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THE CONVERGENCE OF SUMS OF TRANSITION PROBABILITIES OF A POSITIVE RECURRENT MARKOV CHAIN

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1. Introduction.

Let $X = \{X_n\}_{n \geqq 0}$ be an aperiodic, positive recurrent Harris chain on a general measurable state space $(E, \mathscr{E})$. We shall follow throughout this paper Revuz' [8] and Nummelin's [6] notation and terminology.

In Nummelin [6] (cf. also Cogburn [2] and Griffeath [3]) it was proved that, if λ and μ are *regular* probability measures on $(E, \mathscr{E})$, then the sum $\sum (\lambda P_n - \mu P_n)$ converges absolutely in total variation norm:

$$(1.1) \qquad \sum_0^\infty \|\lambda P_n - \mu P_n\| < \infty .$$

Recall that λ is called regular, provided that

$$\boldsymbol{E}_\lambda S_A < \infty \qquad \text{for all } A \in \mathscr{E}^+ .$$

Here $S_A = \inf\{n > 0 : X_n \in A\}$, $\mathscr{E}^+ = \{A \in \mathscr{E} : m(A) > 0\}$ and m denotes the invariant probability measure of the chain.

Recall from Cogburn [2] the definition of a strongly uniform set: $F \in \mathscr{E}$ is called *strongly uniform*, provided that

$$\sup_{x \in F} \boldsymbol{E}_x S_A < \infty, \qquad \text{for all } A \in \mathscr{E}^+ .$$

Cogburn [2, Theorem 5.1] proved that a set $F \in \mathscr{E}$ is strongly uniform, if and only if

$$(1.2) \qquad \sup_{x, y \in F,\, N \geqq 1} \left\| \sum_1^N (P_n(x, \cdot) - P_n(y, \cdot)) \right\| < \infty .$$

The main purpose of the present paper is to strengthen these results in the following way: we show that a weaker convergence than (1.1) (respectively (1.2)) is sufficient for λ to be regular (respectively F to be strongly uniform), and conversely, that the regularity of λ and μ (respectively the strong uniformity of F) implies a stronger convergence than (1.1) (respectively (1.2)). As an

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application we prove a convergence result for renewal processes. We shall also state some characterization results for regular measures and strongly uniform sets. In particular, in the case when E is a topological space a sufficient condition is given for compact sets to be strongly uniform (cf. Section 4 of Cogburn [2]).

We shall assume throughout this paper that the *Minorization Assumption* (M) of Nummelin [6] holds: *for some* $k \geqq 1$, $h \in b\mathscr{E}_+$ *with* $m(h) > 0$ *and a probability measure* ν *on* $(E, \mathscr{E})$,

$$(\mathrm{M}) \qquad P_k \geqq h \otimes \nu .$$

Note that, if $\mathscr{E}$ is countably generated, then by Orey's C-set theorem (see Revuz [8, p. 160]) (M) is satisfied for suitable k, h and ν.

A basic method in the proofs of our results will be the *Splitting Technique* introduced in Nummelin [4]. Suppose for a while that (M) holds with $k=1$, i.e.

$$P \geqq h \otimes \nu .$$

Let $\{Y_n\}_{n \geqq 0}$ be a sequence of random variables taking values zero or one. Suppose that X_0 has initial distribution λ. The probability law governing the evolution of the bivariate process $\{X_n^*\} \overset{\text{def}}{=} \{(X_n, Y_n)\}$ is determined by the conditions

$$(1.3\text{a}) \quad \boldsymbol{P}_\lambda\{X_{n+1} \in A,\ Y_n = 1 \mid X_0, \ldots, X_n, Y_0, \ldots, Y_{n-1}\} = h(X_n)\nu(A) ,$$

$$(1.3\text{b}) \quad \boldsymbol{P}_\lambda\{X_{n+1} \in A,\ Y_n = 0 \mid X_0, \ldots, X_n, Y_0, \ldots, Y_{n-1}\} = P(X_n, A) - h(X_n)\nu(A) .$$

Clearly $\{X_n^*\}$ is also a Markov chain. From (1.3a) we see that the conditional distribution of X_{n+1}^* given $X_n = x$ and $Y_n = 1$ equals ν, in particular it is independent of x. This means that the set $B = E \times \{1\}$ is an *atom* for $\{X_n^*\}$. If $\{X_n\}$ is recurrent in the sense of Harris, then $Y_n = 1$ infinitely often with probability one, i.e. the atom B is recurrent. This allows the use of renewal theory in the analysis of general state space Markov chains, a tool, which has turned out to be powerful in the analysis of Markov chains on a countable state space. The reader is referred to Section 2 of Nummelin [6] for a detailed study of the splitting method.

