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# ON HSIAO'S CONJECTURE ON HECKE GROUPS

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## 1.

Let $q \geqq 3$ be a rational integer, and put

$$\zeta_q = \exp(\pi i/q), \quad \lambda_q = \zeta_q + \zeta_q^{-1} = 2\cos\frac{\pi}{q}.$$

The Hecke group $G(\lambda_q)$ is generated by the transformations

$$U: \tau \mapsto \tau + \lambda_q \quad \text{and} \quad T: \tau \mapsto -\tau^{-1}$$

of the upper half plane. These groups were introduced by E. Hecke [1, Nr. 33] when he found his famous correspondence between Dirichlet series with functional equation on the one hand and entire modular forms for $G(\lambda_q)$ on the other hand. Recently Hong-Jen Hsiao [2] studied pairs of Dirichlet series which are related by a functional equation. He proved that these pairs correspond to entire modular forms for the subgroup $H(\lambda_q)$ of $G(\lambda_q)$ which is generated by $U$ and

$$L = TUT: \tau \mapsto \tau/(-\lambda_q \tau + 1).$$

The relation $TU^k = L^k T$ shows that $H(\lambda_q)$ has index at most 2 in $G(\lambda_q)$. If $q$ is odd then the formula

$$T = (TU)^q T = L(UL)^{(q-1)/2} \in H(\lambda_q)$$

proves $H(\lambda_q) = G(\lambda_q)$. (This formula can be obtained from [1, p. 613].) Using some formulae from M. Knopp [3] Hsiao proved

$$[G(\lambda_q): H(\lambda_q)] = 2 \tag{1}$$

if $q = 4$ or $q = 6$, and he conjectures this to be true for all even $q \geqq 4$. In 2 below I give a simple proof of this conjecture. As in [2] this result yields examples of pairs of Dirichlet series $\varphi \neq \psi$ which are related by a functional equation; this is shown in 3.

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**2.**

It is convenient to use the matrices

$$\tilde{U} = \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}, \quad \tilde{T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tilde{L} = \tilde{T}\tilde{U}\tilde{T} = \begin{pmatrix} -1 & 0 \\ \lambda_q & -1 \end{pmatrix}.$$

Let $\tilde{G}(\lambda_q)$ be the group which is generated by $\tilde{U}$ and $\tilde{T}$, and let $\tilde{H}(\lambda_q)$ be the group which is generated by $\tilde{U}$ and $\tilde{L}$. Then $G(\lambda_q)$ and $H(\lambda_q)$ are obtained from $\tilde{G}(\lambda_q)$ and $\tilde{H}(\lambda_q)$ by factoring out the center

$$\left\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}.$$

We introduce the set $S_q$ which consists of all matrices

$$\pm\begin{pmatrix} 1+\lambda_q^2 p_1(\lambda_q^2) & \lambda_q p_2(\lambda_q^2) \\ \lambda_q p_3(\lambda_q^2) & 1+\lambda_q^2 p_4(\lambda_q^2) \end{pmatrix}$$

with arbitrary polynomials $p_1, p_2, p_3, p_4$ in $\mathbf{Z}[X]$. Obviously, $\tilde{U}$ and $\tilde{L}$ belong to $S_q$, and the product of any two matrices in $S_q$ is again in $S_q$. Therefore,

$$\tilde{H}(\lambda_q) \subset S_q .$$

We assume that $q \geqq 4$ is even, and we consider the field $K_{2q} = \mathbf{Q}(\zeta_q)$ of $(2q)$-th roots of unity and its maximal real subfield $K_{2q}^* = K_{2q} \cap \mathbf{R} = \mathbf{Q}(\lambda_q)$. Let us suppose that $\tilde{T} \in S_q$. Then $\lambda_q \cdot p_2(\lambda_q^2) = 1$ for some $p_2 \in \mathbf{Z}[X]$, whence $\lambda_q \in \mathbf{Q}(\lambda_q^2)$, and $K_{2q}^* = \mathbf{Q}(\lambda_q^2)$. But $\mathbf{Q}(\lambda_q^2) = \mathbf{Q}(\zeta_q^2 + \zeta_q^{-2}) = K_q^*$ is the maximal real subfield of $K_q = \mathbf{Q}(\zeta_q^2)$. From

