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EMBEDDING $C(K)$ IN $B(X)$

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Summary.

We establish a one-to-one correspondence between the embeddings of a space $C(K)$ of continuous functions on a compact Hausdorff space in the space $B(X)$ of bounded operators on a Banach space X and a class of mappings of the elements of X into the lattice of closed subsets of K.

1. Introduction.

Many characteristic sub-algebras of $B(X)$, the Banach algebra of bounded operators on a Banach space X, can be shown to be (isometrically) isomorphic to a $C(K)$, i.e. to the Banach algebra of continuous functions on a suitably chosen compact Hausdorff space with the supremum norm. If this is shown by means of an abstract representation theorem, then it is often not clear what K looks like. However in certain concrete cases (e.g. the centre of a C*-algebra, the centraliser of a Banach space) it has been shown that K can be constructed from certain subspaces of X equipped with the Jacobson topology. In honour of a now classic theorem [4] these are usually called theorems of Dauns–Hofmann type. Elliott has proved a quite general theorem of this type [5] and in this paper we shall use basically his methods to establish a correspondence which holds for all continuous homomorphic embeddings of a $C(K)$ in a $B(X)$.

2. Capacities and support representations.

In order to present a unified treatment we first show the equivalence of three different ways of looking at the problem of connecting a Banach space X with a compact Hausdorff space K.

NOTE. Throughout this paper X is a fixed Banach space and K a compact Hausdorff topological space. The letters F and H (with or without suffixes) are reserved for closed sets while the letter G always denotes open sets. Unless explicitly stated, the scalars can be either real or complex.

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DEFINITION 1. A mapping $J\colon F \mapsto J(F)$ which maps each closed subset F of K onto a closed subspace $J(F)$ of X in such a manner that:

i) $J(\emptyset)=\{0\}$, $J(K)=X$,
ii) $J(\bigcap F_\alpha)=\bigcap J(F_\alpha)$ for arbitrary collections,
iii) $F\subseteqq F_1^\circ \cup F_2^\circ \Rightarrow J(F)\subseteqq J(F_1)+J(F_2)$ where $^\circ$ denotes the interior

is called a *capacity* (on K taking values in X).

It is called a co-*capacity* if instead:

1) $J(\emptyset)=X$, $J(K)=\{0\}$,
2) $J(F_1\cap F_2)=J(F_1)+J(F_2)$,
3) $J(F)=\sup\{J(F_\alpha) : F_\alpha^\circ \supseteqq F\}$ in the lattice of closed subspaces of X,
4) $F\subseteqq F_1^\circ \cup F_2^\circ \Rightarrow J(F_1)\cap J(F_2)\subseteqq J(F)$.

Note that iii) and 4) can be extended to arbitrary finite collections by induction, since if $F\subseteqq F_1^\circ\cup\ldots\cup F_n^\circ$ we can find $H_2,\ldots,H_{n-1}$ satisfying $F\subseteqq F_1^\circ\cup H_2^\circ$, $H_2\subseteqq F_2^\circ \cup H_3^\circ$ and so on.

A capacity or co-capacity is said to be *bounded* if there is a constant M such that every decomposition $x=x_1+x_2$ implied by iii) or 2) can be performed so that $\|x_1\|, \|x_2\|\leqq M\|x\|$.

EXAMPLES. The map $F\mapsto J(F):=\{f : f\in C(K),\ f\equiv 0 \text{ on } F\}$ is clearly a bounded (with $M=1$) co-capacity on K taking values in $C(K)$. More generally let X be a function module on K (see [3] for definitions). Then the map

$$F\mapsto J(F) := \{x : x\in X,\ x(k)=0 \text{ all } k \text{ in } F\}$$

is a bounded co-capacity on K taking values in X.

Let T be a decomposable operator on X (see [2] for definitions), then $F\mapsto X_T(F)$ is a capacity on $\sigma(T)$ taking values in X, though not in general bounded. That iii) holds has been shown by Radjabalipour [7].

It can be seen already from these examples that iii) and 3) do not generally hold if we replace '$F\subseteqq F_1^\circ\cup F_2^\circ$' by '$F\subseteqq F_1\cup F_2$'.