We shall use the notation $\boldsymbol{P}_B$ (respectively $\boldsymbol{E}_B$) for the probability law (respectively expectation operator) corresponding to the initial condition $Y_0 = 1$. Note that the law $\boldsymbol{P}_B$ does not depend on the initial distribution of X_0. For $n \geqq 1$ we write

$$S_B = \inf\{n \geqq 1 : Y_n = 1\}, \quad a_x(n) = \boldsymbol{P}_x\{S_B = n\}, \quad \psi_g(n) = \boldsymbol{E}_B[g(X_n) ; S_B \geqq n\} ,$$

$$u(n) = \boldsymbol{P}_B\{Y_n = 1\} = \nu P_{n-1} h, \quad u(0) = 1 .$$

The sequence $\{u(n)\}_{n\geqq 0}$ is the renewal sequence corresponding to the successive visits of $\{X_n^*\}$ to the atom B. The following first-entrance-last-exit decomposition is of basic importance: For any $n\geqq 1$, $x \in E$, $g \in \mathscr{E}_+$,

$$P_n g(x) = \boldsymbol{E}_x[g(X_n)\,;\,S_B\geqq n]+a_x*u*\psi_g(n)\,. \tag{1.4}$$

We denote

$$G = \sum_{n=0}^{\infty} (P_k-h\otimes v)^n, \qquad \hat{G} = G\sum_{i=0}^{k-1} P_i\,.$$

By Theorem 3 of Nummelin [6], $vG=(m(h))^{-1}m$. Note that, if $k=1$, then

$$G = \hat{G} = \sum_{n=0}^{\infty} (P-h\otimes v)^n\,,$$

and G has the following probabilistic interpretation:

$$Gg(x) = \boldsymbol{E}_x\left[\sum_{n=0}^{T_B} g(X_n)\right], \qquad T_B \overset{\text{def}}{=} \inf\{n\geqq 0\,:\,Y_n=1\}\,.$$

The kernels G and Q (cf. Nummelin [6]) are connected to each other by the relation $Q=PG$.

Note that, by recurrence, $Gh(x)=\boldsymbol{P}_x\{T_B<\infty\}\equiv 1$. Since the Harris recurrence is preserved when turning from the original Markov chain $\{X_n\}$ to the k-step Markov chain $\{X_{nk}\}$, $Gh\equiv 1$ also in the general case $k\geqq 1$ (cf. Lemma 2.1 of Nummelin [6].

LEMMA 1.1. (i) $I+\hat{G}P=\hat{G}+1\otimes v$.
(ii) $\sum_0^{k-1} P_i+P_k\hat{G}=\hat{G}+k(m(h))^{-1}h\otimes m$.

PROOF. We have

$$\begin{aligned} I+\hat{G}P &= I+G\sum_{i=1}^{k} P_i \\ &= I+G\left[\sum_{i=1}^{k-1} P_i+(P_k-h\otimes v)+h\otimes v\right] \\ &= I+G\sum_{i=1}^{k-1} P_i+\sum_{n=1}^{\infty} (P_k-h\otimes v)^n+1\otimes v \\ &= \hat{G}+1\otimes v\,, \end{aligned}$$

which proves (i). The proof of (ii) is similar.

LEMMA 1.2. *Let* $u=\{u(n)\}_{n\geqq 0}$ *be an aperiodic positive recurrent (i.e., there exists* $\lim u(n)=u(\infty)>0$) *zero delay renewal sequence and let* $v=b*u$ *be a* *delayed renewal sequence with delay distribution* $b=\{b(n)\}_{n\geqq 1}$. *Denote by* β *the mean of the delay,*

$$\beta = \sum_1^\infty nb(n) .$$

There is a finite constant γ, *depending only on the sequence* u, *such that*

$$\sum_0^\infty |v(n)-u(n)| \leqq \gamma\beta . \tag{1.5}$$

Proof. Pitman [7] proved that, if β is finite, then the left hand side of (1.5) is finite. He did not explicitely state that even (1.5) holds, although the same proof yields also this result. In the following we shall only write the basic inequalities, which lead to (1.5). The reader is referred to Pitman's paper for details omitted here.

We denote by τ the first simultaneous renewal epoch of a zero delay renewal process and of an independent, delayed (with delay distribution b) renewal process. We write $\tilde{\beta}$ for the expectation of τ and define

$$F_N = \sum_1^N (u(n)-v(n)) ,$$

$$G_N = \sum_1^N (u(n)^2-u(n)v(n)) .$$

Then by Lemma 6.11 of Pitman $\lim_{N\to\infty} G_N = u^2(\infty)\tilde{\beta}-1$ and $|F_N|\leqq\beta$ for all N, whence

$$\begin{aligned} u(\infty)^2\tilde{\beta}-1 &\leqq \sup_N |G_N| \\ &= \sup_N \left| F_N u(N) + \sum_1^{N-1} F_n(u(n)-u(n+1)) \right| \\ &\leqq \left(1+\sum_1^\infty |u(n)-u(n+1)|\right)\beta . \end{aligned}$$

The final assertion (1.5) now follows from the coupling inequality

$$|v(n)-u(n)| \leqq 2P\{\tau>n\} .$$

2. The total variation and $\mathscr{L}^1$-convergence of the sum $\sum (P_n f(x)-P_n f(y))$.

The following definition of g-regular measures generalizes the concept of a regular measure defined in Section 1.

