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CHARACTERISATIONS OF EXTREMELY AMENABLE SEMIGROUPS

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1. Introduction.

Let S be a semigroup and $m(S)$ the Banach space of all bounded functions on S with supremum norm. Define $\mathcal{N}_0(S)$ to be the set of all $f \in m(S)$ such that $0 \in$ norm (or weak) closure of the convex set $M_0(S) \odot f$ where $M_0(S)$ consists of all discrete probability measures on S and

$$\mu \odot f(s) = \int f(ts)\, d\mu(t), \qquad s \in S .$$

(In particular, if ε_a is the Dirac measure at $a \in S$, $\varepsilon_a \odot f = l_a f$, the left translate of f by a). It is shown in Wong and Riazi [10] that the following conditions on S are equivalent:

(a) $m(S)$ has a left invariant mean (i.e., S is left amenable).
(b) $\mathcal{N}_0(S)$ is closed under addition.
(c) For any $\mu_1, \mu_2 \in M_0(S)$, $d(\mu_1 * M_0(S), \mu_2 * M_0(S)) = \inf\{\|\mu_1 * \mu - \mu_2 * \nu\| : \mu, \nu \in M_0(S)\} = 0$.

Here the operation $*$ is the convolution of the measure algebra $M(S)$ (S with the discrete topology) which coincides with the semigroup algebra $l_1(S)$. (See also Emerson [2] for the case of groups.)

The main purpose of this paper is to show that somewhat "similar" results hold true for multiplicative left invariant means on $m(S)$, with the set $\mathcal{N}_0(S)$ replaced by the set $\mathcal{N}_l(S)$ of all $f \in m(S)$ such that $0 \in$ norm closure of $\{l_s f : s \in S\}$ (which is no longer convex) and with $M_0(S)$ replaced by the Dirac measures $M_D(S) = \{\varepsilon_a : a \in S\}$. We also consider the situation where $M_0(S)$ is replaced by the set $M_l(S)$ of all $f \in m(S)$ such that $0 \in \sigma(\mathcal{M}, m(S))$ closure of $\{l_s f : s \in S\}$ where $\mathcal{M}$ is the set of all multiplicative means on $m(S)$. Some of our results are also valid for closed subalgebras of $m(S)$. In any case, since the sets $\{l_s f : s \in S\}$ and $M_D(S)$ are no longer convex, we have to employ a different

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approach than that in Wong and Riazi [10] and Emerson [2]. In fact, we make use of Granirer's results and arguments in [3] and [4]. Some interesting consequences of our characterisations and their relation with known results are also discussed.

2. Main results.

For notations and definitions on invariant and multiplicative invariant means, the reader is referred to Day [1] and Granirer [3] and [4]. Let S be a semigroup, $m(S)$ the Banach algebra of all bounded real functions on S with supremum norm and pointwise operations and $M_D(S)=\{\varepsilon_a : a \in S\}$ the Dirac measures on S. As in Granirer [4], let A be a norm closed left invariant subalgebra of $m(S)$ containing the constants. Define H_A as the ideal of all $h \in A$ of the form

$$h = \sum_{i=1}^{n} f_i(l_{a_i}g_i - g_i) \quad \text{for } f_i, g_i \in A,\ a_i \in S,\ 1 \leqq i \leqq n,\ n=1,2,\ldots$$

and $\mathcal{N}_l(A)$ as the set of all $f \in A$ such that $0 \in$ norm closure of the set $\{l_s f : s \in S\}$ in A. Note that $\mathcal{N}_l(A)$ is closed under scalar multiplication and $A\mathcal{N}_l(A) \subset \mathcal{N}_l(A)$ since $\|l_s(gf)\| \leqq \|g\| \cdot \|l_s f\|$. If A has a multiplicative left invariant mean and $f \in A$, we say that f is A-multiplicative left almost convergent (A-mlac) to β if $m(f)=\beta$ for any multiplicative left invariant mean m on A.

