} \in S(\alpha; \mathfrak{g})$, then

$$\mathscr{D}'_{\alpha,\mathfrak{k}} := \{u \in \mathscr{D}'(G) \mid Xu = -i\alpha(X)u, \ \forall X \in \mathfrak{k}\}$$

and

$$\mathscr{E}_{\alpha,\mathfrak{k}} := \mathscr{E}(G) \cap \mathscr{D}'_{\alpha,\mathfrak{k}}$$

are closed subspaces of $\mathscr{D}'(G)$ and $\mathscr{E}(G)$ respectively, and they are invariant under the action of G by left translations. Using the methods of [1, Chap. VIII, § 2 No. 3], we find that the action of G restricts to differentiable representations $\Lambda_{\alpha,\mathfrak{k}}$ and $\lambda_{\alpha,\mathfrak{k}}$ of G on $\mathscr{D}'_{\alpha,\mathfrak{k}}$ and $\mathscr{E}_{\alpha,\mathfrak{k}}$ respectively.

2.1. Definition. Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$. We say that an ordered set $\{X_1,\ldots,X_n\}$ of elements from $\mathfrak{g}$ is a *coexponential basis* mod $\mathfrak{k}$ for $\mathfrak{g}$ if the following two conditions are satisfied:

(a) $\{X_1+\mathfrak{k},\ldots,X_n+\mathfrak{k}\}$ is a basis for $\mathfrak{g}/\mathfrak{k}$.

(b) $\mathfrak{g}_j := \operatorname{span}\{X_j,\ldots,X_n,\mathfrak{k}\}$ is an ideal of codimension 1 in $\mathfrak{g}_{j-1}$ for $j = 1,\ldots,n+1$.

A coexponential basis mod $\{0\}$ for $\mathfrak{g}$ is said to be a *coexponential basis for* $\mathfrak{g}$.

2.2 Lemma. *If* $\mathfrak{k}$ *is a subalgebra of a nilpotent Lie algebra* $\mathfrak{g}$, *then there exists a coexponential basis* mod $\mathfrak{k}$ *for* $\mathfrak{g}$.

Proof. If $\mathfrak{k} \neq \mathfrak{g}$ then take X_n as any element off $\mathfrak{k}$ in the normaliser of $\mathfrak{k}$. Proceed inductively.

2.3. Proposition. *Let* G *be a simply connected, nilpotent Lie group. Let* $\{X_1,\ldots,X_n\}$ *be a coexponential basis for* $\mathfrak{g}$ mod *a subalgebra* $\mathfrak{k}$. *Then the map*

$$(x_1,\ldots,x_n) \mapsto \exp(x_1X_1)\ldots\exp(x_nX_n)\exp(\mathfrak{k})$$

is a diffeomorphism of $\mathbf{R}^n$ *onto* $G/\exp(\mathfrak{k})$.

Proof. The case $\mathfrak{k}=(0)$ is well-known (Theorem 3.18.11 of [10]). The general case follows easily from this when we supplement the given basis with a coexponential basis for $\mathfrak{k}$.

On basis of the Proposition we identify $G/\exp(\mathfrak{k})$ and $\mathbf{R}^n$ when a coexponential basis mod $\mathfrak{k}$ for $\mathfrak{g}$ is given. We will write $g \cdot x$ for the action of $g \in G$ on $x \in \mathbf{R}^n = G/\exp(\mathfrak{k})$.

2.4. DEFINITION. A representation on a topological vector space is said to be *scalar* (resp. *operator*) *irreducible* if the only continuous (resp. densely defined and closed) intertwining operators are scalar multiples of the identity operator.

3. Eigenspaces on nilpotent Lie groups.

In this section we realize our representations $\lambda_{\alpha,\mathfrak{k}}$ and $\Lambda_{\alpha,\mathfrak{k}}$ on $\mathbf{R}^n$ via coexponential bases.

3.1. THEOREM. *Let G be a simply connected, connected nilpotent Lie group, and let $\mathfrak{k} \in S(\alpha; \mathfrak{g})$ for some $\alpha \in (\mathfrak{g}^*)^{\mathbf{C}}$. Let $\{X_1, \ldots, X_n\}$ be a coexponential basis* mod $\mathfrak{k}$ *for $\mathfrak{g}$, and let $s^*\colon \mathscr{E}(G) \to \mathscr{E}(\mathbf{R}^n)$ be the corresponding map given by*

$$(s^*f)(x_1, \ldots, x_n) := f(\exp(x_1X_1) \ldots \exp(x_nX_n))$$

$$\text{for } f \in \mathscr{E}(G) \quad \text{and} \quad (x_1, \ldots, x_n) \in \mathbf{R}^n .$$

Then

(a) *s^* restricts to a topological isomorphism $s^*\colon \mathscr{E}_{\alpha,\mathfrak{k}} \to \mathscr{E}(\mathbf{R}^n)$ which extends uniquely to a topological isomorphism $s^*\colon \mathscr{D}'_{\alpha,\mathfrak{k}} \to \mathscr{D}'(\mathbf{R}^n)$.*

(b) *s^* makes $\lambda_{\alpha,\mathfrak{k}}$ equivalent to a representation λ of G on $\mathscr{E}(\mathbf{R}^n)$ of the form $[\lambda(g)f](x) = \exp[i\alpha(p(g,x))]f(g^{-1}\cdot x)$ for $g \in G$, $f \in \mathscr{E}(\mathbf{R}^n)$, $x \in \mathbf{R}^n$, where $p\colon G \times \mathbf{R}^n \to \mathfrak{k}$ is a certain polynomial map.*
The same formula holds for the representation Λ of G on $\mathscr{D}'(\mathbf{R}^n)$ corresponding to $\Lambda_{\alpha,\mathfrak{k}}$.