DEFINITION 2.1. *Let* $g \in \mathscr{L}^1_+(m)$ be arbitrary. A probability measure λ is called *g-regular*, provided that g is λ-integrable and

$$\lambda U_A g = E_\lambda \sum_1^{S_A} g(X_n) < \infty \qquad \text{for all } A \in \mathscr{E}^+ .$$

Hence, λ is g-regular, if and only if $E_\lambda \sum_0^{S_A} g(X_n)$ is finite for all $A \in \mathscr{E}^+$. Note that in Definition 3.1 of Nummelin [6] the λ-integrability of g was not demanded. Note also that 1-regular means the same as regular. We shall henceforth use the notation $\mathscr{R}_g$ for the set of g-regular probability measures. We define the norms $\|\cdot\|_g$, $g \in \mathscr{E}_+$, in the usual way: for a signed measure ξ such that $g \in \mathscr{L}^1_+(|\xi|)$,

$$\|\xi\|_g = \sup_{|f| \leq g} |\xi(f)| = |\xi|(g) .$$

Note that $\|\cdot\| = \|\cdot\|_1$. In the space of λ-integrable measurable functions $\mathscr{L}^1(\lambda)$ we have the usual norm defined by $\|f\|_\lambda = \lambda(|f|)$. We shall denote by $\mathscr{K}$ the set of signed transition kernels on $(E, \mathscr{E})$ and define for any probability measure λ and any $g \in \mathscr{E}_+$,

$$\mathscr{K}_{\lambda,g} = \left\{ N \in \mathscr{K} : \sup_{|\xi| \leq \lambda, |f| \leq g} |\xi N f| < \infty \right\},$$

and for any $N \in \mathscr{K}_{\lambda,g}$,

$$\|N\|_{\lambda,g} = \sup_{|\xi| \leq \lambda, |f| \leq g} |\xi N f| = \sup_{|f| \leq g} \|Nf\|_\lambda = \sup_{|\xi| \leq \lambda} \|\xi N\|_g .$$

Then $(\mathscr{K}_{\lambda,g}, \|\cdot\|_{\lambda,g})$ clearly becomes a normed space.

LEMMA 2.1. (i) *Write* $\bar{g} = \sum_0^{k-1} P_i g$. *A probability measure* λ *is g-regular, if and only if it is* $\bar{g}$*-regular with respect to the k-step chain* $\{X_{nk}\}_{n \geq 0}$.

(ii) λ *is g-regular, if and only if* $\lambda \hat{G} g$ *is finite.*

(iii) *If* λ *is g-regular, then* λ *is* $P_n g$*-regular for all n. Hence, in particular,* $\lambda P_n g$ *is finite for all n.*

PROOF. (i) Use Lemma 3.2 of Nummelin [6].

(ii) By (i) there is no loss of generality in assuming that $k=1$. Suppose first that $\lambda G g = E_\lambda \sum_0^{T_B} g(X_n)$ is finite. Let $A \in \mathscr{E}^+$ be arbitrary. By Lemma 3.4 of Nummelin [6] there is $x_0 \in E$ such that $h(x_0) > 0$ and $U_A g(x_0) < \infty$. Now the finiteness of $E_\lambda \sum_0^{S_A} g(X_n)$ follows from the inequalities

$$h(x_0) E_B \sum_{n=1}^{S_A} g(X_n) = E_{x_0}\left[\sum_{n=1}^{S_A} g(X_n) \,;\, Y_0 = 1 \right] \leq U_A g(x_0)$$

and

$$E_\lambda \sum_{n=0}^{S_A} g(X_n) \leqq E_\lambda \sum_{n=0}^{T_B} g(X_n) + E_B \sum_{n=1}^{S_A} g(X_n) .$$

Conversely, suppose that $E_\lambda \sum_0^{S_A} g(X_n)$ is finite for all $A \in \mathscr{E}^+$. Since $\nu Gg = m(g)$ is finite, there is $A_0 \in \mathscr{E}^+$ such that $Gg(x) = E_x \sum_0^{T_B} g(X_n)$ is bounded on A_0. The assertion now follows from the inequality

$$E_\lambda \sum_{n=0}^{T_B} g(X_n) \leqq E_\lambda \sum_{n=0}^{S_{A_0}} g(X_n) + \sup_{x \in A_0} E_x \sum_{n=0}^{T_B} g(X_n) .$$

(iii) Use part (ii) and Lemma 1.1 (i).

REMARK. Note that in the proof of Lemma 2.1 the assumption of positive recurrence was not needed (cf. Section 4).

Recall from Revuz [8] the definition of special functions; a function $g \in \mathscr{L}^1_+(m)$ is called *special*, provided that

$$\sup_{x \in E} E_x \left[\sum_1^{S_A} g(X_n) \right] < \infty \qquad \text{for all } A \in \mathscr{E}^+ ,$$

(cf. also Lemma 5.6 of Nummelin [4]). Note that, g is bounded and special, if and only if every probability measure λ on $(E, \mathscr{E})$ is g-regular. Hence, as is seen by Lemma 2.1 (ii), g is bounded and special, if and only if the function $\hat{G}g$ is bounded.

The following theorem is one of our main results. It strengthens the earlier results (see Nummelin [6, Theorem 5]) in two ways. The convergence is uniform over a larger class of functions and we have $\mathscr{L}^1$-convergence with respect to the initial probability measures.