THEOREM 2.1. *Let A be a norm closed left invariant subalgebra of $m(S)$ containing the constants. Consider the following statements*
(1) *$\mathcal{N}_l(A)$ is closed under addition and $\mathcal{N}_l(A)$ contains H_A.*
(2) *A has a multiplicative left invariant mean.*

In general (1) *implies* (2). *If $A = m(S)$, or* AP (S), *the strongly almost periodic functions on S, then* (1) *and* (2) *are equivalent. In this case, $\mathcal{N}_l(A)=\{f \in A : f$ is A-mlac to $0\} = H_A^-$.*

PROOF. Suppose $\mathcal{N}_l(A)$ is closed under addition and $\mathcal{N}_l(A)$ contains H_A. Then $\mathcal{N} = \mathcal{N}_l(A)$ is an ideal in A with the following properties:

(i) $l_a f - f \in \mathcal{N}$ for any $f \in A, a \in S$ and
(ii) $f \in A$ and $\inf\{f(s): s \in S\} > 0$ implies $f \notin \mathcal{N}$.

(Since $\|l_s f\| \geqq \inf\{f(t): t \in S\}$). Therefore $\inf\{h(s): s \in S\} \leqq 0$ for any $h \in H_A$. By Granirer [4, Theorem 2, p. 101], A has a multiplicative left invariant mean and (1) implies (2). Observe that each $f \in \mathcal{N}_l(A)$ is clearly A-mlac to 0. Also if

$f \in A$ is A-mlac to 0, then $f \in H_A^-$. Otherwise $f \notin H_A^-$ implies there is a multiplicative left invariant mean m on A such that $m(f) \neq 0$ by a standard argument on maximal ideals of the commutative Banach algebra A as in Granirer [4, Lemma 3, (d) implies (a) p. 99]. Therefore to show that (2) implies (1), it is sufficient to show that $H_A^- \subset \mathcal{N}_l(A)$. If $A = m(S)$ has a multiplicative left invariant mean, then by Granirer [3, Theorem 3, p. 187], every finite subset of S has a common right zero. If $h \in H_A$ is of the form $h = \sum_{i=1}^n g_i(l_{a_i} f_i - f_i)$, $a_i \in S$, $g_i, f_i \in A$, there is some $b \in S$ such that $a_i b = b$, $1 \leqq i \leqq n$. Therefore $l_b h = 0$ or $h \in \mathcal{N}_l(A)$ which is clearly norm closed. Hence

$$\mathcal{N}_l(A) = \{f \in A : f \text{ is } A\text{-mlac to } 0\} = H_A^- .$$

In particular $\mathcal{N}_l(A)$ is closed under addition for $A = m(S)$. If $A = \mathrm{AP}(S)$ has a multiplicative left invariant mean, then there is a net $s_\alpha \in S$ such that $\varepsilon_a * \varepsilon_{s_\alpha} - \varepsilon_{s_\alpha} \to 0$ in the topology $\sigma(\mathrm{AP}(S)^*, \mathrm{AP}(S))$ for each $a \in S$. Hence for any $t \in S$,

$$|l_{s_\alpha} h(t)| \leqq \sum_{i=1}^{n} \|g_i\| \cdot |(\varepsilon_{a_i} * \varepsilon_{s_\alpha} - \varepsilon_{s_\alpha})(r_t f_i)| \to 0$$

(since AP (S) is right invariant). That is $l_{s_\alpha} h \to 0$ pointwise hence in norm and $h \in \mathcal{N}_l(A)$. The proof now proceeds as before. This completes the proof.

REMARKS. (1) In general, $\mathcal{N}_l(A)$ does not satisfy condition (i) above. For if G is a non-trivial group, then for any $f \in A = m(G)$, $a, s_\alpha \in G$, we have $\|l_{s_\alpha}(l_a f - f)\| = \|l_a f - f\|$. Hence if $l_a f - f \in \mathcal{N}_l(A)$, f must be constant, which is a contradiction.

(2) If we define $\mathcal{N}_0(A) = \{f \in A : 0 \text{ norm closure of } M_0(s) \odot f\}$, then $\mathcal{N}_0(A)$ satisfies both conditions (i) and (ii) above but unlike $\mathcal{N}_l(A)$, $\mathcal{N}_0(A)$ need not be an ideal in the multiplicative semigroup of A.