DEFINITION 2. A map $x\mapsto \operatorname{supp} x$ which maps each element x of X onto a closed subset $\operatorname{supp} x$ of K is said to be a *support representation* (of X on K) if and only if:

I) $\operatorname{supp} x=\emptyset \Leftrightarrow x=0$,
II) $\operatorname{supp} \lambda x=\operatorname{supp} x$ all scalars $\lambda\neq 0$,
III) $\operatorname{supp}(\sum x_n)\subseteqq[\bigcup \operatorname{supp} x_n]^-$ for all convergent series $\sum x_n$ in X
IV) $\operatorname{supp} x\subseteqq G_1\cup G_2$ implies that x can be written as x_1+x_2 with $\operatorname{supp} x_1\subseteqq G_1$ and $\operatorname{supp} x_2\subseteqq G_2$ (remember — G's are open sets!).

A support representation is said to be *bounded* if there is a constant M such that in IV) x_1, x_2 can always be chosen with $\|x_1\|, \|x_2\| \leqq M\|x\|$.

EXAMPLES. As the name suggests the map $f \mapsto \operatorname{supp} f$ (here the normal support of a function) is a (bounded with $M=1$) support representation of $C(K)$ on K for any K.

The usual support is also a support representation of $C^1[0,1]$ on $[0,1]$ but *not* a bounded one.

If T is a decomposable operator on X then $x \mapsto \sigma(x;\, T)$ (the local spectrum of x) is a support representation of X on $\sigma(T)$ though not necessarily bounded.

PROPOSITION 1. a) *If J is a bounded capacity then*

$$\operatorname{supp} x := \bigcap \{F : x \in J(F)\}$$

defines a bounded support representation.

b) *If J is a bounded* co-capacity *then*

$$\operatorname{supp} x := \bigcap \{K \setminus F^\circ : x \in J(F)\}$$

defines a bounded support representation.

c) *If* supp *is a bounded support representation then*

$$J(F) := \{x : \operatorname{supp} x \subseteqq F\} \quad \text{and} \quad J_{\mathrm{co}}(F) := \{x : \operatorname{supp} x \cap F = \varnothing\}^-$$

define a bounded capacity and co-*capacity respectively.*

d) *These constructions establish a one-to-one correspondence between bounded capacities,* co-*capacities and support representations for given K and X.*

PROOF. a) By ii) of Definition 1 we have $x \in J(\operatorname{supp} x)$ for all x. Thus I) of definition 2 follows immediately from i) of Definition 1. II) is trivial and III) is a direct consequence of the fact that $J([\bigcup \operatorname{supp} x_n]^-)$ is a closed subspace of X. Finally IV), including boundedness, follows from iii) by setting $F = \operatorname{supp} x$ and choosing closed sets F_1 and F_2 with $F_1 \subseteqq G_1$, $F_2 \subseteqq G_2$ and $F \subseteqq F_1^\circ \cup F_2^\circ$.

b) Note that when $F \cap \operatorname{supp} x = \varnothing$ we have a finite number of closed sets $F_1, \ldots, F_n$ with $F \cap \bigcap (K \setminus F_i^\circ) = \varnothing$ and $x \in J(F_i)$ for each i. But then $F \subseteqq \bigcup F_i^\circ$ and $x \in \bigcap J(F_i)$. By 4) x lies in $J(F)$. I) is now trivial as is II). For III) note that any point k in the complement of $[\bigcup \operatorname{supp} x_n]^-$ can be separated from it by a closed set F with $k \in F^\circ$. But then $\operatorname{supp} x_n \cap F = \varnothing$ for each n and thus $x_n \in J(F)$. Since $J(F)$ is closed $\sum x_n$ also lies in it and therefore $\operatorname{supp}(\sum x_n) \subseteqq K \setminus F^\circ$ so $k \notin \operatorname{supp}(\sum x_n)$. IV) follows immediately from 2) with the same bound.

c) That J is a bounded capacity is immediate. Also 1) is trivial for J_{co}. For 2) note that

$$J_{co}(F_1 \cap F_2) \supseteqq J_{co}(F_1)+J_{co}(F_2)$$

is immediate from the definition. Now, if $x \in J_{co}(F_1 \cap F_2)$, let $\sum x_n$ be a series converging to x with

$$\operatorname{supp} x_n \cap (F_1 \cap F_2) = \emptyset \quad \text{for all } n \quad \text{and} \quad \sum \|x_n\| \leqq 2\|x\| .$$