THEOREM 1. *For any m-integrable non-negative g, any $g \vee 1$-regular λ and μ,*

$$\sum_{n=0}^{\infty} \sup_{|f| \leqq g} \iint \lambda(dx)\mu(dy)|P_n f(x) - P_n f(y)| < \infty . \tag{2.1}$$

In particular, if λ and μ are regular, then

$$\sum_{n=0}^{\infty} \sup_{|f| \leqq 1} \iint \lambda(dx)\mu(dy)|P_n f(x) - P_n f(y)| < \infty . \tag{2.2}$$

PROOF. First of all, note that by Lemma 2.1 (iii) the integrand is well defined and finite for λ-almost all x and μ-almost all y. Without loss of generality we can replace $g \vee 1$ by g and assume $g \geqq 1$. Suppose first that $k = 1$ in (M).

According to the first-entrance-last-exit decomposition (1.4) we have, for any $|f| \leqq g$,

$$|P_n f(x) - \psi_f * u(n)| \leqq \mathbf{E}_x[g(X_n)\,;\, S_B \geqq n] + |a_x * u - u| * \psi_g(n)\,,$$

whence

$$\sum_0^\infty \sup_{|f| \leqq g} \int \lambda\,(dx)|P_n f(x) - \psi_f * u(n)| \leqq \mathbf{E}_\lambda\left[\sum_0^{S_B} g(X_n)\right]$$

$$+ \int \lambda\,(dx) \sum_{n=1}^\infty |a_x * u(n) - u(n)| m(g)/m(h)\,.$$

By Lemma 2.1 (ii) the first term on the right hand side is finite. By Lemma 1.2 the second term is majorized by $\gamma \int \lambda(dx) \mathbf{E}_x S_B$, γ a finite constant. Since by Lemmas 1.1 (ii) and 2.1 (ii) $\mathbf{E}_\lambda S_B = \lambda PG1 \leqq \lambda G1 + (m(h))^{-1} < \infty$, the second term is finite. Similar calculations for μ and the triangle inequality lead to (2.1).

Let now k in (M) be arbitrary. By Lemma 2.1 (i), for any $0 \leqq i < k$, λ and μ are $P_i g$-regular with respect to $\{X_{nk}\}$. We can now apply (2.1) to the k-step chain $\{X_{nk}\}$ with g replaced by $P_i g$ (note that $|f| \leqq g$ implies $|P_i f| \leqq P_i g$):

$$\sum_{n=0}^\infty \sup_{|f| \leqq g} \iint \lambda(dx)\mu(dy)|P_{nk+i} f(x) - P_{nk+i} f(y)|$$

$$\leqq \sum_{n=0}^\infty \sup_{|f| \leqq P_i g} \iint \lambda(dx)\mu(dy)|P_{nk} f(x) - P_{nk} f(y)| < \infty\,.$$

The final assertion is obtained from this inequality by summing i from 0 to $k-1$.

COROLLARY 2.2. *For any* $g \in \mathscr{L}^1_+(m)$, λ *and* $\mu \in \mathscr{R}_{g \vee 1}$,

$$\sum_{n=0}^\infty \|P_n - 1 \otimes \mu P_n\|_{\lambda, g} = \sum_{n=0}^\infty \sup_{|f| \leqq g} \int \lambda(dx)|P_n f(x) - \mu P_n f| < \infty;$$

in particular,

$$\sum_{n=0}^\infty \|\lambda P_n - \mu P_n\|_g \leqq \infty\,.$$

By Proposition 3.5 of Nummelin [6], for any $g \in \mathscr{L}^1_+(m)$, m-almost all $x \in E$, the measure ε_x assigning unit mass to the point x is g-regular and $Gg(x)$ is finite (cf. Lemma 2.1 (ii)). This gives us the following corollary.

COROLLARY 2.3. *For any* $g \in \mathscr{L}^1_+(m)$, *m-almost all* $x, y \in E$,

$$\sum_0^\infty \|P_n(x, \cdot) - P_n(y, \cdot)\|_g < \infty\,.$$

Next we shall apply the preceding results to an interesting subclass of positive recurrent Markov chains. We call the Markov chain $X=\{X_n\}$ *ergodic of degree* 2 provided that for some strongly uniform $A \in \mathscr{E}^+$

$$\int_A m(dx)\boldsymbol{E}_x[S_A^2] < \infty . \tag{2.3}$$

Cogburn [2] proved that, if X is ergodic of degree 2, then (2.3) holds for all $A \in \mathscr{E}^+$ and the invariant probability measure m is regular, and conversely, if m is regular then the chain is ergodic of degree 2. Moreover, Cogburn [2, Theorem 5.3] proved that, if X is ergodic of degree 2, then the series

$$\sum_0^\infty (P_n - 1 \otimes m) f(x)$$

converges uniformly over $f \in \mathscr{U} = \{|f| \leqq 1\}$ for each regular x and it converges in $\mathscr{L}^1(m)$ uniformly over $f \in \mathscr{U}$. Note that by Lemma 2.1(ii) and Cogburn's results quoted above, X is ergodic of degree 2, if and only if $\nu G^2 1 = mG1$ is finite. From our Theorem 1 we obtain the following corollary strengthening Cogburn's Theorem 5.3 cited above.