THEOREM 2.2. *The following statements are equivalent*:

(1) *$m(S)$ has a multiplicative left invariant mean* (Extreme Amenability)

(2) *There is a net $s_\alpha \in S$ such that $\|\varepsilon_a * \varepsilon_{s_\alpha} - \varepsilon_{s_\alpha}\| \to 0$ for each $a \in S$* (Strong Extreme Amenability)

(3) *For any $a, b \in S$, there exists $c \in S$ such that $ac = bc = c$ (any two elements of S have a common right zero)*

(4) *For any $a, b \in S$, there exists $c \in S$ such that $ac = bc$ and $\mathcal{N}_l(S)$ contains $l_s f - f$ for any $s \in S$, $f \in m(S)$.*

(5) *For any $a, b \in S$, there exist $c, d \in S$ such that $ac = bd$ and $\mathcal{N}_l(S)$ contains $l_s f - f$ for any $s \in S$, $f \in m(S)$.*

PROOF. The equivalence of (1), (2) and (3) is due to Granirer [3, Theorem 3,

p. 187]. Assume (3) and take $s \in S$, $f \in m(S)$. There is ome $t \in S$ such that $st = t$. Hence $l_t(l_s f - f) = 0$ and $l_s f - f \in \mathscr{N}_l(S)$. Hence (3) implies (4). Clearly (4) implies (5). Finally, assume (5). We show that $\mathscr{N}_l(S)$ is closed under addition. Take $f, g \in \mathscr{N}_l(S)$. For each n, there are some $a_n, b_n \in S$ such that $\|l_{a_n} f\| \leqq 1/n$ and $\|l_{b_n} g\| \leqq 1/n$. By (5), there are $c_n, d_n \in S$ such that $a_n c_n = b_n d_n$. Then

$$\begin{aligned} \|l_{a_n c_n}(f+g)\| &= \|l_{c_n}(l_{a_n} f) + l_{d_n}(l_{b_n} g)\| \\ &\leqq \|l_{a_n} f\| + \|l_{b_n} g\| \leqq \frac{2}{n} \to 0 \end{aligned}$$

as $n \to \infty$. Hence $f+g \in \mathscr{N}_l(S)$. By Theorem 2.1, (5) implies (1) which is equivalent to (3). This completes the proof.

REMARKS. (1) Let $a, b \in S$ be fixed. The condition that $ac = bd$ for some $c, d \in S$ is equivalent to $\|\varepsilon_a * \varepsilon_c - \varepsilon_b * \varepsilon_d\| = 0$. This in turn is equivalent to $\|\varepsilon_a * \varepsilon_{s_\alpha} - \varepsilon_b * \varepsilon_{t_\alpha}\| \to 0$ for some nets s_α, t_α in S (since $\|\varepsilon_s - \varepsilon_t\| = 0$ or 2 and is 0 iff $s = t$). That is, $0 \in$ norm closure of $\varepsilon_a * M_D(S) - \varepsilon_b * M_D(S)$ or $d(\varepsilon_a * M_D(S), \varepsilon_b * M_D(S)) = 0$ where d is the metric induced by the norm. Similarly, the condition that $ac = bc$ for some $c \in S$ is equivalent to $0 \in$ norm closure of $(\varepsilon_a - \varepsilon_b) * M_D(S)$ or $d(0, (\varepsilon_a - \varepsilon_b) * M_D(S)) = 0$. These are the kind of conditions used in Wong and Riazi [10] and Emerson [2].

(2) It is known that the existence of a multiplicative left invariant mean on $m(S)$ is equivalent to S having the common fixed point property for action of S on compacta (Granirer [3, Theorem 3]).

We now deduce some interesting consequences from the preceeding Theorem. By definition, a semigroup S is of class π if every element of S has a right zero (Ljapin [8]). In [3, p. 188], Granirer mentioned that each extremely left amenable semigroup is of class π but the converse is false. Here, we have found the missing link:

COROLLARY 2.3. *Let S be a semigroup of class π, then S is extremely left amenable iff for any $a, b \in S$, there are some $c, d \in S$, such that $ac = bd$.*

PROOF. Let $s \in S$, $f \in m(S)$. Then there is some $t \in S$ such that $st = t$ and so $l_t(l_s f - f) = l_{st} f - l_t f = 0$ or $l_s f - f \in \mathscr{N}_l(S)$. The Corollary now follows from Theorem 2.2. (Of course, a direct proof is also quite simple).