By IV) with $G_1 = K \setminus F_1$ and $G_2 = K \setminus F_2$ we can write each x_n as $y_n + z_n$ with $y_n \in J_{co}(F_1)$ and $z_n \in J_{co}(F_2)$. Then $\sum y_n$ and $\sum z_n$ converge to y and z in $J_{co}(F_1)$ and $J_{co}(F_2)$ respectively. Now

$$\|y\| \leqq \sum \|y_n\| \leqq M \cdot \sum \|x_n\| \leqq 2M\|x\|$$

and similarly for z where M is the bound for supp. Thus 2) is satisfied with the bound $2M$. 3) is immediate from the definition since $\operatorname{supp} x \cap F = \emptyset$ implies $\operatorname{supp} x \cap F_\alpha = \emptyset$ for some F_α with $F_\alpha^\circ \supseteqq F$. 4) follows by noting that $x \in J_{co}(F)$ implies $\operatorname{supp} x \subseteqq K \setminus F^\circ$ by virtue of III.

d) That the correspondence supp $\leftrightarrow J$ for capacities is one-to-one is a simple consequence of the definition. The correspondence for co-capacities is somewhat more tricky. Suppose J is a bounded co-capacity and J_{co} is the bounded co-capacity constructed by c) from supp which is itself constructed by b) from J. We must show that $J(F) = J_{co}(F)$ for all F. The inclusion $J_{co}(F) \subseteqq J(F)$ was already shown in the proof of b). Now take x in $J(F)$. Then, by 3) $x = \lim x_n$ where $x_n \in J(F_n)$ with $F_n^\circ \supseteqq F$ for each n. But $\operatorname{supp} x_n \subseteqq K \setminus F_n^\circ$ and therefore also in $K \setminus F$ for each n. Thus $x_n \in J_{co}(F)$ snd, since $J_{co}(F)$ is closed, so also is x. It is interesting to note that this is the only point at which the property 3) is used.

Suppose on the other hand we start with a bounded support representation supp and construct J_{co} as in c) and then supp′ from J_{co} as in b). Then for each x we have

$$\operatorname{supp}' x := \bigcap \{K \setminus F^\circ : x = \lim x_n,\ \operatorname{supp} x_n \subseteqq K \setminus F \text{ for all } n\} .$$

Then $\operatorname{supp}' x \subseteqq \operatorname{supp} x$ is immediate and $\operatorname{supp}' x \supseteqq \operatorname{supp} x$ follows from the fact that $\operatorname{supp} x_n \subseteqq K \setminus F$ for all n implies $\operatorname{supp} x \subseteqq \overline{K \setminus F} \subseteqq K \setminus F^\circ$ by III).

In view of this proposition we can restrict our attention in future to one of these three concepts and for technical reasons we choose to use support representations in the proofs of the main theorems.

The following definition will be used to characterise those operators commuting with a given embedding of $C(K)$ in $B(X)$. Hence the choice of the word "commute".

DEFINITION 3. An operator T in $B(X)$ is said to *commute* with the support representation supp of X on K if and only if

$$\operatorname{supp} Tx \subseteqq \operatorname{supp} x \quad \text{for all } x \text{ in } X .$$

Clearly this is a form of invariance and we have the following simple lemma, whose proof is straightforward and left to the reader.

LEMMA. *Let* supp *be a bounded support representation of X on K and T an operator in $B(X)$. The following are equivalent:*

a) *T commutes with* supp,

b) *$J(F)$ is invariant under T for every closed subset F of K,*

c) *$J_{co}(F)$ is invariant under T for all closed subsets F of K where J and J_{co} are the capacity and* co-*capacity of Proposition* 2.

3. The main results.

In view of the length of the proof of the main theorem we state it in this section without proof and delay its proof till the end of the paper.