$$[K_{2q} \colon K_{2q}^*] = 2, \quad [K_q \colon K_q^*] = 2, \quad K_q \subset K_{2q}$$

and $K_q^* = K_{2q}^*$ we conclude that $K_q = K_{2q}$. This is impossible since $q$ is even. This contradiction shows that $\tilde{T} \notin S_q$, and hence that $\tilde{T} \notin \tilde{H}(\lambda_q)$. Thus Hsiao's conjecture (1) is proved for all even $q \geqq 4$.

We note that the identity and $T$ represent the cosets of $H(\lambda_q)$ in $G(\lambda_q)$. This will be used in the last section.

**3.**

The groups $G(\lambda_q)$ and $H(\lambda_q)$ meet the requirements in [4, chapter VIII]. Therefore ([4, p. 282]), for every even $k \geqq 4$, the Eisenstein series $E_k(\tau)$ and $F_k(\tau)$ for $G(\lambda_q)$ and $H(\lambda_q)$, respectively, converge on the upper half plane and define entire modular forms of weight $k$ for these groups. They are defined as follows. For any

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{R}),$$

put

$$j_M(\tau) = \frac{d(M\tau)}{d\tau} = (c\tau+d)^{-2}.$$

The subgroups of translations in $G(\lambda_q)$ and in $H(\lambda_q)$ are both generated by $U$; we denote this group by $G_\infty$. Then

$$E_k(\tau) = \sum_{M:G_\infty \setminus G(\lambda_q)} (j_M(\tau))^{k/2}, \quad F_k(\tau) = \sum_{M:G_\infty \setminus H(\lambda_q)} (j_M(\tau))^{k/2},$$

where $M$ runs through a complete set of representatives of cosets $G_\infty M$ in $G(\lambda_q)$ and in $H(\lambda_q)$, respectively. These functions are not identically 0 because of $E_k(i\infty) = F_k(i\infty) = 1$.

Now, for $k \geqq 4$ even, choose $r = -i^k$ and define

$$F_k^*(\tau) = r\left(\frac{\tau}{i}\right)^{-k} F_k\left(-\frac{1}{\tau}\right) = -\sum_{M:G_\infty \setminus H(\lambda_q)} (j_{MT}(\tau))^{k/2}.$$

Then, by the remark at the end of section 2, $F_k - F_k^* = E_k \neq 0$. Let $\varphi$ and $\psi$ denote the Dirichlet series associated with $F_k$ and $F_k^*$, respectively. Then $\varphi \neq \psi$, and the functional equation

$$(2\pi/\lambda_q)^{-s}\Gamma(s)\varphi(s) = r(2\pi/\lambda_q)^{-(k-s)}\Gamma(k-s)\psi(k-s)$$

holds, as explained in [2].

In closing this note I would like to call the reader's attention to [5] and [6] and related work on Hecke groups which is listed there.

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## REFERENCES

1. E. Hecke, *Mathematische Werke*, Vandenhoeck und Ruprecht, Göttingen, 1959, 2. Aufl. 1970.
2. Hong-Jen Hsiao, *A remark on Hecke's theory on Dirichlet series with functional equation*, Chinese J. of Math. 6 (1978), 147–152.
3. M. Knopp, *Modular functions in analytic number theory*, Markham, Chicago, 1970.
4. J. Lehner, *Discontinuous groups and automorphic functions*, Math. Surveys no. 7, Amer. Math. Soc., Providence, 1964.
5. J. Wolfart, *Eine Bemerkung über Heckes Modulgruppen*, Arch. Math. (Basel), 29 (1977), 72–77.
6. J. Wolfart, *Transzendente Zahlen als Fourierkoeffizienten von Heckes Modulformen*, Acta Arithmetica, erscheint demnächst.

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