The condition that for any $a, b \in S$, $ac = bd$ for some $c, d \in S$ is always satisfied if S is commutative. Therefore we have

COROLLARY 2.4. *A commutative semigroup is extremely (left) amenable iff $\mathscr{N}_l(S)$ contains $l_s f - f$ for any $s \in S$ and $f \in m(S)$.*

PROOF. Trivial.

It follows that each commutative semigroup of class π is always extremely amenable.

Theorem 2.2 has its counterpart for $A = \mathrm{AP}(S)$.

THEOREM 2.5. *The following statements are equivalent:*

(1) $\mathrm{AP}(S)$ *has a multiplicative left invariant means.*

(2) *There is a net* $s_\alpha \in S$ *such that* $\varepsilon_a * \varepsilon_{s_\alpha} - \varepsilon_{s_\alpha} \to 0$ *in the topology* $\sigma(\mathrm{AP}(S)^*, \mathrm{AP}(S))$ *for each* $a \in S$.

(3) *For any* $a, b \in S$, *there is a net* $s_\alpha \in S$ *such that* $(\varepsilon_a - \varepsilon_b) * \varepsilon_{s_\alpha} \to 0$ *in the topology* $\sigma(\mathrm{AP}(S)^*, \mathrm{AP}(S))$ *and* $\mathcal{N}_l(\mathrm{AP}(S))$ *contains* $l_s f - f$ *for any* $s \in S$, $f \in \mathrm{AP}(S)$.

(4) *For any* $a, b \in S$, *there are nets* $s_\alpha, t_\alpha \in S$ *such that* $\varepsilon_a * \varepsilon_{s_\alpha} - \varepsilon_b * \varepsilon_{t_\alpha} \to 0$ *in the topology* $\sigma(\mathrm{AP}(S)^*, \mathrm{AP}(S))$ *and* $\mathcal{N}_l(\mathrm{AP}(S))$ *comtains* $l_s f - f$ *for any* $s \in S$, $f \in \mathrm{AP}(S)$.

PROOF. It is well-known that (1) and (2) are equivalent. To show that (2) implies (3), let s_α be a net in S such that $\varepsilon_a * \varepsilon_{s_\alpha} - \varepsilon_{s_\alpha} \to 0$ in the topology $\sigma(\mathrm{AP}(S)^*, \mathrm{AP}(S))$ for any $a \in S$. Then $(\varepsilon_a - \varepsilon_b) * \varepsilon_{s_\alpha} = \varepsilon_a * \varepsilon_{s_\alpha} - s_\alpha + s_\alpha - \varepsilon_b * \varepsilon_{s_\alpha} \to 0$ in the same topology. Moreover, for each $t \in S$,

$$l_{s_\alpha}(l_s f - f) = (\varepsilon_s * \varepsilon_{s_\alpha} - \varepsilon_{s_\alpha})(r_t f) \to 0$$

since $r_t f \in \mathrm{AP}(S)$. That is $l_{s_\alpha}(l_s f - f) \to 0$ pointwise, hence in norm or $l_s f - f \in \mathcal{N}(\mathrm{AP}(S))$. Clearly (3) implies (4). Finally, assume (4). We want to show that $\mathcal{N}_l(\mathrm{AP}(S))$ is closed under addition. Take $f_1, f_2 \in \mathcal{N}_l(\mathrm{AP}(S))$. There are nets (even sequences) $a(m)$, $b(m)$, $(m \in D)$ such that

$$\|l_{a(m)} f_1\| \to 0, \qquad \|l_{b(m)} f_2\| \to 0 .$$

By assumption, for each $m \in D$, there are nets $s(m, n)$, $t(m, n)$, $(n \in E_m)$ such that

$$\varepsilon_{a(m)} * \varepsilon_{s(m,n)} - \varepsilon_{b(m)} * \varepsilon_{t(m,n)} \to 0 \quad \text{with respect to } n ,$$

in the topology $\sigma(\mathrm{AP}(S)^*, \mathrm{AP}(S))$. Define, for each (m, f) in the product directed set $D \times \prod \{E_m : m \in D\}$, $R(m, f) = (m, f(m))$ and put

$$S(m, n) = l_{a(m)s(m,n)} f_2 - l_{b(m)t(m,n)} f_2, \qquad m \in D,\ n \in E_m .$$

