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SPREADING BASIC SEQUENCES AND SUBSPACES OF JAMES' QUASI-REFLEXIVE SPACE

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Abstract.

We prove that every sequence from J having no nonzero weak cluster point has a subsequence equivalent to either the unit vector basis of l_2 or to a spreading basis for J. This implies that J embeds isomorphically in each of its non-reflexive subspaces, and that the only spreading models of J with basic fundamental sequence are J and l_2.

1. Introduction.

We discuss the existence and number of spreading basic sequences in James' quasi-reflexive Banach space J, and make applications of this study to spreading models of J and subspaces of J.

In section 2 we show that in one sense, J has many spreading basic sequences, yet only two, up to equivalence. Namely, any seminormalized sequence with no weak cluster point has a subsequence which is a spreading basic sequence. Moreover, the subsequence can be chosen to have complemented span and to be equivalent either to the unit vector basis of l_2 or to a certain basis for J. This latter result implies that J has, up to equivalence, precisely two spreading basic sequences. Since J is primary [4], it possesses exactly two spreading basic sequences in an essential manner, in that it is not the direct sum of two spaces each having a unique spreading basic sequence.

In section 3 we use the results of section 2 to study subspaces of J, and show that every non-reflexive subspace of J contains a complemented isomorph of J. This extends results of Casazza [4].

In section 4 we show that J has precisely two spreading models, itself and l_2. Here we assume that the fundamental sequence defined in [2] is a Schauder basis.

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We use the standard notation of Banach space theory. If $(z_n) \subset Z$, we denote the closed linear span of (z_n) by $[(z_n)]$, and say (z_n) is seminormalized if there exists a constant M such that $M^{-1} \leqq \|z_n\| \leqq M$ for all n.

Recall that sequences (y_n) and (z_n) in Banach spaces Y and Z are *equivalent* if there exists a constant K such that for any scalar sequence (a_n),

$$K^{-1}\|\sum a_n y_n\| \leqq \|\sum a_n z_n\| \leqq K\|\sum a_n y_n\| .$$

A basic sequence (y_n) in a Banach space Y is said to be *spreading* if (y_n) is equivalent to each of its subsequences. If, in addition, (y_n) is unconditional, it is said to be *subsymmetric*.

The notion of spreading model is due to Brunel and Sucheston [3]. They showed that if (y_n) is a bounded sequence from a Banach space Y, then there exists a subsequence (y'_n) such that for every scalar sequence $(a_i)_{i=1}^n$, and every choice of integers $\{k_1 < \ldots < k_n\}$, the limit

$$L((a_i)) = \lim_{k_1 \to \infty} \left\| \sum_{i=1}^{n} a_i y'_{k_i} \right\| \tag{1}$$

exists. In the event that (y_n) has no Cauchy subsequences, formula (1) defines a norm $|\cdot|$ on the space S of all finite real sequences by

$$\left| \sum_{i=1}^{n} a_i f_i \right| = L((a_i)_{i=1}^n) . \tag{2}$$

The sequence of unit vectors (f_i) is clearly spreading. The completion of S under this norm is called the *spreading model of Y with fundamental sequence (f_n) based on (y'_n)*.

James' space J [8], [9], [10], is the Banach space of all null sequences of scalars for which the squared-variation norm

$$\left\| \sum_{i=1}^{\infty} a_i e_i \right\| = \sup_{\substack{n \\ p_0 < \ldots < p_n}} \left[\sum_{i=1}^{n} |a_{p_i} - a_{p_{i-1}}|^2 \right]^{\frac{1}{2}} \tag{3}$$

is finite, and the second conjugate J^{**} is the space of all sequences for which the squared-variation norm is finite [7]. Notice that $\|\sum a_i e_i\| < \infty$ implies $\lim_{i \to \infty} a_i$ exists. We shall reserve the notation (e_i) for the unit vector basis in James' space, (e_i^*) for the biorthogonal sequence, and P_n for the natural projections associated with (e_i). We will regard e_i^* and P_n as defined on J and also on J^{**}.

The sequence $(x_n)_{n=1}^{\infty} \subset J$ defined by $x_n = \sum_{i=1}^{n} e_i$ is known to be a basis for J, and the norm may be computed as

$$\|\sum b_n x_n\| = \sup_{\substack{n \\ p_0 < \ldots < p_n}} \left[\sum_{i=1}^{n} \left| \sum_{j=p_{i-1}}^{p_i - 1} b_j \right|^2 \right]^{\frac{1}{2}} . \tag{4}$$

From either (3) or (4) it follows that (x_n) is a spreading sequence. We shall reserve the notation (x_n) for this spreading basis of J.