Suppose we have a Banach space X and a bounded support representation supp of X on some compact Hausdorff space K. Let f be a continuous function on K (complex-valued if X is a complex space). We wish to associate an embedding of $C(K)$ in $B(X)$ with supp. Thus we need to define an operator T_f corresponding to the function f. Since supp x is supposed, in some sense, to be the "support" of the element x, it is clear that $T_f x$ should only depend on the values of f on supp x, that is $f \equiv g$ on supp x must imply $T_f x = T_g x$ if the correspondence between the embedding and supp is to make any sense. Since the constant functions map onto multiples of the identity it follows that when f has the constant value λ on supp x then $T_f x$ must be λx. If however f is not constant on supp x there is no obvious way of defining what $T_f x$ should be. Nevertheless there is an unique embedding of $C(K)$ in $B(K)$ satisfying this simple property. This is the content of our central result which we now state without proof.

THEOREM 1. *Let X be a real or complex Banach space and* supp *a bounded support representation of X on a compact Hausdorff space K. There is an unique continuous homomorphism of unital Banach algebras from $C(K)$ into $B(X)$, $f \mapsto T_f$ satisfying $T_f x = T_g x$ whenever $f \equiv g$ on* supp x. *Moreover this mapping actually satisfies $T_f x = T_g x$ if and only if $f \equiv g$ on* supp x.

An operator A in $B(X)$ commutes with the support representation supp *if and only if it commutes with every operator T_f.*

PROOF. See section 5.

Although the proof of this result is difficult and lengthy, the following converse is quite easy.

THEOREM 2. *Let* $f \mapsto T_f$ *be a continuous homomorphism of unital Banach algebras from* $C(K)$ *into* $B(X)$. *Then there is an unique bounded support representation* supp *of* X *on* K *satisfying* $T_f x = T_g x$ *if and only if* $f \equiv g$ *on* supp x.

PROOF. Set

$$\operatorname{supp} x := \bigcap \{f^{-1}(0) : T_f x = 0\} \text{ for all } x \text{ in } X .$$

Then clearly $T_f x = 0$ implies $f \equiv 0$ on supp x. On the other hand suppose that $f \equiv 0$ on supp x. For any natural number n we can choose a function g_n in $C(K)$ with $|g_n - f| < 1/n$ and $\operatorname{supp} g_n \cap \operatorname{supp} x = \emptyset$ (here supp denotes also the normal support of a function). Then by the definition of supp x and compactness there are function $f_1, \ldots, f_m$ with $T_{f_i} x = 0$ all i and

$$\operatorname{supp} g_n \cap f_1^{-1}(0) \cap \ldots \cap f_m^{-1}(0) = \emptyset .$$

Then there is a function h in $C(K)$ with

$$h(|f_1|^2 + \ldots + |f_m|^2) = g_n .$$

Since $T_{|f_i|^2} x = 0$ all i follows from $|f_i|^2 = \bar{f}_i f_i$ and $T_{f_i} x = 0$, we have $T_{g_n} x = 0$ and thus by continuity $T_f x = 0$. This shows that supp satisfies $T_f x = 0$ if and only if $f \equiv 0$ on supp x which is obviously equivalent to $T_f x = T_g x$ if and only if $f \equiv g$ on supp x. It remains to show that supp is indeed a bounded support representation.

I) $\operatorname{supp} x = \emptyset \Leftrightarrow 1 \equiv 0$ on supp $x \Leftrightarrow x = T_1 x = T_0 x = 0$.

II) Is trivial.

III) If $f \equiv 0$ on $[\bigcup \operatorname{supp} x_n]^-$ then $f \equiv 0$ on every supp x_n, that is $T_f x_n = 0$ for all x_n. By continuity $T_f(\sum x_n) = 0$. Thus $\operatorname{supp}(\sum x_n) \subseteq [\bigcup \operatorname{supp} x_n]^-$.

IV) If $\operatorname{supp} x \subseteq G_1 \cup G_2$, let f_1, f_2 be in $C(K)$ with $\|f_1\|, \|f_2\| \leqq 1$, with $\operatorname{supp} f_1 \subseteq G_1$, $\operatorname{supp} f_2 \subseteq G_2$ and $f_1 + f_2 \equiv 1$ on supp x. Then

$$x = T_{f_1 + f_2} x = T_{f_1} x + T_{f_2} x \quad \text{and} \quad \operatorname{supp} T_{f_1} x \subseteq \operatorname{supp} f_1 \subseteq G_1$$

and similarly for f_2. Moreover $\|T_{f_i} x\| \leqq M \|f_i\| \cdot \|x\| \leqq M \|x\|$ where M is the norm of the map $f \mapsto T_f$.