(c) *The endomorphisms $d\lambda(X)\colon \mathscr{E}(\mathbf{R}^n) \to \mathscr{E}(\mathbf{R}^n)$, and $d\Lambda(X)\colon \mathscr{D}'(\mathbf{R}^n) \to \mathscr{D}'(\mathbf{R}^n)$, $X \in \mathfrak{g}$, are differential operators with polynomial coefficients.*

PROOF. Let K be the analytic subgroup corresponding to $\mathfrak{k}$, and let χ be the character on K given by

$$\chi(\exp X) := e^{i\alpha(X)} \quad \text{for } X \in \mathfrak{k} .$$

Let $s\colon \mathbf{R}^n \to G$ be the map

$$s(x_1, \ldots, x_n) := \exp(x_1X_1) \ldots \exp(x_nX_n) \quad \text{for } (x_1, \ldots, x_n) \in \mathbf{R}^n .$$

Now $f \in \mathscr{E}_{\alpha,\mathfrak{k}}$ *iff*

$$(*) \qquad f(s(x)k) = (f \circ s)(x)\chi(k)^{-1} \quad \text{for } x \in \mathbf{R}^n \text{ and } k \in K .$$

(a) It follows from (*) that s^* is a vector space isomorphism between $\mathscr{E}_{\alpha,\mathfrak{k}}$ and $\mathscr{E}(\mathbf{R}^n)$. It is clearly continuous and the continuity of its inverse is a consequence of the open mapping theorem.

Let us normalize the Haar measure dk on K in such a way that

$$\int_G f(g)\,dg = \int_{G/K}\int_K f(s(m)k)\,dk\,dm, \qquad \text{where } dm \text{ is Lebesgue measure on } G/K = \mathsf{R}^n\,,$$

and choose $\theta \in \mathscr{D}(K)$ such that

$$\int_K \theta(k)\,dk = 1\,.$$

We define the continuous linear map i: $\mathscr{D}(G/K) \to \mathscr{D}(G)$ by

$$(i\varphi)(s(m)k) := \varphi(m)\chi(k)\theta(k) \qquad \text{for } \varphi \in \mathscr{D}(G/K),\ m \in G/K,\ k \in K\,,$$

and let Φ denote the restriction to $\mathscr{D}'_{\alpha,\mathfrak{k}}$ of the transpose map i^t to i. So Φ: $\mathscr{D}'_{\alpha,\mathfrak{k}} \to \mathscr{D}'(G/K)$ is continuous and linear. It is easy to check that Φ is an extension of s^* on $\mathscr{E}_{\alpha,\mathfrak{k}}$.

It is left to show that Φ has a continuous inverse. For that purpose we introduce the continuous linear map β: $\mathscr{D}(G) \to \mathscr{D}(G/K)$ given by

$$(\beta\varphi)(m) := \int_K \varphi(s(m)k)\chi(k)^{-1}\,dk \qquad \textit{for } \varphi \in \mathscr{D}(G),\ m \in G/K\,.$$

As is easily seen, its transpose β^t: $\mathscr{D}'(G/K) \to \mathscr{D}'(G)$ satisfies the relation

$$R(k)\beta^t = \chi(k)^{-1}\beta^t \qquad \text{for all } k \in K\,,$$

i.e. β^t maps $\mathscr{D}'(G/K)$ into $\mathscr{D}'_{\alpha,\mathfrak{k}}$.

Since $\beta \circ i$ is the identity map on $\mathscr{D}(G/K)$, we get that $\Phi\circ\beta^t$ is the identity on $\mathscr{D}'(G/K)$ and so β^t is a right inverse of Φ. On the other hand,

$$\beta^t s^* f = f \qquad \text{for all } f \in \mathscr{E}_{\alpha,\mathfrak{k}}\,,$$

so $\beta^t s^*$ is the identity map on the subspace $\mathscr{E}_{\alpha,\mathfrak{k}}$ of $\mathscr{D}'_{\alpha,\mathfrak{k}}$.
Since $\mathscr{E}_{\alpha,\mathfrak{k}}$ contains the Gårding space for $\Lambda_{\alpha,\mathfrak{k}}$, it is dense in $\mathscr{D}'_{\alpha,\mathfrak{k}}$. By continuity $\beta^t\circ\Phi$ is the identity on all of $\mathscr{D}'_{\alpha,\mathfrak{k}}$, so that β^t is a left inverse, too.

(b) The formula for λ is a consequence of (*) with p: $G \times \mathsf{R}^n \to \mathfrak{k}$ determined by $g^{-1}s(x) = s(g^{-1}\cdot x)\exp[p(g,x)]$ for $g \in G$, $x \in \mathsf{R}^n$.

Since $\mathscr{E}(\mathsf{R}^n)$ is dense in $\mathscr{D}'(\mathsf{R}^n)$, we get the corresponding formula for Λ by continuity.

(c) Differentiate the formula in (b).