The following lemma is obtained easily from (3) and (4). Proofs may be found in [4], [5].

LEMMA 1.1. a) *For any scalar sequence* (b_n),

$$\|\sum b_n x_n\| \geqq [\sum |b_n|^2]^{\frac{1}{2}} .$$

(b) *If* $w_n = \sum_{q_n}^{r_n} a_i e_i$, $\|w_n\| = 1$ *for all* n, *and* $r_{n-1} + 1 < q_n \leqq r_n$ *for all* n, *then for any scalar sequence* (b_n),

$$[\sum b_n^2]^{\frac{1}{2}} \leqq \|\sum b_n w_n\| \leqq \sqrt{2}\, [\sum b_n^2]^{\frac{1}{2}} .$$

2. Spreading basic sequences in J.

In [5], Casazza, Lin, and Lohman proved that every subsymmetric basic sequence in J is equivalent to the unit vector basis of l_2. One consequence of the main result of this section is that any spreading basic sequence in J is equivalent either to the unit vectors in l_2 or to the spreading basis (x_n) of J.

The main result is

THEOREM 2.1. *Let* (z_k) *be a seminormalized sequence from* J *having no non-zero weak cluster point. Then*

a. *if* (z_k) *has a weakly null subsequence, then there is a subsequence* (z_{n_k}) *equivalent to the unit vectors in* l_2 *with* $[(z_{n_k})]$ *complemented in* J *or*

b. *if* (z_n) *has no weak cluster point, then there is a subsequence* (z_{n_k}) *equivalent to* (x_k) *with* $[(z_{n_k})]$ *complemented in* J.

We present the proof in a sequence of propositions, and will use the following standard perturbation argument [11].

PROPOSITION 2.2. *Let* (y_n) *be a basic sequence in a Banach space* Y *having basis constant* M, *and assume* $[(y_n)]$ *is complemented by a projection* P. *If* $(z_n) \subset Y$ *satisfies*

$$\sum_{n=1}^{\infty} \|y_n - z_n\| < \frac{1}{8M\|P\|} ,$$

then (z_n) *is a basic sequence equivalent to* (y_n) *and* $[(z_n)]$ *is complemented in* Y.

In the case that (z_n) has a weakly null subsequence, the result follows from the argument of [5].

PROPOSITION 2.3. *If* $(z_n) \subset J$ *is a seminormalized sequence having a weakly null* *subsequence, then there is a subsequence (z_{n_k}) having span complemented in J and equivalent to the unit vector basis of l_2.*

PROOF. Assume $\|z_n\| = 1$ for all n and $z_n \to 0$ weakly. Let (ε_k) be a sequence of positive real numbers such that $\sum \varepsilon_k < 2^{-7}$ and choose an increasing sequence of integers (n_k) such that with $z'_{n_k} = (P_{n_k} - P_{n_{k-1}})z_{n_k}$, we have

$$\tfrac{1}{2} \leqq \|z'_{n_k}\| \leqq 2$$

and

$$\|z_{n_k} - z'_{n_k}\| < \varepsilon_k .$$

Then by Lemma 1.1b, $(z'_{n_{2k}})$ is $4\sqrt{2}$-equivalent to the unit vectors in l_2, and is hence a spreading basic sequence with basis constant at most $4\sqrt{2}$. By Theorem 10 of [5], $[(z'_{n_{2k}})]$ is complemented in J by a projection P of norm at most $2\sqrt{2}$. Since

$$\sum \|z'_{n_{2k}} - z_{n_{2k}}\| < \sum \varepsilon_k < 2^{-7} < \frac{1}{8M\|P\|},$$

it follows from Proposition 2.2 that $(z_{n_{2k}})$ is equivalent to the unit vectors in l_2 and has complemented span.

REMARK 2.4. Suppose now that (z_n) has no weakly null subsequence. By regarding (z_n) as a sequence in J^{**} and passing to a subsequence, (z_n) has a weak* limit z in J^{**}. Since $(z_n) \subset J$ has no weak limit, $z \notin J$. The proof of Theorem 2.1b. is based on regarding the sequence (z_n) as arising from its weak* limit in J^{**}.

We begin with a simple yet important case.