Thus supp is a bounded support representation of X on K with the desired property. Clearly this property determines supp uniquely.

Putting these two theorems together with Proposition 1 we obtain:

THEOREM 3. *Let K be a compact Hausdorff space and X a real or complex Banach space. There is a $1-1$ correspondence between the following:*

i) *the bounded capacities on K taking values in X,*
ii) *the bounded* co-*capacities on K taking values in X,*
iii) *the bounded support representations of X on K,*
iv) *the continuous homomorphisms of $C(K)$ into $B(X)$ which map the function 1 onto I.*

Furthermore, for an operator A in $B(X)$ the following are equivalent:

i) *A commutes with every operator T_f,*
ii) *$J(F)$ is invariant under A for all F,*
iii) *$J_{co}(F)$ is invariant under A for all F,*
iv) *A commutes with* supp *where $f \mapsto T_f$, J, J_{co} and* supp *correspond.*

4. Miscellaneous results.

In this section we prove some simple technical results and more importantly a result on duality. We assume throughout that supp, J, J_{co} and $f \mapsto T_f$ are given and correspond as in Theorem 3.

PROPOSITION 4. *For a closed subset F of K the following are equivalent:*

i) $\operatorname{supp} x \subseteq F$ *for all x,*
ii) $J(F)=X$,
iii) $J_{co}(F)=\{0\}$,
iv) *$f \in F^{\perp}$ (that is $f \equiv 0$ on F) implies $T_f=0$.*

Further $f \mapsto T_f$ is $1-1$ precisely when K is the only closed subset with these properties.

PROOF. i) $\Leftrightarrow$ ii) is trivial.

i) $\Rightarrow$ iii) since $J_{co}(F)=\{x : \operatorname{supp} x \cap F=\varnothing\}^{-}$.

iii) $\Rightarrow$ i) If F_1 is any closed set with $F_1^{\circ} \supseteq F$ then we have $x=x_1+x_2$ with $\operatorname{supp} x_1 \subseteq F_1$ and $\operatorname{supp} x_2 \subseteq K \setminus F$. But then $\operatorname{supp} x_2 \cap F=\varnothing$ so $x_2 \in J_{co}(F)=\{0\}$. Thus $x=x_1 \in J(F_1)$. Since F_1 was arbitrary with $F_1^{\circ} \supseteq F$ we have $\operatorname{supp} x \subseteq F$.

i) $\Rightarrow$ iv) $f \in F^{\perp}$ implies $f \equiv 0$ on $\operatorname{supp} x$ all x and therefore $T_f x=0$ for all x, that is, $T_f=0$.

iv) $\Rightarrow$ i) For all f with $f \equiv 1$ on F we have $T_f=I$ and $T_f x=x$ for all x. Thus $\operatorname{supp} x \subseteq \operatorname{supp} f$ (Lemma 8) and since f was arbitrary $\operatorname{supp} x \subseteq F$.

If the map is $1-1$ then iv) and thus all of i)–iv) can be true only for $F=K$. On the other hand if $T_f=0$ for some $f \neq 0$ then $T_f x=0$ for each x so that $f \equiv 0$

on all sets $\operatorname{supp} x$. Thus $f^{-1}(0)$ is a non-trivial closed subset of K with the properties i)–iv).

Note that if $f \mapsto T_f$ is not $1-1$ then we can factor $f \mapsto T_f$ through $C(K_0)$ where

$$K_0 = [\bigcup \{\operatorname{supp} x : x \in X\}]^- ,$$

i.e. the smallest closed set satisfying the above conditions. By reducing our attention now to K_0 we obtain a $1-1$ mapping $C(K_0) \to B(X)$.

PROPOSITION 5. *For all x in X and f in $C(K)$ we have*

$$T_f x \in J(\operatorname{supp} x) \cap J_{co}(f^{-1}(0)) .$$

PROOF. Let g be a function in $C(K)$ with $g \equiv 1$ on $\operatorname{supp} x$. Then

$$T_f x = T_f T_g x = T_g(T_f x) \quad \text{so that} \quad \operatorname{supp} T_f x \subseteqq \operatorname{supp} g .$$

Since g was arbitrary with $g \equiv 1$ on $\operatorname{supp} x$ we have $\operatorname{supp} T_f x \subseteqq \operatorname{supp} x$. $T_f x \in J_{co}(f^{-1}(0))$ is Lemma 8 i) (see proof of main theorem).

