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UNIQUE SOLUTIONS OF SOME VOLTERRA INTEGRAL EQUATIONS

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1. Introduction and statement of results.

The purpose of this paper is to study the uniqueness of the trivial solution $x(t) \equiv 0$ of the integral inequality

$$0 \leqq x(t) \leqq \int_0^t k(t,s,x(s))\,ds, \qquad t \geqq 0\,. \tag{1.1}$$

This inequality arises if one considers the equations

$$y_i(t) = f(t) + \int_0^t g(t,s,y_i(s))\,ds, \qquad t \geqq 0,\ i=1,2$$

in some Banach space and takes $x(t) = \|y_1(t) - y_2(t)\|$ and assumes that

$$\|g(t,s,u_1) - g(t,s,u_2)\| \leqq k(t,s,\|u_1 - u_2\|)\,.$$

But the uniqueness of the trivial solution of (1.1) is also crucial in some other cases, e.g. in certain studies concerning the existence of solutions of abstract nonlinear Volterra equations, see [5].

One well-known approach to be uniqueness problem for (1.1) is to assume that the function k is Lipschitz-continuous in its third argument, for details see [3, Chap. II]. Here we will not make assumptions concerning Lipschitz-continuity, but the main idea still remains that the trivial solution is unique if and only if the function k is small enough in some sense. For another approach to the uniqueness of solutions of Volterra integral equations that relies on monotonicity methods, see e.g. [1, 2]. If the function k in (1.1) does not depend on t, then (1.1) is just an integrated form of a differential inequality, see [6] for this case.

The following well-known comparison theorem is, of course, very important since it allows us to replace (1.1) by another equation that may be easier to study. A proof of this theorem can be found in [3, Chap. II], (see also [4]).

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COMPARISON THEOREM. *Assume that $T>0$ and that*

(1.2) $k_i(t,s,u)$ *is nonnegative and measurable on* $[0,T]\times[0,T]\times\mathbf{R}$, $i=1,2$,

(1.3) $k_i(t,s,u) = 0$ *if* $s>t$ *or* $u\leqq 0$, $i=1,2$,

(1.4) *for every fixed* $(t,s)\in[0,T]\times[0,T]$, $k_i(t,s,u)$ *is a continuous function of* u, $i=1,2$,

(1.5) *for every* $M>0$ *there exists a measurable function* $m(t,s)$ *on* $[0,T]\times[0,T]$ *so that* $k_2(t,s,u)\leqq m(t,s)$ *if* $u\leqq M$, $(t,s)\in[0,T]\times[0,T]$ *and* $\sup_{t\in[0,T]}\int_0^T m(t,s)\,ds<\infty$,

(1.6) $\lim_{t\to t_0,\, t\in[0,T]}\sup\{\int_0^T |k_2(t,s,y(s))-k_2(t_0,s,y(s))|\,ds \mid y\in C([0,T];\mathbf{R});\ y(t)\leqq M,\ t\in[0,T]\}=0$ *for all* $M>0$ *and* $t_0\in[0,T]$,

(1.7) *there exists* $M_0>0$ *such that* $k_1(t,s,u_1)\leqq k_2(t,s,u_2)$ *if* $(t,s)\in[0,T]\times[0,T]$ *and* $u_1\leqq u_2\leqq M_0$,

(1.8) *the function* $x\in C([0,T];\mathbf{R})$ *satisfies* $x(t)\leqq\int_0^t k_1(t,s,x(s))\,ds$, $t\in[0,T]$.

Then there exists a number $T_0\in(0,T]$ *and a function* $y\in C([0,T_0];\mathbf{R})$ *such that*

$$y(t) = \int_0^t k_2(t,s,y(s))\,ds, \qquad t\in[0,T_0] \tag{1.9}$$

and

$$x(t) \leqq y(t), \qquad t\in[0,T_0]\,. \tag{1.10}$$

Observe that this theorem can be used in two different ways. If it is known that the equation (1.9) has only the trivial solution $y(t)\equiv 0$ then it follows that $x(t)\leqq 0$ on $[0,T_0]$. If on the other hand it is known that $x(t)>0$ when $t\in(0,T]$ then equation (1.9) must have a nontrivial solution.

The next theorem gives a large class of equations that can be used for comparison purposes, since it is possible to give precise information about the uniqueness of the trivial solution for these equations.