PROPOSITION 2.5. *Suppose $y \in J^{**} - J$ and $(n_k) \subset \mathbf{N}$ is an increasing sequence. Let $y_k = P_{n_k} y$, and suppose (n_k) is such that $y_k \neq y_{k+1}$ for any k. Then*

a. *(y_k) is a spreading basic sequence equivalent to (x_k). The constant of this eqtivalence has bound depending on y but not on (n_k).*

b. *$[(y_k)]$ is complemented in J by a projection P, and $\|P\| < B$, where B depends on y but not on (n_k).*

PROOF. The proof is simplest in the case where $e_j^*(y) \neq 0$ for all j, and we present this case first. Let $T\colon J \to J$ be the operator defined by

$$T(\sum a_i e_i) = \sum e_i^*(y) a_i e_i .$$

That is, T is coordinatewise multiplication by $y \in J^{**}$. Since J^{**} is the multiplier algebra of J [1], T is bounded.

Moreover, $Tx_{n_k}=y_k$, so that for any scalar sequence (a_n),

$$\begin{aligned}\|\sum a_n y_n\| &\leqq \|T\|\,\|\sum a_k x_{n_k}\|\\ &= \|T\|\,\|\sum a_k x_k\|\\ &\leqq 2\|y\|\,\|\sum a_k x_k\|\,.\end{aligned}$$

Since $y \in J^{**}-J$ and $e_j^*(y)\neq 0$ for any j, we have that

$$\lim_j e_j^*(y) \neq 0 \quad \text{and} \quad \max_j\,[|e_j^*(y)|^{-1}] < \infty\,.$$

It follows that the sequence

$$y' = (e_j^*(y)^{-1}) \in J^{**} \quad \text{and} \quad \|y'\| \leqq \max_j\,[|e_j^*(y)|^{-1}]^2\|y\|\,.$$

Thus multiplication by y', which is T^{-1}, is a bounded operator on J, and hence

$$\|\sum a_k x_k\| \leqq \|T^{-1}\|\,\|\sum a_k y_k\|$$

for all scalar sequences (a_k). Thus (x_k) is equivalent to (y_k) and the constant of the equivalence is

$$\|T\|\,\|T^{-1}\| \leqq 4\|y\|^2 \max\,(|e_j^*(y)|^{-2})\,.$$

Casazza [4] has proved that for any sequence (n_k), (x_{n_k}) is complemented by a contractive projection P. But then TPT^{-1} is a projection from J onto $[(y_k)]$ with norm independent of (n_k).

In the case that there do exist j with $e_j^*(y)=0$, since $y \in J^{**}-J$ it follows that $\{j : e_j^*(y)=0\}$ is finite, say $\{j_1<j_2<\ldots<j_l\}$. The permutation τ of $\mathbf{N}$ defined by $\tau(j)=i$ if $j=j_i$, $\tau(j)=j$ for $j>j_l$, and the requirement that τ preserves order in the remaining cases induces an automorphism U of J [1]. Denoting by S the operator on J defined by $Se_n=e_{n+l}$, the sequence (y_k) is equivalent to the sequence Uy_k which has no nonzero coordinates when regarded as a sequence in the space SJ, which is isometric to J. By the preceding arguments, (Uy_k) is equivalent to (x_k) and complemented in SJ. It follows that (y_k) is equivalent to (x_k) and is complemented in J. The constant of the equivalence and norm of the projection now depend on U, but U is determined by y and is independent of (n_k).

We shall use the following lemma.

LEMMA 2.6. *Let (q_i), (r_i) be sequences of natural numbers such that for all i, $q_i<r_i<r_i+1<q_{i+1}$, and define a projection Q on J^{**} by*

$$e_j^*(Qy) = \begin{cases} e_{r_i+1}^*(y) & q_i \leqq j \leqq r_i \\ e_j^*(y) & \text{otherwise}\,. \end{cases}$$

Then $\|Q\|=1$ and $Qy \in J$ if and only if $y \in J$.

PROOF. Any estimate by (3) of $\|Qy\|$ is also an estimate of $\|y\|$. Hence $\|Qy\| \leqq \|y\|$ for any $y \in J$. For $y \in J^{**}$, $\|y\| = \lim_k \|P_k y\|_J$, so Q has norm one on J^{**} also.