COROLLARY 2.4. *Assume that X is ergodic of degree 2. Let λ be a regular probability measure and $g \in \mathscr{L}^1_+(m)$ be such that λ and m are g-regular. Then*

$$\sum_0^\infty \|P_n - 1 \otimes m\|_{\lambda, g} < \infty ,$$

and

$$\sum_0^\infty \|P_n - 1 \otimes \lambda P_n\|_{m, g} < \infty .$$

In Arjas, Nummelin, and Tweedie [1, Theorem 6] the total variation convergence result (1.1) was used when proving a total variation convergence result for renewal processes on the real line. Similarly, the strengthened version of (1.1), that is (2.2), can be used to obtain the following result. Note that this result could also be derived from Stone's [9] decomposition results for renewal measures.

COROLLARY 2.5. *Assume that F is a distribution on $\mathbf{R}_+ = [0, \infty)$ with finite mean and such that, for some $n \geqq 1$, the n-th convolution F^{n*} has an absolutely continuous component with respect to Lebesgue measure. Denote $U = \sum_0^\infty F^{n*}$. Let G be an arbitrary distribution on $\mathbf{R}_+$ with finite mean. Then we have*

$$\int_{\mathbf{R}_+} \int_{\mathbf{R}_+} G(dt) |U(dv - t) - U(dv)| < \infty .$$

PROOF. Similar to the proof of Theorem 6 of Arjas, Nummelin and Tweedie [1].

The following theorem identifies the limit of the sum $\sum (\lambda P_n f - \mu P_n f)$. The proof of Theorem 2 is the same as that of Theorem 6(i) of Nummelin [6], and is therefore omitted. Theorem 6(i) of Nummelin [6] is a special case ($g \equiv 1$) of Theorem 2.

THEOREM 2. *For any* $g \in \mathscr{L}^1_+(m)$, $\lambda, \mu \in \mathscr{R}_{g \vee 1}$

$$\sum_0^N (P_n - 1 \otimes \mu P_n) \to (I - 1 \otimes \mu)\hat{G}(I - 1 \otimes m) \quad \text{in } \|\cdot\|_{\lambda, g}\text{-norm as } N \to \infty;$$

in particular,

$$\sum_0^\infty (\lambda P_n g - \mu P_n g) = \lambda \hat{G} g + m(g) \mu \hat{G} 1 - (\mu \hat{G} g + m(g) \lambda \hat{G} 1). \tag{2.4}$$

By using Proposition 3.5 of Nummelin [6] we obtain the following corollary.

COROLLARY 2.6. *For any* $g \in \mathscr{L}^1_+(m)$, *m-almost all* x, y,

$$\sum_{n=0}^N [P_n(x, \cdot) - P_n(y, \cdot)] \to [G(I - 1 \otimes m)(x, \cdot) - G(I - 1 \otimes m)(y, \cdot)]$$

$$\text{in } \|\cdot\|_g\text{-norm as } N \to \infty.$$

If X is ergodic of degree 2, then by Theorem 2 and Corollary 2.4, and since $(P - 1 \otimes m)^n = P_n - 1 \otimes m$ for $n \geq 1$, we have for any $g \in \mathscr{L}^1_+(m)$, any regular λ such that λ and m are g-regular:

$$\lim_{N \to \infty} \left\| \sum_0^N (P - 1 \otimes m)^n - Z \right\|_{\lambda, g} = 0, \tag{2.5}$$

where $Z \in \mathscr{K}_{\lambda, g}$ is defined by

$$Z = 1 \otimes m + (I - 1 \otimes m) G (I - 1 \otimes m).$$

By using (2.5) and noting that λ is Pg-regular whenever λ is g-regular (use Lemma 1.1(i) and Lemma 2.1(ii)) we can conclude that Z is the inverse operator of $I - P + 1 \otimes m$ in the following sense: for any regular λ and any $f \in \mathscr{L}^1(m)$ such that λ and m are $|f|$-regular

$$\lambda Z (I - P + 1 \otimes m) f = \lambda (I - P + 1 \otimes m) Z f = \lambda(f).$$

The following theorem states that the equation (2.4) holds even in the case when the right hand side is infinite provided only that it is well defined.

THEOREM 3. *For any* $g \in \mathscr{L}^1_+(m)$, *regular* λ *and* g*-regular* μ *the formula* (2.4) *holds. Both sides of* (2.4) *are finite, provided that in addition* λ *is* g*-regular and* μ *is regular; if* λ *is not* g*-regular or* μ *is not regular, then both sides of* (2.4) *are equal to* $+\infty$.

REMARK. Of course, by symmetry, a similar statement is valid for g-regular λ and regular μ.

PROOF. (i) Assume first that $\lambda \in \mathscr{R}_1$ and $\mu \in \mathscr{R}_g \setminus \mathscr{R}_1$. By (1.4), if $k=1$ in (M), then for any $N \geqq 1$,

$$\sum_0^N (\lambda P_n g - \mu P_n g) = E_\lambda\left[\sum_0^{N \wedge S_B} g(X_n)\right] - E_\mu\left[\sum_0^{N \wedge S_B} g(X_n)\right] + \sum_1^N (a_\lambda - a_\mu) * \psi_g * u(n)\,. \tag{2.6}$$

The first term on the right hand side is non-negative. The second terms tends to the limit $-E_\mu[\sum_0^{S_B} g(X_n)]$, which is finite by the g-regularity of μ. Since $E_\lambda S_B < \infty$ and $E_\mu S_B = \infty$, we have by Lemma 3.11 of Pitman [7]:

$$\sum_1^\infty (a_\lambda - a_\mu) * \psi_g * u(n) = \infty\,.$$

Hence
$$\sum_0^\infty (\lambda P_n g - \mu P_n g) = \infty\,.$$

The case of general k follows by using Lemma 2.1 (i).