THEOREM 1. *Assume that* $\alpha>0$ *and that*

(1.11) $h\colon\mathbf{R}\to\mathbf{R}$ *is continuous on* $(0,\infty)$, *positive and nonincreasing on* $(0,\delta]$ *for some* $\delta>0$ *and* $h(u)=0$ *if* $u\leqq 0$,

(1.12) *for each* $p>0$ *there exists* $\delta_p\in(0,\delta]$ *such that* $uh(u)^p$ *is nondecreasing on* $[0,\delta_p]$.

Then

$$(1.13) \qquad x(t) = \int_0^t (t-s)^{\alpha-1} h(x(s))x(s)\,ds, \qquad t \in [0,T]$$

where $x \in C([0,T];\mathsf{R})$ *for some* $T>0$ *implies that* $x(t) \equiv 0$ *on* $[0,T]$ *if and only if*

$$(1.14) \qquad \int_0^\delta (uh(u)^{1/\alpha})^{-1}\,du = +\infty .$$

In the Comparison Theorem it is assumed that the function k_2 dominates the function k_1 for all t,s, see (1.7). In some special cases this assumption is unnecessarily strong as is seen from the following two theorems that also give us another class of equations that can be used for comparison purposes.

THEOREM 2a. *Assume that* $T>0$, $\alpha>0$, $p>1$, (1.11), (1.12) *and* (1.14) *hold and that*

(1.15) $k(t,s)$ *is nonnegative and measurable on* $[0,T]\times[0,T]$, $k(t,s)=0$ *if* $s>t$,

$$(1.16) \qquad \sup_{t\in[0,T]} \int_0^t k(t,s)^p (t-s)^{p(1-\alpha)-1}\,ds < \infty .$$

If $x \in C([0,T];\mathsf{R})$ *satisfies the inequality*

$$(1.17) \qquad x(t) \leqq \int_0^t k(t,s)h(x(s))x(s)\,ds, \qquad t \in [0,T]$$

then $x(t) \leqq 0$ *on* $[0,T]$.

THEOREM 2b. *Assume that* $T>0$, $\alpha>0$, $p>0$, (1.11), (1.12) *and* (1.15) *hold but* (1.14) *does not hold and that*

$$(1.18) \qquad \sup_{t\in[0,T]} \int_0^t k(t,s)\,ds < \infty ,$$

$$(1.19) \qquad \lim_{t\to t_0,\, t\in[0,T]} \int_0^T |k(t,s)-k(t_0,s)|\,ds = 0 \quad \textit{for all } t_0 \in [0,T]$$

$$(1.20) \qquad \sup_{t\in[0,T]} \int_0^t k(t,s)^{-p}(t-s)^{p(\alpha-1)-1}\,ds < \infty .$$

Then there exists a number $T_0>0$ *and a function* $x \in C([0,T_0];\mathsf{R})$ *such that*

$$x(t) = \int_0^t k(t,s)h(x(s))x(s)\,ds, \qquad t \in [0, T_0] \tag{1.21}$$

and

$$x(t) > 0, \qquad t \in (0, T_0]\,. \tag{1.22}$$

2. Proof of Theorem 1.

First we establish some useful properties of the function h.

LEMMA. *Assume that* (1.11) *and* (1.12) *hold and let* $p>0$ *be arbitrary. Then*

$$\int_0^u h(s)^p\,ds \leqq 2uh(u)^p \qquad \text{if } u \in [0, \delta_{2p}]\,. \tag{2.1}$$

Moreover, h *is locally Lipschitz-continuous on* $(0, \delta_p)$ *and*

$$d/du(uh(u)^p) \geqq 2^{-1}h(u)^p \qquad \text{for a.e. } u \in (0, \delta_{2p})\,. \tag{2.2}$$

PROOF. To establish (2.1) we have only to note that $u^{\frac{1}{2}}h(u)^p$ is nondecreasing on $[0, \delta_{2p}]$ and hence

$$\int_0^u h(s)^p\,ds \leqq u^{\frac{1}{2}}h(u)^p \int_0^u s^{-\frac{1}{2}}\,ds = 2uh(u)^p, \qquad u \in [0, \delta_{2p}]\,.$$

From (1.12) we know that $u^q h(u)$ is nondecreasing on $[0, \delta_{1/q}]$ and therefore it follows from (1.11) that for every $\varepsilon > 0$

$$|h(u+\varepsilon) - h(u)| \leqq (u+\varepsilon)^{-q}((u+\varepsilon)^q - u^q)h(u), \; u \in (0, \delta_{1/q} - \varepsilon)$$

and so

$$|h'(u)| \leqq qu^{-1}h(u), \qquad \text{a.e. } u \in (0, \delta_{1/q})\,.$$

Using this inequality we easily deduce that (2.2) holds.