Since $\lim_j e_j^*(Qy) = \lim_j e_j^*(y)$, it follows that $Qy \in J$ if and only if $y \in J$. The proof of Theorem 2.1 is now completed by

PROPOSITION 2.7. *Let (z_n) be a seminormalized sequence from J having no weak cluster point. Then (z_n) has a subsequence (z_{n_k}) equivalent to (x_k) and such that $[(z_{n_k})]$ is complemented in J.*

PROOF. Assume $M^{-1} \leqq \|z_n\| \leqq M$ for all n. Regarding (z_n) as a sequence from J^{**}, and passing to a subsequence we may assume, by Remark 2.4, that (z_n) has a weak* limit $y \in J^{**} - J$. Let (ε_k) be a sequence of reals decreasing to zero, and choose a subsequence, also denoted by (z_k), and an increasing sequence of integers (n_k) such that

$$\|P_{n_k} z_k - z_k\| < \varepsilon_k$$

and

$$\tfrac{1}{2} M^{-1} \leqq \|P_{n_k} z_k\| \leqq M .$$

Let $z_k' = P_{n_k} z_k$. The sequence $(z_k') \subset J^{**}$ converges in the weak* topology to y, so we may select an increasing sequence of natural numbers (m_k) such that

$$\|P_{n_{m_{k-1}}+2}(z_{m_k}' - y)\| < \varepsilon_k .$$

Define

$$\begin{aligned} z_k'' &= (P_{n_{m_{k-1}}+2})y + (I - P_{n_{m_{k-1}}+2})(z_{m_k}') \\ &= y_k + w_k . \end{aligned}$$

We may write

$$\begin{aligned} w_k &= \sum_{j=q_k}^{r_k} b_j e_j, \quad \text{with} \\ q_k &= n_{m_{k-1}} + 3, \quad \text{and} \qquad (5) \\ r_k &= n_{m_k} , \end{aligned}$$

so that the sequences (q_i) and (r_i) satisfy the hypotheses of Lemmas 1.1.b and 2.6. Since $\|w_k\| \leqq 2M$ for all k, it follows from Lemma 1.1 that

$$\|\sum a_k w_k\| \leqq 2M[\sum |a_k|^2]^{\frac{1}{2}}$$

for all scalar sequences (a_k). Hence for any sequence (a_k)

$$
\begin{aligned}
\|\sum a_k z_k''\| &\leqq \|\sum a_k y_k\| + \|\sum a_k w_k\| \\
&\leqq 2\|y\| \, \|\sum a_k x_k\| + 2M[\sum |a_k|^2]^{\frac{1}{2}} \\
&\leqq 2[\|y\| + M]\|\sum a_k x_k\| ,
\end{aligned} \tag{6}
$$

by Proposition 2.5 and Lemma 1.1.

Let Q be the projection defined in Lemma 2.6 using sequences (r_i) and (q_i) defined in (5). Let $y' = Qy \in J^{**} - J$, and let $y_k' = (P_{n_{m_{k-1}}+2})y$. Since

$$e_{r_i+1}^*(w_k) = 0$$

for all i and k, it follows that

$$Qz_k'' = (P_{n_{m_{k-1}}+2})(y') = y_k' . \tag{7}$$

Thus for any scalar sequence (a_k),

$$
\begin{aligned}
\|\sum a_k z_k''\| &\geqq \|Q \sum a_k z_k''\| \\
&= \|\sum a_k y_k'\| \\
&\geqq A\|\sum a_k x_k\|
\end{aligned} \tag{8}
$$

for some $A > 0$ which depends on y'. It follows from (6) and (8) that (z_k'') is equivalent to (x_k). The basis constant of (z_k'') is the constant A_1 of this equivalence. We also have that for some constant A_2, (z_k'') is A_2-equivalent to (y_k'), so that from (7) we see that $Q|_{[(z_k'')]}$ is an isomorphism from $[(z_k'')]$ onto $[(y_k')]$. By Proposition 2.5 there is a constant A_3 such that any subsequence of (y_k') has span complemented by a projection P with $\|P\| \leqq A_3$. Thus any subsequence of (z_k'') is complemented by a projection $Q|_{[(z_k'')]}{}^{-1}PQ$ of norm smaller than $A_2 A_3$. Now

$$
\begin{aligned}
\|z_k'' - z_{m_k}\| &\leqq \|z_k'' - z_{m_k}'\| + \|z_{m_k}' - z_{m_k}\| \\
&\leqq \|P_{n_{m_{k-1}}+2}(z_{m_k}' - y)\| + \|z_{m_k}' - z_{m_k}\| \\
&< \varepsilon_k + \varepsilon_{m_k} < 2\varepsilon_k ,
\end{aligned}
$$

so that if k_l is chosen so that

$$\sum \varepsilon_{k_l} < \frac{1}{16 A_1 A_2 A_3} ,$$

it follows from Proposition 2.2 that $(z_{m_{k_l}})$ is equivalent to (x_l) and has complemented span. This completes the proof of Proposition 2.7 and Theorem 2.1.