Since

$$(l_{a(m)s(m,n)} f_2 - l_{b(m)t(m,n)} f_2)(x) = (\varepsilon_{a(m)} * \varepsilon_{s(m,n)} - \varepsilon_{b(m)} * \varepsilon_{t(m,n)})(r_x f_2) \xrightarrow{n} 0$$

for each $x \in S$ and $m \in D$, it follows that $\lim_n S(m, n) = 0$ in the pointwise topology of $\mathrm{AP}(S)$. Therefore, the double limit

$$\lim_m \lim_n S(m,n) = 0 ,$$

in the same topology. By Kelley [6, Theorem 4, p. 69],

$$\lim_{(m,f)} S\circ R(m,f) = 0$$

in the pointwise topology of AP (S). In other words

$$\lim_{(m,f)} l_{a(m)s(m,f(m))}f_2 - l_{b(m)t(m,f(m))}f_2 = 0 \quad \text{pointwise} .$$

But

$$\|l_{a(m)s(m,f(m))}f_1\| \leqq \|l_{a(m)}f_1\|, \quad \|l_{b(m)t(m,f(m))}f_2\| \leqq \|l_{b(m)}f_2\| .$$

Hence

$$\lim_{(m,f)} l_{a(m)s(m,f(m))}f_1 = 0 \quad \text{and} \quad \lim_{(m,f)} l_{b(m)t(m,f(m))}f_2 = 0$$

both in norm hence pointwise. Consequently,

$$\begin{aligned} &l_{a(m)s(m,f(m))}(f_1+f_2) \\ &= l_{a(m)s(m,f(m))}f_1 + l_{b(m)t(m,f(m))}f_2 + \\ &\qquad + l_{a(m)s(m,f(m))}f_2 - l_{b(m)t(m,f(m))}f_2 \xrightarrow{(m,f)} 0 \end{aligned}$$

pointwise, hence in norm (since $f_1+f_2 \in$ AP (S)). That is $f_1+f_2 \in \mathcal{N}_l(\text{AP }(S))$. By Theorem 2.1, (4) implies (1). This completes the proof.

COROLLARY 2.6. *If S is commutative, then* AP (S) *has a multiplicative left invariant mean iff $\mathcal{N}_l(\text{AP }(S))$ contains l_sf-f for amy $s \in S$, $f \in$* AP (S).

COROLLARY 2.7. *If S is of class π, then* AP (S) *has a multiplicative left invariant mean iff for any $a, b \in S$, there are nets s_α, t_α in S such that $\varepsilon_a * \varepsilon_{s_\alpha} - \varepsilon_b * \varepsilon_{t_\alpha} \to 0$ in the topology $\sigma(\text{AP }(S)^*, \text{AP }(S))$.*

RMARK. It is also known that AP (S) has a multiplicative left invariant mean iff S has the common fixed point property for equicontinuous actions of S on compacta (Lau [7, Theorem 3.6]).

3. A second approach.

DEFINITION 3.1. Again, let A be a norm closed left invariant subalgebra of $m(S)$ containing the constants. Define $\mathcal{M}_l(S)$ as the set of all $f \in A$ for which there is a net s_α in S such that $m(l_{s_\alpha}f) \to 0$ for any m in the set $\mathcal{M}$ of all multiplicative means on A. That is, $f \in \mathcal{M}_l(A)$ iff $0 \in \sigma(\mathcal{M}, A)$ closure of the set $\{l_sf : s \in S\}$.

Clearly $\mathscr{M}_l(A) \supset \mathscr{N}_l(A)$. Moreover, the set $\mathscr{M}_l(A)$, like $\mathscr{N}_l(A)$ has the following properties:

(i) $\mathscr{M}_l(A)$ is closed under scalar multiplication,
(ii) $A\mathscr{M}_l(A) \subset \mathscr{M}_l(A)$ (since $|m(l_s(fg))| = |m(l_s f)| \cdot |m(l_s g)| \leqq \|f\| \cdot |m(l_s(g))|$ for any $m \in \mathscr{M}$) and
(iii) $f \in A$, $\inf\{f(s) : s \in S\} > 0$ implies $f \notin \mathscr{M}_l(A)$ (since $m(l_s f) \geqq \inf\{f(t) : t \in S\}$ for any $m \in \mathscr{M}$).