Assume that $x \in C([0, T]; \mathbf{R})$ satisfies (1.13) but $x(t)$ is not zero for all $t \in [0, T]$. Then we may without loss of generality assume that $x(t) > 0$ on $(0, T]$. Observe that (1.12) and an application of the Comparison Theorem show that we can assume that x is nondecreasing on $[0, T_1]$ where $T_1 \in (0, T]$. Moreover, we choose T_1 to be such that $x(t) \leqq \delta_1$ on $[0, T_1]$. We are going to show that we get a contradiction if (1.14) holds.

The first case we will consider is when $\alpha = n+1$ where n is a nonnegative integer. The equation (1.13) is just the differential equation

$$x^{(n+1)}(t) = n!\,h(x(t))x(t), \; t \in [0, T_1], \qquad x^{(i)}(0) = 0, \; i = 0, 1, \ldots, n\,. \tag{2.3}$$

If $n=0$, then it is well-known that (1.14) implies that $x(t)\equiv 0$. Let $n>0$. We are going to show by induction that

$$(2.4)\qquad x^{(n+1-j)}(t) \leqq d_j(h(x(t))x(t))^{1/(j+1)}(x^{(n-j)}(t))^{j/(j+1)}, \qquad t\in[0,T_1]\,,$$

where d_j is a certain constant and $j=0,1,\ldots,n$. This inequality holds when $j=0$ by (2.3). Assume that (2.4) holds for a certain integer j and multiply both sides in the inequality by $(x^{(n-j)}(t))^{1/(j+1)}$ and integrate. This gives since $x^{(n-j)}(0)=x^{(n-j-1)}(0)=0$ and $h(x(t))x(t)$ is nondecreasing on $[0,T^1]$

$$(j+1)/(j+2)(x^{(n-j)}(t))^{(j+2)/(j+1)} \leqq d_j(h(x(t))x(t))^{1/(j+1)}x^{(n-j-1)}(t),\ t\in[0,T_1]$$

and we see that (2.4) holds with j replaced by $j+1$. If we take $j=n$ in (2.4) we have

$$x'(t) \leqq d_n h(x(t))^{1/(n+1)}x(t),\ t\in[0,T_1]$$

and an application of the Comparison Theorem shows that again we have to consider the case $n=0$. If (1.14) holds we get a contradiction as $\alpha=n+1$.

Next we assume that $\alpha=n+1-\beta$ where n is a nonnegative integer, $\beta\in(0,1)$ if $n\geqq 1$ and $\beta\in(0,2^{-1})$ if $n=0$. By (1.11) and (1.13) we conclude that

$$(2.5)\qquad x'(t) \leqq \int_0^t (t-s)^{n-\beta}h(x(s))x'(s)\,ds, \qquad \text{a.e. } t\in[0,T_1]$$

and

$$(2.6)\qquad x^{(n+1)}(t) \leqq c_1\int_0^t (t-s)^{-\beta}h(x(s))x'(s)\,ds \qquad \text{a.e. } t\in[0,T_1]$$

where c_1 is a certain constant. We substitute the inequality (2.5) on the right side in (2.6) and use the fact that $h(x(t))$ is nonincreasing on $[0,T_1]$. Hence there exists a constant c_2 such that

$$(2.7)\quad x^{(n+1)}(t) \leqq c_2\int_0^t (t-s)^{n+1-2\beta}h(x(s))^2x'(s)\,ds, \qquad \text{a.e. } t\in[0,T_1]\,.$$