Theorem 2.1 may be used to extend a result of Casazza, Lin, and Lohman [5].

COROLLARY 2.8. *If* (z_n) *is a spreading basic sequence in* J, *then*

(a) *if* z_n *converges weakly to zero, then* (z_n) *is equivalent to the unit vector basis of* l_2.

(b) *If* $z_n \not\to 0$ *weakly, then* (z_n) *is equivalent to* (x_n).

PROOF. Since (z_n) is a basic sequence, (z_n) has no non-zero weak cluster point. Thus, by Theorem 2.1, if (z_n) converges weakly to zero, (z_n) has a subsequence equivalent to the unit vector basis of l_2. Since (z_n) is spreading, (z_n) is itself equivalent to the l_2 basis. In the case that (z_n) is not weakly null, Theorem 2.1 implies that (z_n) has a subsequence equivalent to (x_n), from which it follows that (z_n) is equivalent to (x_n).

3. Subspaces of J.

In this section we use Theorem 2.1 to obtain results concerning subspaces of J.

THEOREM 3.1. *If* $X \subset J$, *then* X *is isomorphic to* $Y \oplus R$ *where* R *is reflexive,* Y *is complemented in* J, *and* Y *is either trivial or isomorphic to* J.

PROOF. If X is reflexive, we choose $Y = \{0\}$ and $R = X$.

Otherwise, there exists a sequence $(z_n) \subset X$, $\|z_n\| = 1$ for all n such that (z_n) has no weak cluster point in J. By Theorem 2.1b, there is a subsequence (z_{n_k}) equivalent to (x_k) with span complemented in J by a projection P. We take $Y = [(z_{n_k})]$, and $R = (I-P)X$. Then Y is isomorphic to J, and hence [5], $(I-P)J$ is reflexive. Since $R \subset (I-P)J$, R is also reflexive.

Theorem 3.1 specializes to

COROLLARY 3.2. *If* $X \subset J$ *is non-reflexive, then there exists a subspace* $Y \subset X$ *such that* Y *is isomorphic to* J *and* Y *is complemented in* J.

4. Spreading Models of J.

In this section we show that if a Banach space X is a spreading model of J, then either X is isomorphic to l_2 or X is isomorphic to J. In fact, if a basis (f_n) is a fundamental sequence based on $(y_n) \subset J$ then either (f_n) is equivalent to the unit vectors in l_2 or (f_n) is equivalent to the spreading basis (x_n) for J. We assume here that the fundamental sequence is Schauder basis.

We shall use a result of Guerre and Lapresté [7], THEOREM 4.1. *Let (f_n) be the fundamental sequence of a spreading model of a Banach space X, based on a sequence $(y_n) \subset X$. Then*

a. *If $f_n \to 0$ weakly, then $y_n \to 0$ weakly.*

b. *If (f_n) is weakly Cauchy but not weakly convergent, then (y_n) is weakly Cauchy but not weakly convergent.*

We have then

THEOREM 4.2. *Let a basis (f_n) be the fundamental sequence of a spreading model of J, based on a sequence $(y_n) \subset J$. Then either (f_n) is equivalent to the unit vectors in l_2 or (f_n) is equivalent to the basis (x_n) for J.*

PROOF. We consider three cases.

1. If (f_n) converges weakly to zero, then the same is true of (y_n) by Theorem 4.1a. By Theorem 2.1 we may pass to a subsequence (y_{n_k}) equivalent to the unit vector basis in l_2. But this implies that (f_n) is equivalent to the unit vector basis in l_2.

2. If (f_n) is weakly Cauchy, yet not weakly convergent, then the same is true of (y_n) by Theorem 4.1b. By Theorem 2.1 there is a subsequence (y_{n_k}) equivalent to the spreading basis (x_n) for J. But this implies that (f_n) is itself equivalent to (x_n).

3. Since (f_n) is assumed to be a basis, the only remaining case is that when (f_n) is not weakly Cauchy. Since (f_n) is spreading, it follows from a result of Rosenthal [12] that (f_n) is equivalent to the unit vector basis in l_1. Thus it is sufficient to show that l_1 is not a spreading model of J.

Now Proposition I.2 of [2] asserts that a Banach space X not containing l_1 has l_1 as a spreading model if and only if X fails the Banach–Saks–Rosenthal (BSR) property. Recall that X has BSR if every weakly null sequence has a subsequence with norm convergent Cesaro means. Now $l_1 \not\subset J$, and J has BSR since every weakly null sequence in J has a subsequence equivalent to the unit vectors in l_2. It follows that J does not have l_1 as a spreading model.

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