3.2. Lemma. *Let G be a simply connected, connected, nilpotent Lie group. Let $\mathfrak{k} \in S(\alpha, \mathfrak{g})$ for some $\alpha \in (\mathfrak{g}^*)^{\mathsf{C}}$, and assume given two coexponential bases* mod $\mathfrak{k}$ *for* $\mathfrak{g}$. *Define as in Theorem* 3.3 *corresponding maps s_1^* and s_2^*.*

Then $s_1^*\circ(s_2^*)^{-1}$: $\mathscr{E}(\mathsf{R}^n) \to \mathscr{E}(\mathsf{R}^n)$ transforms the algebra of all differential operators with polynomial coefficients on R^n onto itself.

PROOF. An explicit calculation reveals that $s_1^* \circ (s_2^*)^{-1}$ is given by

$$[s_1^* \circ (s_2^*)^{-1} f](x) = \exp\big(i\alpha(q(x))\big) f(p(x)) \quad \text{for } f \in \mathscr{E}(\mathsf{R}^n) \text{ and } x \in \mathsf{R}^n\,,$$

where $p\colon \mathsf{R}^n \to \mathsf{R}^n$ and $q\colon \mathsf{R}^n \to \mathfrak{k}$ are certain polynomial maps.

The lemma now follows because $(s_1^* \circ (s_2^*)^{-1})^{-1} = s_2^* \circ s_1^{*\,-1}$ has the same form (with p replaced by p^{-1} and q by $-q \circ p^{-1}$).

4. The algebra $d\lambda(\boldsymbol{D}(G))$.

Below we present our key result. Its unitary analogue was proved by Kirillov (Theorem 7.1 of [7]). A crucial point in his proof is use of equivalence of certain representations. Section 7 below shows that the corresponding representations are not equivalent here, so the classical approach has been modified here.

4.1. THEOREM. *Let G be a simply connected, connected, nilpotent Lie group, let $\alpha \in \mathsf{C}\mathfrak{g}^* \setminus \{0\}$ and let $\mathfrak{k} \in M(\alpha;\, \mathfrak{g})$. Let there be given a coexponential basis* mod $\mathfrak{k}$ *for* $\mathfrak{g}$. *Then the corresponding realization λ of the representation $\lambda_{\alpha,\mathfrak{k}}$ on $\mathscr{E}(\mathsf{R}^n)$ has the property that $d\lambda(\boldsymbol{D}(G))$ is the algebra of all differential operators with polynomial coefficients on* R^n.

4.2. REMARKS. (i) λ is described in Theorem 3.1 where it is also shown that $d\lambda(\boldsymbol{D}(G))$ consists of polynomial differential operators.

(ii) The theorem clearly remains true if $\lambda_{\alpha,\mathfrak{k}}$ is replaced by $\Lambda_{\alpha,\mathfrak{k}}$.

PROOF. In the proof we work with suitably chosen coexponential bases. That is allowed by Lemma 3.2.

The proof goes by induction on $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 1$, then λ is a representation on $\mathscr{E}(\mathsf{R}^0)$ and $d\lambda(X) = i\alpha(X)$ for $X \in \mathfrak{g}$. Since $\alpha \neq 0$, the theorem holds in this special case.

The induction hypothesis is that the theorem is true whenever $\dim \mathfrak{g} \leqq N$, so we shall show that it holds when $\dim \mathfrak{g} = N+1$.

Let $\mathfrak{z}$ denote the center of $\mathfrak{g}$ and observe that $\mathfrak{z} \subseteqq \mathfrak{k}$ by the maximality of $\mathfrak{k}$. We divide the proof into three parts (A), (B) and (C), of which (A) and (B) can be handled as by Kirillov in his proof of Theorem 7.1 of [7]. The proofs of (A) and (B) are therefore deleted.

(A) We may and will assume that $\ker \alpha \cap \mathfrak{z} = \{0\}$.

From now on, $\dim \mathfrak{z} = 1$ because $\mathfrak{h} := \ker \alpha \cap \mathfrak{k}$ has codimension 1 in $\mathfrak{k}$ and $\mathfrak{z} \subseteqq \mathfrak{k}$. Fix an element $Z \in \mathfrak{z} \setminus \{0\}$, and note that there exists an element $Y \in \mathfrak{g}$ such that $[\mathfrak{g}, Y] = \mathsf{R}Z$. The subspace $\mathfrak{g}_0 := \{V \in \mathfrak{g} \mid [V, Y] = 0\}$ is an ideal of codimension 1 in $\mathfrak{g}$, so the induction hypothesis applies to $\mathfrak{g}_0$.

(B) $\mathfrak{k} \subseteqq \mathfrak{g}_0$.

(C) $\mathfrak{k} \nsubseteq \mathfrak{g}_0$.

In this case there is an element $X \in \mathfrak{h}$ such that $[X, Y] = Z$. Furthermore $Y \notin \mathfrak{k}$ because $\mathfrak{k} \in S(\alpha, \mathfrak{g})$ and $\alpha(Z) \neq 0$.

Let $\mathfrak{h}_0 := \mathfrak{h} \cap \mathfrak{g}_0$. Since $X \notin \mathfrak{g}_0$, we get the following identities of direct sums of vector spaces:

$$\mathfrak{g} = \mathsf{R}X + \mathfrak{g}_0 \quad \text{and} \quad \mathfrak{k} := \mathsf{R}Z + \mathsf{R}X + \mathfrak{h}_0 .$$

Let $\mathfrak{k}' := \mathsf{R}Z + \mathsf{R}Y + \mathfrak{h}_0$. Then $\mathfrak{k}'$ is a direct sum of vector spaces, it is a subalgebra of $\mathfrak{g}$ which is a subordinate α, and finally $\mathfrak{k}'$ is of maximal dimension subordinate α because $\dim \mathfrak{k}' = \dim \mathfrak{k}$. Since $\mathfrak{k}' \subseteqq \mathfrak{g}_0$, it follows from (B) that the representation $\lambda_{\alpha, \mathfrak{k}'}$ satisfies the conclusion of the theorem.

$$\mathscr{D}(G;\, K) := \{\varphi \in \mathscr{E}(G) \mid \varphi \text{ has compact support modulo } K\}$$

is a subspace of $\mathscr{E}(G)$, invariant under left translations.