(Recall that $\int_0^t (t-s)^a s^b\,ds=\Gamma(a+1)\Gamma(b+1)/\Gamma(a+b+2)t^{a+b+1}$, $a,b>-1$). Choose $T_2\in(0,T_1]$ so that $x(t)\leqq\delta_{2(n+1)/\beta}$ on $[0,T_2]$. Invoking (2.2) we deduce that an integrating by parts in (1.13) yields

$$(2.8)\qquad x(t) \geqq c_3\int_0^t (t-s)^{n+1-\beta}h(x(s))x'(s)\,ds, \qquad t\in[0,T_2]$$

where c_3 is a certain constant. Now we apply Hölder's inequality on the right side in (2.7) and use (2.1) and (2.8), (recall that $n+1-\beta>n+1-2\beta>0$). This yields

$$x^{(n+1)}(t) \leqq c_2\left(\int_0^t (t-s)^{n+1-\beta}h(x(s))x'(s)\,ds\right)^{(n+1-2\beta)/(n+1-\beta)} \times$$

$$\times\left(\int_0^t h(x(s))^{(n+1)/\beta}x'(s)\,ds\right)^{\beta/(n+1-\beta)} \leqq c_4 h(x(t))^{(n+1)/(n+1-\beta)}x(t)\,,$$

$$\text{a.e. } t \in [0, T_2]$$

for a certain constant c_4. Now it follows from the Comparison Theorem (recall that $x^{(i)}(0)=0$, $i=0,1,\ldots,n$ and $x^{(n)}$ is absolutely continuous) that we have reduced the problem to the case when $\alpha-1$ is a nonnegative integer and that case we already treated above. Hence we deduce that $x(t)\equiv 0$ if (1.14) holds.

Next we consider the case when $\alpha=1-\beta$, $\beta \in [2^{-1},1)$. Assume that we have already shown that if $\alpha>2^{-m}$ then (1.14) implies that the trivial solution of (1.13) is unique, (this is the case when $m=1$), and let $\alpha=1-\beta$, $\alpha>2^{-(m+1)}$. Since $h(x(t))$ is nonincreasing on $[0,T_1]$ it follows from (1.13) that

$$x(t) = \int_0^t (t-s)^{-\beta}h(x(s))\int_0^s (s-r)^{-\beta}h(x(r))x(r)\,dr\,ds$$

$$\leqq c_5\int_0^t (t-s)^{1-2\beta}h(x(s))^2x(s)\,ds, \qquad t \in [0,T_1]\,,$$

where c_5 is a certain constant. Since $1-2\beta>-(1-2^{-m})$ we can invoke the Comparison Theorem and the induction hypothesis to show that we must have $x(t)\equiv 0$ if (1.14) holds. Hence an induction argument completes the proof of the first part of Theorem 1.

Now we proceed to show that if (1.14) does not hold, then there exist nontrivial solutions of (1.13). First we consider the case when $\alpha=1-\beta$, $\beta \in (2^{-1},1)$. For every $\varepsilon \in (0,1)$, let x_ε be a continuous nondecreasing solution of the equation

$$x_\varepsilon(t) = \varepsilon t+\int_0^t (t-s)^{-\beta}h(x_\varepsilon(s))x_\varepsilon(s)\,ds, \qquad t \in [0,T_3] \tag{2.9}$$

where $T_3>0$ is independent of ε, (it is easy to see that such a solution always exists, cf. [3, Chap. II]). We can choose T_3 to be so small that $x_\varepsilon(t)\leqq\delta_2$ on $[0,T_3]$ for all $\varepsilon \in (0,1)$. Then we have by (2.2) and (2.9)

$$x'_\varepsilon(t) > 2^{-1}\int_0^t (t-s)^{-\beta}h(x_\varepsilon(s))x'_\varepsilon(s)\,ds, \qquad \text{a.e. } t \in [0,T_3]\,. \tag{2.10}$$

We can solve $h(x_\varepsilon(t))x_\varepsilon(t)$ from (2.9) and since $\varepsilon>0$ we get for some constant c_6

$$h(x_\varepsilon(t))x_\varepsilon(t) < c_6\int_0^t (t-s)^{\beta-1}x'_\varepsilon(s)\,ds, \quad \text{a.e. } t \in [0,T_3]\,.$$