The algebra A is called multiplicative left introverted (left M-introverted) in $m(S)$ if the function $s \to m(l_s f)$ always belongs to A for any $f \in A$, and any multiplicative mean m on A. (See Mitchell [9]). For such A, we have

THEOREM 3.2. *Let A be a multiplicative left introverted norm closed left invariant subalgebra of $m(S)$ containing the constants. The following statements are equivalent*:
(1) *$\mathscr{M}_l(A)$ is closed under addition and $\mathscr{M}_l(A)$ contains H_A.*
(2) *A has a multiplicative left invariant mean.*
In this case, $\mathscr{M}_l(A) = \{f \in A : f$ is A-mlac to $0\} = H_A^-$.

PROOF. As in the proof of Theorem 2.1, (1) implies (2). Conversely assume (2). We have $\mathscr{M}_l(A) \subset \{f \in A : f$ is A-mlac to $0\} \subset H_A^-$. We need only show that $H_A^- \subset \mathscr{M}_l(A)$. Let s_α be a net in S such that for each $a \in S$,

$$\varepsilon_a * \varepsilon_{s_\alpha} - \varepsilon_{s_\alpha} \to 0$$

in the topology $\sigma(A^*, A)$. If $f \in H_A$ is of the form

$$h = \sum_{i=1}^{n} g_i(l_{a_i} f_i - f_i) ,$$

$a_i \in S$, $f_i, g_i \in A$. Then for any multiplicative mean m on A, we have

$$\begin{aligned} |m(l_{s_\alpha} h)| &\leqq \sum_{i=1}^{n} |m(l_{s_\alpha} g_i)| \cdot |(\varepsilon_{a_i} * \varepsilon_{s_\alpha} - \varepsilon_{s_\alpha})(m_l(f_i))| \\ &\leqq \sum_{i=1}^{n} \|g_i\| \cdot |(\varepsilon_{a_i} * \varepsilon_{s_\alpha} - \varepsilon_{s_\alpha})(m_l(f_i))| \end{aligned}$$

where $m_l(f_i)(s) = m(l_s f_i)$, $s \in S$. Hence $m(l_{s_\alpha} h) \to 0$ since $m_l(f_i) \in A$, $1 \leqq i \leqq n$, by multiplicative left introvertedness of A. That is, $H_A \subset \mathscr{M}_l(A)$ which is norm closed. (For example, use the double limit argument in the proof of Theorem 2.5). Consequently, $\mathscr{M}_l(A) = \{f \in A : f$ is A-mlac to $0\} = H_A^-$ which is certainly closed under addition. This completes the proof.

REMARKS. (1) Theorem 3.2 is stronger than Theorem 2.1. However, it is only valid for any multiplicative left introverted norm closed left invariant subalgebras A of $m(S)$ containing the constants, including $A=m(S)$ or AP (S). For $A=$ AP (S), there is nothing new since $\mathcal{N}_l(A)=\mathcal{M}_l(A)$. For if $f \in$ AP (S), the topology $\sigma(\mathcal{M}, A)$ coincides with the norm topology on $\{l_s f : s \in S\}^-$ (closure in norm topology). However, we are unable to obtain a version of Theorem 3.2 with $\mathcal{M}_l(S)$ in place of $\mathcal{N}_l(S)$. On the other hand, if $m(S)$ has a multiplicative left invariant mean, then $\mathcal{N}_l(S)=\mathcal{M}_l(S)$, because they both coincide with H^- by Theorems 2.1 and 3.2.

(2) There are many other multiplicative left introverted norm closed left invariant subalgebras of $m(S)$ which contain the constants. For example, the weakly periodic functions WAP (S), the left uniformly continuous functions LUC (S) and the left multiplicatively continuous functions LMC (S) of a topological semigroup S (see Mitchell [9]).

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