Since there is a coexponential basis for $\mathfrak{g}$ mod $\mathfrak{k}$ with Y as a member, it follows that the function $y \to \varphi(\exp(yY))$ belongs to $\mathscr{D}(\mathsf{R})$ for any $\varphi \in \mathscr{D}(G;\, K)$. The prescription

$$(F\varphi)(g) := \int_{-\infty}^{\infty} \varphi(g \exp(yY))\, dy \quad \text{for } g \in G,\ \varphi \in \mathscr{D}(G;\, K)$$

clearly defines a linear mapping $F\colon \mathscr{D}(G;\, K) \to \mathscr{E}(G)$. It is easily checked that $F(\mathscr{D}_{\alpha,\mathfrak{k}}) \subseteqq \mathscr{E}_{\alpha,\mathfrak{k}'}$, where $\mathscr{D}_{\alpha,\mathfrak{k}} := \mathscr{E}_{\alpha,\mathfrak{k}} \cap \mathscr{D}(G;\, K)$. Furthermore F commutes with left translations so that

$$(*) \qquad F\lambda_{\alpha,\mathfrak{k}}(g)\varphi = \lambda_{\alpha,\mathfrak{k}'}(g)F\varphi \quad \text{for all } g \in G,\ \varphi \in \mathscr{D}_{\alpha,\mathfrak{k}} .$$

To see what are the properties of F, we introduce coordinates:

Note that the direct sum of vector spaces

$$\tilde{\mathfrak{g}} := \mathsf{R}Y + \mathsf{R}Z + \mathsf{R}X + \mathfrak{h}_0$$

is in fact a subalgebra of $\mathfrak{g}$ of codimension $n \geqq 0$. So if we let $\{X_1, \ldots, X_n\}$ denote a coexponential basis mod $\tilde{\mathfrak{g}}$ for $\mathfrak{g}$, then $\{X_1, \ldots, X_n, Y\}$ is a coexponential basis mod $\mathfrak{k}$ for $\mathfrak{g}$ while $\{X_1, \ldots, X_n, X\}$ is a coexponential basis mod $\mathfrak{k}'$ for $\mathfrak{g}$.

From Theorem 3.1 (a) we get the topological isomorphism

$$\Psi := s^*\colon \mathscr{E}_{\alpha,\mathfrak{k}}(G) \to \mathscr{E}(\mathsf{R}^{n+1})$$

given by

$$(\Psi f)(t_1, \ldots, t_n, y) = f(\exp(t_1 X_1) \ldots \exp(t_n X_n) \exp(yY)) .$$

As is easily seen, Ψ restricts to an isomorphism,

$$\Psi: \mathscr{D}_{\alpha,\mathfrak{t}} \to \mathscr{D}(\mathsf{R}^{n+1}).$$

Similarly we get another topological isomorphism

$$\Psi' := s'^*: \mathscr{E}_{\alpha,\mathfrak{t}'}(G) \to \mathscr{E}(\mathsf{R}^{n+1})$$

given by

$$(\Psi' f)(t_1,\ldots,t_n,x) = f(\exp(t_1X_1)\ldots\exp(t_nX_n)\exp(xX)).$$

These two topological isomorphisms define the representations λ and λ' corresponding to $\lambda_{\alpha,\mathfrak{t}}$ and $\lambda_{\alpha,\mathfrak{t}'}$ from the statement of the theorem.

An explicit calculation reveals that the operator

$$\Psi' F \Psi^{-1}: \mathscr{D}(\mathsf{R}^{n+1}) \to \mathscr{E}(\mathsf{R}^{n+1})$$

equals $F_{\alpha(Z)}$, where

$$F_c: \mathscr{D}(\mathsf{R}^{n+1}) \to \mathscr{E}(\mathsf{R}^{n+1}) \quad \text{for any } c \in \mathsf{C}$$

is given by

$$(F_c\varphi)(t,x) = \int_{-\infty}^{\infty} \varphi(t,y)e^{-icxy}\,dy \quad \text{for } \varphi \in \mathscr{D}(\mathsf{R}^{n+1}).$$

We collect the properties of F_c that we need in a lemma.

4.2. LEMMA. *Let $c \neq 0$. Then F_c is continuous and injective, and to every polynomial differential operator E on R^{n+1} there exists another E' such that*

$$F_cE\varphi = E'F_c\varphi \quad \textit{for all } \varphi \in \mathscr{D}(\mathsf{R}^{n+1}).$$

PROOF. The continuity is an immediate consequence of the closed graph theorem.

To prove the injectivity, we assume that $F_c\varphi = 0$. Then for any fixed $\underline{t} \in \mathsf{R}^n$ the function

$$z \mapsto \int_{-\infty}^{\infty} \varphi(\underline{t},y)e^{-iczy}\,dy$$

is entire and vanishes on R, so it is identically 0. In particular, the ordinary Fourier transform of $y \mapsto \varphi(\underline{t},y)$ is 0, and so $\varphi = 0$.