We apply Hölder's inequality on the right side in the inequality above and use (2.10), (recall that $\beta > 1-\beta > 0$). Since $h(x_\varepsilon(t))$ is nonincreasing on $[0, T_3]$ and $x_\varepsilon(0) = 0$ it follows that

$$h(x_\varepsilon(t))x_\varepsilon(t) < c_6\left(\int_0^t (t-s)^{-\beta} h(x_\varepsilon(s))x_\varepsilon'(s)\,ds\right)^{(1-\beta)/\beta} \times$$

$$\times \left(\int_0^t h(x_\varepsilon(s))^{(\beta-1)/(2\beta-1)} x_\varepsilon'(s)\,ds\right)^{(2\beta-1)/\beta}$$

$$< c_7 x_\varepsilon'(t)^{(1-\beta)/\beta} h(x_\varepsilon(t))^{(\beta-1)/\beta} x_\varepsilon(t)^{(2\beta-1)/\beta}, \quad \text{a.e. } t \in [0, T_3]$$

or

$$(2.11) \qquad x_\varepsilon'(t) > c_8 h(x_\varepsilon(t))^{1/(1-\beta)} x_\varepsilon(t), \quad \text{a.e. } t \in [0, T_3]$$

where c_7 and c_8 are certain constants.

Since we assume that (1.14) does not hold, there exists a function $v \in C^1([0, T_3]; \mathrm{R})$ such that

$$(2.12) \qquad v'(t) = c_8 h(v(t))^{1/(1-\beta)} v(t),\ t \in [0, T_3]\,,$$

$$v(t) > 0,\ t \in (0, T_3]\,,$$

(let v be the solution of the equation $\int_0^{v(t)} (h(u)^{1/\alpha} u)^{-1}\,du = c_8 t$). Since $v'(t) \to 0$ as $t \to 0$ but $x_\varepsilon(t) \geqq \varepsilon t$ there exists for every $\varepsilon \in (0,1)$ a number $T_\varepsilon \in (0, T_3]$ so that $x_\varepsilon(t) \geqq v(t)$ on $[0, T_\varepsilon]$. But then it follows from (1.12), (2.11) and (2.12) that there exists a number $T_4 \in (0, T_3]$ such that $T_\varepsilon \geqq T_4$ for all $\varepsilon \in (0,1)$. Now it is easy to see that a subsequence of the functions x_ε must converge uniformly on $[0, T_4]$ to a solution x of (1.13) as $\varepsilon \to 0$ (see [3, Chap. II]). Since we must have $x(t) \geqq v(t)$ on $[0, T_4]$ it follows that we have found a nontrivial solution of (1.13).

Assume next that $\alpha = 2n+2-2\beta$ where n is a nonnegative integer, $\beta \in [0,1)$, (1.14) does not hold and that we have already established the second part of Theorem 1 if α takes the value $n+1-\beta$, (this is the case when $n=0$ and $\beta \in (2^{-1}, 1)$). Hence we may assume that there exists a nondecreasing and continuous function v such that

$$(2.13) \qquad v(t) = c_9^{\frac{1}{2}} \int_0^t (t-s)^{n-\beta} h(v(s))^{\frac{1}{2}} v(s)\,ds,\ t \in [0, T_5],\ v(t) > 0\,,\ t \in (0, T_5]$$

where $T_5 > 0$ and c_9 is a positive constant that will be specified below. We choose T_5 to be so small that for every $\varepsilon \in (0,1)$ we have $x_\varepsilon(t) \leqq \delta_1$ on $[0, T_5]$ where x_ε is a continuous nondecreasing solution of the equation

$$(2.14) \quad x_\varepsilon(t) = \varepsilon t^{n+1-\beta} + \int_0^t (t-s)^{2n+1-2\beta} h(x_\varepsilon(s)) x_\varepsilon(s)\,ds, \qquad t \in [0, T_5]\,.$$

From this equation we obtain

$$(2.15)\qquad x_\varepsilon(t) > c_9 \int_0^t (t-s)^{n-\beta} \int_0^s (s-r)^{n-\beta} h(x_\varepsilon(r))x_\varepsilon(r)\,dr\,ds\,, \quad t \in (0, T_5]$$

where c_9 is a certain constant. Observe that by (2.13) $v(t)=o(t^{n+1-\beta})$ as $t \to 0$ and by (2.14) $x_\varepsilon(t) > \varepsilon t^{n+1-\beta}$ on $(0, T_5]$. Hence there exists for every $\varepsilon \in (0,1)$ a number $T_\varepsilon \in (0, T_5]$ such that $x_\varepsilon(t) \geqq v(t)$ on $[0, T_\varepsilon]$. From this fact combined with (1.11) and (1.12) we conclude that