(ii) Assume now that $\lambda \in \mathscr{R}_1 \setminus \mathscr{R}_g$ and that $\mu \in \mathscr{R}_g$. With these assumptions the first term on the right hand side of (2.6) converges to $+\infty$, whereas the second term tends to a finite limit. The third term tends to a finite limit or to $+\infty$ according as μ is regular or non-regular (use Pitman's Theorem 6.11 and Lemma 3.11).

(iii) The rest follows from Lemma 2.1 (ii) and Theorem 2.

As stated in Theorem 1, if the probability measures λ and μ are regular, then the sum $\sum (P_n - 1 \otimes \mu P_n)$ converges absolutely in $\|\cdot\|_{\lambda,g}$-norm. Theorem 3 gives an answer to the following converse problem: What kind of convergence of the transition probabilities is sufficient for regularity of the initial probability measure λ? Combining Theorem 1 and 3 we also get a solidarity result for the convergence of sums of transition probabilities.

COROLLARY 2.7. *Let* λ *and* μ *be two probability measures on* $(E, \mathscr{E})$. *Assume that* μ *is regular. If for some* $g \in \mathscr{L}^1_+(m)$ *such that* λ *is* g*-regular* (*in particular, for some bounded special function* $g \in \mathscr{L}^1_+(m)$)

$$\limsup_{N\to\infty} \sum_{0}^{N} (\lambda P_n g - \mu P_n g) > -\infty ,$$

then λ is regular and μ is g-regular, whence

$$\sum_{0}^{\infty} \sup_{|f|\leqq g} \iint \lambda(dx)\mu(dy)|P_n f(x) - P_n f(y)| < \infty .$$

Theorem 3 yields the following criterion for X to be ergodic of degree 2.

COROLLARY 2.8. *If for some regular probability measure λ, some $g \in \mathscr{L}^1_+(m)$ such that λ and m are g-regular (in particular, some bounded special g) we have*

$$\liminf_{N\to\infty} \sum_{0}^{N} (\lambda P_n g - m(g)) < \infty ,$$

then the chain is ergodic of degree 2.

3. Uniform convergence of the sum $\sum(P_n f(x) - P_n f(y))$.

The following theorem strengthens Cogburn's [2, Theorem 5.1] mentioned in Section 1. We shall assume throughout this section that λ is a fixed regular probability measure on $(E, \mathscr{E})$. Note that by (1.1) when studying the uniform convergence (over x and y) of the sum $\sum (P_n f(x) - P_n f(y))$, $f \in b\mathscr{E}$, it is equivalent to consider the uniform convergence (over x) of the sum $\sum (P_n f(x) - \lambda P_n f)$.

THEOREM 4. *Let $F \in \mathscr{E}^+$ be arbitrary. The following three conditions are equivalent:*

(i) *F is strongly uniform;*

(ii) $\sup_{x\in F} \sum_{0}^{\infty} \|P_n(x, \cdot) - \lambda P_n\| < \infty;$

(iii) $\limsup_{N\to\infty} \left[\inf_{x\in F} \sum_{0}^{N} (P_n g(x) - \lambda P_n g)\right] > -\infty$

for some special $g \in b\mathscr{E}_+$ such that $m(g) > 0$.

When proving the theorem we shall need the following lemma, the proof of which is similar to that of Lemma 2.1 (ii).

LEMMA 3.1. *A set $F \in \mathscr{E}$ is strongly uniform, if and only if $\hat{G}1(x)$ is bounded on F.*

PROOF OF THEOREM 4. (i) $\Rightarrow$ (ii): By Lemma 2.1(iii), Corollary 2.2 and since ν is regular, it suffices to show that

$$\sup_{x \in F} \sum_0^\infty \|P_{n+1}(x, \cdot) - \nu P_n\| < \infty .$$

Assume first that $k=1$ in (M). Note that $\nu P_{n-1} f = \boldsymbol{E}_B f(X_n) = u * \psi_f(n)$ (cf. Section 1). By (1.4) we have for any $f \in b\mathscr{E}$ such that $|f| \leqq 1$ and any $x \in E$,

$$\begin{aligned} \sum_0^\infty |P_{n+1} f(x) - \nu P_n f| &= \sum_1^\infty |\boldsymbol{E}_x[f(X_n) 1_{\{S_B \geqq n\}}] + a_x * u * \psi f(n) - u * \psi_f(n)| \\ &\leqq \boldsymbol{E}_x[S_B] + [m(h)]^{-1} \sum_1^\infty |a_x * u(n) - u(n)| , \end{aligned}$$

since

$$|\psi_f(n)| \leqq \psi_1(n) \quad \text{and} \quad \sum_1^\infty \psi_1(n) = \frac{m(E)}{m(h)} = \frac{1}{m(h)} .$$

If F is strongly uniform, then

$$\sup_{x \in F} \boldsymbol{E}_x[S_B] = \sup_{x \in F} \sum_1^\infty n a_x(n) < \infty .$$

The final assertion now follows by Lemma 1.2.