Since F_c commutes with differential operators only involving the $\underline{t}$-variables, it suffices to prove the last statement in the cases $E = y$ and $E = \partial/\partial y$. But here $E' = (-ic)^{-1}\partial/\partial x$ and $E' = icx$ work.

We continue with the proof of the Theorem. It follows from (*) and the continuity of F_c, where $c=\alpha(Z)$, that

$$F_c d\lambda(D)\varphi = d\lambda'(D)F_c\varphi \quad \text{for all } \varphi \in \mathscr{D}(\mathsf{R}^{n+1})\,.$$

Let now E be an arbitrary polynomial differential operator on R^{n+1}. Let E' be the polynomial differential operator from the Lemma.

Since $\lambda_{\alpha,\mathfrak{k}'}$ as noted earlier satisfies the conclusions of the theorem, there exists $D \in \boldsymbol{D}(G)$ with $d\lambda'(D)=E'$. Thus

$$F_c d\lambda(D)\varphi = d\lambda'(D)F_c\varphi = E'F_c\varphi = F_c E\varphi$$

for all $\varphi \in \mathscr{D}(\mathsf{R}^{n+1})$. By injectivity

$$d\lambda(D)\varphi = E\varphi \quad \text{for all } \varphi \in \mathscr{D}(\mathsf{R}^{n+1})\,.$$

Hence the two differential operators $d\lambda(D)$ and E coincide.

5. Irreducibility statements.

5.1. THEOREM. *Let G be a simply connected, connected, nilpotent Lie group. If $\alpha \in \mathsf{C}\mathfrak{g}^*$ and $\mathfrak{k} \in M(\alpha;\mathfrak{g})$ then the representations $\lambda_{\alpha,\mathfrak{k}}$ and $\Lambda_{\alpha,\mathfrak{k}}$ are both topologically and operator irreducible.*

PROOF. If $\alpha=0$ then the representations are one-dimensional and the theorem is trivially true, so we will in the rest of the proof assume $\alpha \neq 0$.

The representation $\lambda_{\alpha,\mathfrak{k}}$ is by Theorem 3.1 (b) equivalent to a representation λ of G on R^n of the form

$$(*) \qquad [\lambda(g)f](x) = \exp[i\alpha(p(g,x))]\,[\tau(g)f](x)\,,$$

where p is a certain map of $G \times \mathsf{R}^n$ into $\mathfrak{k}$ and where τ denotes the transitive action of G on $\mathsf{R}^n \cong G/\exp(\mathfrak{k})$. Moreover by Theorem 4.1, $d\lambda(\boldsymbol{D}(G))$ is the algebra of all differential operators on R^n with polynomial coefficients. Similarly for $\Lambda_{\alpha,\mathfrak{k}}$.

Now the topological irreducibility of λ:

Let $E \neq \{0\}$ be a closed λ-invariant subspace of $\mathscr{E}(\mathsf{R}^n)$ and let $u_0 \in E^{\perp} \subseteq \mathscr{E}'(\mathsf{R}^n)$. By the Hahn–Banach theorem it suffices to show that $u_0=0$. Since E is invariant under polynomial differential operators we get for all polynomials p and all $e \in E$ that

$$\langle u_0, pe\rangle = 0\,.$$

The polynomials are dense in $\mathscr{E}(\mathsf{R}^n)$ (see f.ex. [9]), so

$$\langle u_0, fe\rangle = 0 \quad \text{for all } f \in \mathscr{E}(\mathsf{R}^n) \text{ and } e \in E\,,$$

and thus

$$eu_0 = 0 \quad \text{for all } e \in E .$$

Now $\tau(g)$ acts transitively on $\mathbf{R}^n$ and $\lambda(g)$ and $\tau(g)$ differ according to the formula (*) only by a nowhere vanishing factor, so for any $x \in \mathbf{R}^n$ there is an $e \in E$ with $e(x) \neq 0$. Consequently $u_0 = 0$.

Minor modifications of the proof just given yields the topological irreducibility of $\Lambda_{\alpha,\mathfrak{k}}$: Replace $\mathscr{E}$ by $\mathscr{D}'$ and observe that $\mathscr{D}$ is reflexive (it is a Montel space).

The operator irreducibility of $\lambda_{\alpha,\mathfrak{k}}$ and $\Lambda_{\alpha,\mathfrak{k}}$ is an immediate consequence of the following lemma.

5.2. LEMMA. *Let A be densely defined, closed linear operator on $\mathscr{E}(\mathbf{R}^n)$ or $\mathscr{D}(\mathbf{R}^n)$ with the property that $PA \subseteq AP$ for every polynomial differential operator P on $\mathbf{R}^n$. Then $A = cI$ for some constant $c \in \mathbf{C}$.*

PROOF. Since the polynomials are dense in $\mathscr{E}(\mathbf{R}^n)$ and A is closed we deduce that the domain of definition $D(A)$ of A is closed under multiplication by C^∞-functions and that A commutes with such multiplications.

A is densely defined, so there exists for each $x \in \mathbf{R}^n$ a function $g \in D(A)$ with $g(x) \neq 0$. Since $D(A)$ is closed under multiplications $|g|^2 \in D(A)$, so we may take $g \geqq 0$, and choose a locally finite partition of unity $\{\psi_k\} \subseteq \mathscr{E}(\mathbf{R}^n)$ and corresponding functions $g_k \in D(A)$ so that

$$f_0(x) := \sum_1^\infty \psi_k(x) g_k(x) > 0 \quad \text{for all } x \in \mathbf{R}^n .$$

The sum $g_0 := \sum_1^\infty \psi_k A g_k \in \mathscr{E}(\mathbf{R}^n)$. Since A commutes with multiplications we get for any $\varphi \in D(A)$ that

$$A\varphi = A\left(\frac{1}{f_0}\sum_1^\infty \psi_k g_k \varphi\right) = \frac{g_0}{f_0}\varphi ,$$

so A is multiplication by $g_0/f_0 \in \mathscr{E}(\mathbf{R}^n)$. But then A is everywhere defined. Furthermore, since A commutes with differentiation the function g_0/f_0 is a constant.