$$h(x_\varepsilon(r))x_\varepsilon(r) \geqq h(v(s))^{\frac{1}{2}} h(v(r))^{\frac{1}{2}} v(r), \qquad 0 \leqq r \leqq s \leqq T_\varepsilon$$

since $v(t)$ is nondecreasing on $[0, T_5]$. Therefore we are able to deduce from (2.13) and (2.15) that $x_\varepsilon(t) > v(t)$ on $(0, T_\varepsilon]$ so that we have $T_\varepsilon = T_5$ for all $\varepsilon \in (0,1)$. Again we see that a subsequence of the functions x_ε converge uniformly on $[0, T_5]$ to a solution of (1.13) (with $\alpha = 2n+2-2\beta$) as $\varepsilon \to 0$. Since we must obviously have $x(t) \geqq v(t)$ on $[0, T_5]$ it follows that we have found a nontrivial solution of (1.13) for this value of α.

To complete the proof of Theorem 1 we can use an induction argument, since all the necessary steps have been established above.

3. Proofs of Theorems 2a and 2b.

Since $h(u)=0$ if $u \leqq 0$ we may clearly assume that the function x is nonnegative on $[0, T]$. Then it follows from the continuity of x and (1.16), (1.17) that $x(0)=0$ and hence we may without loss of generality assume that $x(t) \leqq \min\{1, \delta\}$ on $[0, T]$. From (1.17) we deduce with the aid of Hölder's inequality that $(q = p/(p-1))$

$$(3.1)\qquad x(t)^q \leqq \left(\int_0^t k(t,s)^p (t-s)^{p(1-\alpha)-1}\,ds\right)^{q/p} \int_0^t (t-s)^{q(1/p-(1-\alpha))} h(x(s))^q x(s)^q\,ds, \quad t \in [0, T]\,.$$

Let $y(t) = x(t)^q$ and note that by (1.11) $h(x(t)) \leqq h(y(t))$ on $[0, T]$ since $x(t) \leqq \min\{1, \delta\}$. Therefore it follows from (3.1) that

$$(3.2)\qquad y(t) \leqq c_1^{q/p} \int_0^t (t-s)^{q\alpha-1} h(y(s))^q y(s)\,ds, \quad t \in [0, T]$$

where $c_1 = \sup_{t \in [0,T]} \int_0^t k(t,s)^p (t-s)^{p(1-\alpha)-1}\,ds$. Now we can apply the Comparison Theorem and Theorem 1 to the inequality (3.2). We conclude that $y(t) \equiv 0$ on $[0, T]$ and the proof of Theorem 2a is completed.

To prove Theorem 2b we observe that by Theorem 1 there exists $T_1 \in (0, T]$ and a function $x \in C([0, T_1]; \mathbf{R})$ such that $x(t) > 0$, $t \in (0, T_1]$ and

$$(3.3) \qquad x(t) = c_2^{-1/(p+1)} \int_0^t (t-s)^{\alpha/q-1} h(x(s))^{1/q} x(s)\, ds, \qquad t \in [0, T_1]$$

where now $q=(p+1)/p$ and $c_2 = \sup_{t\in[0,T]} \int_0^t k(t,s)^{-p}(t-s)^{p(\alpha-1)-1}\, ds$. By (3.3) and Hölder's inequality we deduce that

$$(3.4) \quad x(t)^q \leqq c_2^{-q/(p+1)} \left(\int_0^t k(t,s)^{-(p+1)/q} (t-s)^{(p+1)(\alpha/q-1)}\, ds \right)^{q/(p+1)} \times$$

$$\times \int_0^t k(t,s) h(x(s)) x(s)^q\, ds, \; t \in [0, T_1] .$$

Again we let $y(t) = x(t)^q$ and note that $h(x(s)) \leqq h(y(s))$ by (1.11) since we may assume that $x(t) \leqq \min\{1, \delta\}$ on $[0, T_1]$. Then it follows from (3.4) and the definition of c_2 that

$$y(t) \leqq \int_0^t k(t,s) h(y(s)) y(s)\, ds, \qquad t \in [0, T_1] .$$

Now we can apply the Comparison Theorem to complete the proof of Theorem 2b.

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