The case of general k can be treated in a manner analogous to the proof of Theorem 1.

(ii) $\Rightarrow$ (iii): Trivial.

(iii) $\Rightarrow$ (i): We have

$$\begin{aligned} -\infty &< \limsup_{N \to \infty} \left[\inf_{x \in F} \sum_0^N (P_n g(x) - \lambda P_n g) \right] \\ &\leqq \inf_{x \in F} \limsup_{N \to \infty} \sum_0^N (P_n g(x) - \lambda P_n g) \\ &= \inf_{x \in F} [\hat{G} g(x) - m(g) \hat{G} 1(x) - \lambda \hat{G} g + m(g) \lambda \hat{G} 1] \quad \text{by Theorem 3} , \\ &\leqq \sup_{x \in F} \hat{G} g(x) - m(g) \sup_{x \in F} \hat{G} 1(x) + m(g) \lambda \hat{G} 1 . \end{aligned}$$

The function $\hat{G} g$ is bounded, since g is bounded and special, and $\lambda \hat{G} 1$ is finite by the regularity of λ. Hence

$$\sup_{x \in F} \hat{G} 1(x) < \infty ,$$

which by Lemma 3.1 implies that F is strongly uniform.

REMARK. The results of this section could easily be generalized by replacing the total variation norm $\|\cdot\|$ with the more general norms $\| \ \|_g$, $g \in \mathscr{L}^1_+(m)$. In this case one needs the concept of a g-strongly uniform set introduced in Nummelin [4]. A set $F \in \mathscr{E}$ is called *g-strongly uniform*, provided that

$$\sup_{x \in F} E_x\left[\sum_1^{S_A} g(X_n)\right] < \infty \quad \text{for all } A \in \mathscr{E}^+ . \tag{3.1}$$

We state without proof the following result, the proof of which is similar to that of Theorem 4.

THEOREM 5. *For any* $g \in \mathscr{L}^1_+(m)$, $\lambda \in \mathscr{R}_{g \vee 1}$, *any* $g \vee 1$-*strongly uniform set* $F \in \mathscr{E}$,

$$\sup_{x \in F} \sum_1^\infty \|P_n(x, \cdot) - \lambda P_n\|_g < \infty .$$

We shall end this section by considering the special case when the whole state space E is strongly uniform, that is when

$$\sup_{x \in E} E_x S_A < \infty \quad \text{for all } A \in \mathscr{E}^+ , \tag{3.2}$$

or equivalently, when the function 1 (and hence any bounded function) is special. By Lemma 3.1 this is equivalent to saying that the function $\hat{G}1$ is bounded.

Let us call the Markov chain X *uniformly ergodic*, provided that the iterates of P converge to the limit operator $1 \otimes m$ in the operator norm, that is

$$\lim_{n \to \infty} \sup_{x \in E} \|P_n(x, \cdot) - m\| = 0 .$$

It is well known (see e.g. Revuz [8, Proposition 6.4.9]) that X is uniformly ergodic, if and only if (3.2) holds.

Our Theorem 4 applied with $F = E$ yields the following criterion for uniform ergodicity:

COROLLARY 3.2. *If for some regular probability measure* λ, *some special* $g \in b\mathscr{E}_+$ *satisfying* $m(g) > 0$,

$$\lim_{N \to \infty} \sup \left[\inf_{x \in E} \sum_0^N (P_n g(x) - \lambda P_n g)\right] > -\infty ,$$

then the chain is uniformly ergodic.

Recalling the fact that m-almost every $y \in E$ is regular we obtain

COROLLARY 3.3. *If for some $A \in \mathscr{E}^+$, for all $y \in A$ there exists a special function $g \in b\mathscr{E}_+$ with $m(g) > 0$, such that*

$$\limsup_{n\to\infty} \left[\inf_{x\in E} \sum_0^N (P_n g(x) - P_n g(y)) \right] > -\infty ,$$

then the chain is uniformly ergodic.

Note that in the other direction we can not strengthen the existing results, since the operator norm convergence of $P_n - 1 \otimes m$ is known to be exponentially fast, and therefore also the sum $\sum (P_n - 1 \otimes m)$ automatically converges absolutely in operator norm.

4. Criteria for regular measures, strongly uniform sets and special functions.

From the preceding sections one observes that the concepts of regularity, strong uniformity and speciality play a central role when studying the convergence of the sums of transition probabilities. Criteria for these, in terms of the kernel $\hat{G}$, are given in Lemma 2.1 and Lemma 3.1. In fact, however, it is not necessary to calculate explicitely the kernel $\hat{G}$ but only estimate it from above. For these purposes it turns out to be sufficient to consider suitable Poisson-type equations and inequalities (cf. (4.1) and (4.2) below). *We shall drop the assumption of positive recurrence* of the chain and assume only the Harris recurrence and aperiodicity. *We shall also assume throughout this section that the Minorization Assumption* (M) *holds with* $k = 1$. The case of general k can be handled by considering either the k-step Markov chain $\{X_{nk}\}$ or the Markov chain $\{X_n^{(\alpha)}\}$ induced by the transition probability function $P^{(\alpha)} = \sum_1^\infty \alpha(1-\alpha)^n P_n$ ($\alpha \in (0.1)$ fixed), and using results like Lemma 2.1(i) telling us how a certain concept is inherited by the chains $\{X_{nk}\}$ and $\{X_n^{(\alpha)}\}$. The reader is referred to Nummelin [5] for a study of the latter chain.