5.3. REMARK. Theorem 5.1 and hence also Theorem 4.1 are not true in general if $\alpha \in \mathbf{C}\mathfrak{g}^*$ is replaced by $\alpha \in (\mathfrak{g}^*)^{\mathbf{C}}$: Using the notation of [2] we consider $\mathfrak{g} = \mathfrak{g}_{5,4}$, $\mathfrak{k} = \operatorname{span}\{X_3, X_4, X_5\}$, let $\alpha \in (\mathfrak{g}^*)^{\mathbf{C}}$ satisfy $\alpha(X_3) = 0$, $\alpha(X_4) = 1$, $\alpha(X_5) = i$ and choose the coexponential basis $\{X_1, X_2\}$ mod $\mathfrak{k}$ for $\mathfrak{g}$. The center of $\boldsymbol{D}(G)$ contains $D := 2X_1X_5 - 2X_2X_4 + X_3^2$ ([2 p. 326]). Computations show that

$$(d\lambda(D)f](x,y) = 2\frac{\partial f}{\partial x}+2i\frac{\partial f}{\partial y}+y^2 f \quad \text{for } f \in \mathscr{E}(\mathbf{R}^2)\,.$$

It follows that $\lambda_{\alpha,\mathfrak{k}}$ is neither scalar nor topologically irreducible.

6. Representations on homogeneous manifolds.

In this section we apply our results to Helgason's problems, mentioned in the introduction. The answer to (A) which is expressed by Theorem 6.1 below, is due to A. Hole (Theorem 3.7 of [6]). We give an independent proof that connects it to the eigenspace representations.

6.1. THEOREM. *Let G be a simply connected, connected, nilpotent Lie group. Let $\alpha \in \mathfrak{g}^* \setminus (0)$ and $k \in M(\alpha;\, \mathfrak{g})$.*

Then $\mathfrak{h} := \mathfrak{k} \cap \ker\alpha$ is a proper ideal in $\mathfrak{k}$. If H denotes the analytic subgroup of G corresponding to $\mathfrak{h}$ then we have for any $Z \in \mathfrak{k} \setminus \mathfrak{h}$ that

$$\boldsymbol{D}(G/H) = \operatorname{Pol}(\gamma(Z))\,,$$

where $\gamma(Z)$ is the non-zero vector field on G/H given by

$$[\gamma(Z)f](gH) = \left.\frac{d}{dt}\right|_{t=0} f(g\exp(tZ)H) \quad \text{for } f \in \mathscr{E}(G/H),\ g \in G\,.$$

PROOF. $\mathfrak{h}$ is an ideal in $\mathfrak{k}$ (of codim 1) so the formula for $\gamma(Z)$ defines a non-zero element of $\boldsymbol{D}(G/H)$.

We may assume that $[\mathfrak{g}, Z] \subseteqq \mathfrak{h}$ [Let $\mathfrak{i} \subseteqq \mathfrak{g}$ be maximal among the ideals of $\mathfrak{g}$ contained in $\mathfrak{h}$, and let $Z_1 \in \mathfrak{g}$ represent a non-zero element of the center of $\mathfrak{g}/\mathfrak{i}$. Then $[\mathfrak{g}, Z_1] \subseteqq \mathfrak{i} \subseteqq \mathfrak{h}$ and $Z_1 \in \mathfrak{k} \setminus \mathfrak{h}$. Replace Z by Z_1.] Then $\exp(tZ)gH = g\exp(tZ)H$ for all $g \in G$ and $t \in \mathbf{R}$, which implies that $\gamma(Z)$ belongs to the center of $\boldsymbol{D}(G/H)$. In particular, if $D \in \boldsymbol{D}(G/H)$ and $b \in \mathbf{C}$ then D leaves the eigenspace

$$E_b := \{f \in \mathscr{E}(G/H) \mid \gamma(Z)f = bf\}$$

invariant.

Now it is easy to see that the natural representation of G on E_b is equivalent (under the pull back of $\pi\colon G \to G/H$) to $\lambda_{a\alpha,\mathfrak{k}}$, where $a = i\alpha(Z)^{-1}b$. It follows by Theorem 5.1 that D acts as a scalar on each $E_b, b \neq 0$.

We identify G/H with $\mathbf{R}^{n+1}$ via a coexponential basis $\{X_1, \ldots, X_n, Z\}$ for $\mathfrak{g} \bmod \mathfrak{h}$ and denote the corresponding coordinates by $x_1, \ldots, x_n, z$. Then $\gamma(Z) = \partial/\partial z$ and $E_b = \mathscr{E}(\mathbf{R}^n) \otimes e^{bz}$.

The scalar action of D on each E_b implies that D is a polynomial in $\partial/\partial z$.

6.2. COROLLARY. *Let the assumptions and the notation be as in the theorem. Let furthermore χ be a character on the algebra $\mathbf{D}(G/H)$. Then*

(a) *the natural representation L of G on the joint eigenspace*

$$\{f \in \mathscr{E}(G/H) \mid Df = \chi(D)f, \ \forall D \in \mathbf{D}(G/H)\}$$

is equivalent to $\lambda_{a\alpha, \mathfrak{k}}$, where

$$a = i\chi(\gamma(Z))\alpha(Z)^{-1} .$$

(b) *If $\chi \neq 0$, or $\chi = 0$ and $\mathfrak{k} = \mathfrak{g}$, then L is both topologically and operator irreducible.*

(c) *If $\chi = 0$ and $\mathfrak{k} \neq \mathfrak{g}$ then L is neither topologically nor scalar irreducible.*

The corollary remains true if $\mathscr{E}$ is replaced by $\mathscr{D}'$ in the statement.

PROOF. (a) was already noted in the proof of the theorem.

(b) If $\chi \neq 0$ then $a \neq 0$ so the result follows from Theorem 5.1.
If $\chi = 0$ and $\mathfrak{k} = \mathfrak{g}$ then $\mathscr{E}_{a\alpha, \mathfrak{k}} = \mathscr{E}_{0, \mathfrak{g}}$ is one-dimensional.

(c) Here $a\alpha = 0$ so $\lambda_{a\alpha, \mathfrak{k}}$ reduces by Theorem 3.1(b) to the natural representation of G on $\mathscr{E}(G/K)$.
The constants on G/K form an invariant subspace of the infinite dimensional space $\mathscr{E}(G/K)$ so the representation is not topologically irreducible.
It is not scalar irreducible either: Let $X \in \mathfrak{n}(\mathfrak{k}) \setminus \mathfrak{k}$ where $\mathfrak{n}(\mathfrak{k})$ is the normaliser of $\mathfrak{k}$ in $\mathfrak{g}$. Then the corresponding operator $\mathscr{E}(G/K)$ is a continuous, non-trivial, intertwining operator.
The same proof works in the distribution case.

6.3. REMARK. We mention without details how the unitary theory fits into our setup: Let $\alpha \in \mathfrak{g}^*$, let $\mathfrak{k} \in M(\alpha; \mathfrak{g})$, and let χ be the character on $\exp(\mathfrak{k})$ given by

$$\chi(\exp X) = e^{i\alpha(X)} \quad \text{for } X \in \mathfrak{k} .$$

When χ is unitary then the induced unitary representation $\chi \uparrow G$ is a subrepresentation of $\Lambda_{\alpha, \mathfrak{k}}$. Since any continuous, unitary, irreducible representation U is induced from a character of the above form, each such U is equivalent to a subrepresentation of the natural representation on a joint eigenspace of the form

$$\{u \in \mathscr{D}'(G/H) \mid Du = \mu(D)u, \quad \forall D \in \mathbf{D}(G/H)\}$$

from Helgason's problems. In this sense the eigenspace representations contain all the continuous unitary irreducible representations of G.

7. A counterexample.

In the theory of unitary representations of nilpotent Lie groups any two subalgebras $\mathfrak{k}, \mathfrak{k}'$ both of maximal dimension subordinate $\alpha \in \mathfrak{g}^*$, that is $\mathfrak{k}, \mathfrak{k}' \in M(\alpha, \mathfrak{g})$ give rise to equivalent unitary representations of the group. In this section we will show by a simple example that the corresponding result is false for our eigenspace representations, even when the notion "Equivalent" is replaced by "Naimark related".

7.1. EXAMPLE. The 3-dimensional Heisenberg group is a simply connected, connected, nilpotent Lie group G whose Lie algebra $\mathfrak{g}$ is spanned by three elements X, Y and Z satisfying the commutation relations $[X, Y]=Z$, $[X, Z] =[Y, Z]=0$.

Let $\alpha \in \mathfrak{g}^*$ be given by $\alpha(X)=\alpha(Y)=0$ and $\alpha(Z)=1$, and consider

$$\mathfrak{k} := \operatorname{span}\{X, Z\} \quad \text{and} \quad \mathfrak{k}' := \operatorname{span}\{Y, Z\} .$$

Then $\mathfrak{k}, \mathfrak{k}' \in M(\alpha, \mathfrak{g})$.

CLAIM. The eigenspace representations $\Lambda_{\alpha,\mathfrak{k}}$ and $\Lambda_{\alpha,\mathfrak{k}'}$ are not even "Naimark related".

PROOF OF THE CLAIM. Let us first realize the representations on $\mathscr{D}'(\mathbf{R})$:

Let K denote the analytic subgroup of G corresponding to $\mathfrak{k}$. Since $\{Y\}$ is a coexponential basis for $\mathfrak{g} \bmod \mathfrak{k}$ we can identify G/K with $\mathbf{R}$ by means of the diffeomorphism $\exp(yY)K \mapsto y$.