Since $k = 1$, the kernel $\hat{G} = G$ has the form

$$G = \sum_0^\infty (P - h \otimes \nu)^n .$$

According to Lemma 1.1(ii), for any measurable $g \in \mathscr{L}_+^1(m)$ the function $f\colon E \to [0, \infty]$ defined by $f = Gg$ satisfies the equation

$$f + e = Pf + g , \tag{4.1}$$

where $e = (m(g)/m(h))h$ is bounded and special.

If now, for example, λ is a g-regular probability measure on $(E, \mathscr{E})$, then by Lemma 2.1(ii) $\lambda(f)$ is finite, and so in this case there is a bounded special function e such that equation (4.1) possesses a non-negative, λ-integrable solution f. Similarly, if F is a strongly uniform set, then by using Lemma 3.1 we can conclude that there is a bounded special function e such that equation (4.1) possesses a non-negative solution f, which is bounded on F. The following theorem states that these results hold also conversely. It turns out to be sufficient to consider the inequalities

$$f+e \geqq Pf+g\,. \tag{4.2}$$

THEOREM 6. (i) *Let λ be an arbitrary probability measure on $(E,\mathscr{E})$ and g an element of $\mathscr{L}^1_+(m)$. λ is g-regular, if and only if for some bounded $e \in \mathscr{L}^1(m)$ such that λ is e-regular (in particular, for some bounded special e), the inequality* (4.2) *has a non-negative, λ-integrable solution f.*

(ii) *A set F is strongly uniform, if and only if for some bounded special e, the inequality*

$$f+e \geqq Pf+1 \tag{4.3}$$

has a solution f, which is non-negative and bounded on F.

(iii) *A function $g \in b\mathscr{E}_+$ is special, if and only if for some bounded special e, the inequality* (4.2) *has a bounded solution f.*

PROOF. We prove only (i) the proofs of (ii) and (iii) being similar. According to the remarks made above on equation (4.1), if λ is g-regular then (4.2) possesses a solution and even the equality (4.1) holds. So it suffices to prove the converse statement.

We can assume without any loss of generality that e is non-negative, because e can always be replaced by $e \vee 0$. The inequality (4.2) implies

$$f+e \geqq (P-h\otimes v)f+g\,.$$

Iteration of this gives

$$f+\sum_0^N (P-h\otimes v)^n e \geqq \sum_0^N (P-h\otimes v)^n g+(P-h\otimes v)^{N+1}f \geqq \sum_0^N (P-h\otimes v)^n g\,.$$

Letting $N \to \infty$ we obtain by the monotone convergence theorem

$$f+Ge \geqq Gg\,.$$

The final assertion follows from this by integration with respect to λ and by Lemma 2.1 (ii).

COROLLARY 4.1. *If f is non-negative, and if for some bounded special e, f is a solution of the inequality* (4.3), *then for any $x \in E$, $f(x)<\infty$ implies that x is regular.*

COROLLARY 4.2. (cf. Cogburn [2, Section 4]). *Assume that E is a topological space. If for some bounded special e, the inequality* (4.3) *has non-negative, upper semicontinuous solution f, then all compact subsets of E are strongly uniform.*

COROLLARY 4.3. *The chain is uniformly ergodic, if (and only if) for some bounded special e, the inequality* (4.3) *possesses a bounded solution f.*

By recalling that $\nu G = (m(h))^{-1}m$ and imitating the proof of Theorem 6(i) with $\lambda = \nu$ we obtain the following criterion for $g \in \mathscr{E}_+$ to be m-integrable. In particular, setting $g \equiv 1$ we obtain a criterion for the positive recurrence of a Harris recurrent chain.

PROPOSITION 4.4. *For any $g \in \mathscr{E}_+$, g is m-integrable, if and only if there exist $e \in \mathscr{L}^1(m)$ and a measurable, non-negative, not identically infinite function f on E such that the inequality* (4.2) *is satified.*

PROOF. We have to show that $\nu(f)$ is finite. Iteration of (4.2) gives us

$$f \geqq Pf - e \geqq P_{N+1}f - \sum_0^N P_n e \,. \tag{4.4}$$

Let now $y \in E$ and $N \geqq 0$ be such that $f(y) < \infty$ and $P_N h(y) > 0$. This is possible since $\lambda(f) < \infty$, $m(h) > 0$ and the chain is m-irreducible. By using the Minorization Assumption $P \geqq h \otimes \nu$ we get from (4.4)

$$\infty > f(y) \geqq \nu(f) P_N h(y) - \sum_0^N P_n e \,,$$

whence $\nu(f) < \infty$. The rest of the proof is similar to that of Theorem 6(i).

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