It follows via Theorem 3.1(b) that $\Lambda_{\alpha,\mathfrak{k}}$ is equivalent to the representation Λ of G on $\mathscr{D}'(\mathbf{R})$ given by

$$[\Lambda(\exp(x_0 X))u](y) = e^{iyx_0}u(y) ,$$

$$[\Lambda(\exp(y_0 Y))u](y) = u(y-y_0) ,$$

$$[\Lambda(\exp(z_0 Z))u](y) = e^{iz_0}u(y) .$$

Similarly $\Lambda_{\alpha,\mathfrak{k}'}$ is equivalent to the representation Λ' of G on $\mathscr{D}'(\mathbf{R})$ given by

$$[\Lambda'(\exp(x_0 X))u](x) = u(x-x_0) ,$$

$$[\Lambda'(\exp(y_0 Y))u](x) = e^{-ixy_0}u(x) ,$$

$$[\Lambda'(\exp(z_0 Z))u](x) = e^{iz_0}u(x) .$$

Let τ be the left regular representation of $\mathbf{R}$ on $\mathscr{D}'(\mathbf{R})$. Let χ be the representation of $\mathbf{R}$ on $\mathscr{D}'(\mathbf{R})$ given by multiplication with the character $\chi(x_0)$, where $\chi(x_0)(x)=e^{-ixx_0}$. To prove the claim it suffices to show the following result:

7.2. PROPOSITION. *$A=0$ is the only densely defined, closed linear operator on $\mathscr{D}'(\mathsf{R})$ that satisfies the relations*

$$A\tau(x) = \chi(x)A \quad \textit{and} \quad A\chi(-x) = \tau(x)A \quad \text{for all } x \in \mathsf{R}\,.$$

PROOF OF THE PROPOSITION. The integrated representations of τ and χ are given by the formulae

$$\tau(\varphi)u = \varphi * u \quad \text{and} \quad \chi(\varphi)u = \hat{\varphi}u \quad \text{for } \varphi \in \mathscr{D}(\mathsf{R}) \text{ and } u \in \mathscr{D}'(G)\,.$$

From the second relation or rather its equivalent trasposed $\chi(x)A^t = A^t\tau(x)$ we get by integration for any $\varphi \in \mathscr{D}(\mathsf{R})$, $\psi \in D(A^t)$ that

$$\varphi * \psi \in D(A^t) \quad \text{and} \quad A^t(\varphi * \psi) = \hat{\varphi}A^t\psi\,.$$

By commutativity

$$\hat{\varphi}A^t\psi = \hat{\psi}A^t\varphi$$

and in particular

$$\hat{\varphi}(0)(A^t\psi)(0) = \hat{\psi}(0)(A^t\varphi)(0) \quad \text{for all } \varphi, \psi \in D(A^t)\,.$$

Since A^t is densely defined as the transposed of a densely defined, closed operator there exists $\varphi \in D(A^t)$ with $\hat{\varphi}(0)=1$. Then.

$$\langle \delta, A^t\psi \rangle = (A^t\varphi)(0)\langle 1, \psi \rangle \quad \text{for all } \psi \in D(A^t)\,.$$

Now $(A^t)^t = A$ so it follows that

$$\delta \in D(A) \quad \text{and} \quad A\delta = c1 \quad \text{where } c \text{ is a constant}\,.$$

When we integrate the first relation we find for any $u \in D(A)$ and $\varphi \in \mathscr{D}(\mathsf{R})$ that $\varphi * u \in D(A)$ and $A(\varphi * u) = \hat{\varphi}Au$. Substituting $u = \delta$ we get

$$\mathscr{D}(\mathsf{R}) \subseteqq D(A) \quad \text{and} \quad A\varphi = c\hat{\varphi} \quad \text{for all } \varphi \in \mathscr{D}(\mathsf{R})\,.$$

Returning to the operator A^t we get for any $\varphi \in \mathscr{D}(\mathsf{R})$ and $\psi \in D(A^t)$ that

$$\langle \varphi, A^t\psi \rangle = \langle A\varphi, \psi \rangle = c\langle \hat{\varphi}, \psi \rangle = c\langle \varphi, \hat{\psi} \rangle\,.$$

Hence $A^t\psi = c\hat{\psi}$ for all $\psi \in D(A^t)$.

The left hand side is compactly supported since A^t is an operator in $\mathscr{D}(\mathsf{R})$, but if $\psi \neq 0$, then the right hand side is not, unless of course $c=0$. We conclude that $c=0$. Then $A^t=0$ and so $A=0$.

7.3. COROLLARY. *$\lambda_{\alpha,\mathfrak{k}}$ and $\lambda_{\alpha,\mathfrak{k}'}$ are not Naimark related.*

PROOF. The only essential change from the distribution case is the proof of

Proposition 7.2 (with $\mathscr{D}'$ replaced by $\mathscr{E}$): For any $\varphi, \psi \in \mathscr{D}(\mathsf{R})$ and any $f \in D(A)$ we get by integration that

$$\hat{\tilde{\psi}}(\varphi * f) \in D(A) \quad \text{and} \quad A[\hat{\tilde{\psi}}(\varphi * f)] = \psi * (\hat{\varphi} A f)$$

where $\tilde{\psi}(x) = \psi(-x)$ for $x \in \mathsf{R}$.

Since the map $u \mapsto \varphi * u$ is continuous from $\mathscr{D}'(\mathsf{R})$ to $\mathscr{E}(\mathsf{R})$ for fixed $\varphi \in \mathscr{D}(\mathsf{R})$, it follows that A has a closure as an operator in $\mathscr{D}'(\mathsf{R})$. Thus $A = 0$ by the distribution case.

Similar phenomena as in the example above occur in general. In fact, using the results of [8] it is easy to prove the following proposition that we here cite without proof:

7.4. PROPOSITION. *Let G be a simply connected, connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}, \mathfrak{k} \in M(\alpha, \mathfrak{g})$ for some $\alpha \in \mathsf{C}\mathfrak{g}^*$.*

If the representations $\lambda_{\alpha,\mathfrak{h}}$ and $\lambda_{\alpha,\mathfrak{k}}$ are equivalent then $\mathfrak{h} = \mathfrak{k}$.

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