PROPOSITION 6. supp, *J and J_{co} actually satisfy the following stronger forms of* IV), iii) *and* 4) *of the definitions*:

IV) *replace* "$\operatorname{supp} x_1 \subseteqq G_1$" *by* "$\operatorname{supp} x_1 \subseteqq G_1 \cap \operatorname{supp} x$" *and similarly for* x_2.

iii) *replace* "$J(F) \subseteqq J(F_1) + J(F_2)$" *by* "$J(F) = J(F \cap F_1) + J(F \cap F_2)$".

4) *replace* "$J_{co}(F_1) \cap J_{co}(F_2) \subseteqq J_{co}(F)$" *by* "$J_{co}(F_1 \cap F) \cap J_{co}(F_2 \cap F) = J_{co}(F)$".

PROOF. Let f_1, f_2 be functions in $C(K)$ with $\|f_1\| = \|f_2\| = 1$, $\operatorname{supp} f_1 \subseteqq G_1$, $\operatorname{supp} f_2 \subseteqq G_2$ and $f_1 + f_2 \equiv 1$ on $\operatorname{supp} x$. Then $x = T_{f_1} x + T_{f_2} x$ and by Proposition 5,

$$\operatorname{supp} T_{f_i} x \subseteqq \operatorname{supp} x \cap \operatorname{supp} f_i \subseteqq G_i \cap \operatorname{supp} x .$$

The stronger forms of iii) and 4) follows straightforwardly from this, though 4) requires a little care.

Note that these stronger forms are not in general true for non-bounded support representations etc. See, for example [1].

PROPOSITION 7. *A map J_{co} mapping closed subsets of K onto closed subspaces of X is a bounded* co-*capacity if and only if the map $F \mapsto J_{co}(F)^\circ$ is a bounded capacity* (*taking values in X'*).

PROOF. We write $\hat{J}(F)$ to mean $J_{co}(F)^{\circ}$. Assume J_{co} to be a bounded co-capacity with decomposition bound say m. i) and ii) of the definition of bounded capacity are elementary for $\hat{J}$. For iii) take y in $\hat{J}(F)$ and note that since

$$J_{co}(F) \supseteqq J_{co}(F_1) \cap J_{co}(F_2)$$

y is 0 on this intersection. Thus the functional $\hat{y}$ on

$$J_{co}(F_1 \cap F_2) = J_{co}(F_1)+J_{co}(F_2)$$

defined by $\hat{y}(x_1+x_2):=y(x_2)$ where $x_i \in J_{co}(F_i)$, is well-defined. Also $\|\hat{y}\| \leqq m\|y\|$. Let $\tilde{y}$ be an extension of $\hat{y}$ to all of X with the same norm. Then $y=\tilde{y}+(y-\tilde{y})$ is a decomposition of y of the required form and, since

$$\|y-\tilde{y}\| \leqq (m+1)\|y\| ,$$

$\hat{J}$ is a bounded capacity.

Suppose now that $\hat{J}$ is a bounded capacity. That J_{co} satisfies 1) and 4) of Definition 1 follows immediately from i) and iii) for J. 3) is an easy consequence of ii). It remains to prove 2). Let F_1 and F_2 be given and set $F=F_1 \cup F_2$. Choose closed sets F_3 and F_4 with $F_1 \subseteqq F_3^{\circ}$, $F_2 \subseteqq F_4^{\circ}$. Set

$$X_1 = J_{co}(F \cap F_3) \quad \text{and} \quad X_2 = J_{co}(F \cap F_4) .$$

Let x_1 in X_1 be such that the norm of $[x_1]$ in the quotient $X/(X_1 \cap X_2)$ is >1. Then there is an element y in $(X_1 \cap X_2)^{\circ}$ with $\|y\|=1$ and $y(x_1)>1$. Now Proposition 6 shows that $(X_1 \cap X_2)^{\circ}=X_1^{\circ}+X_2^{\circ}$ and y can be written correspondingly as y_1+y_2 with y_i in X_i° and $\|y_i\| \leqq m$, where m is the norm of the algebra homomorphism $C(K) \to B(X')$ corresponding to $\hat{J}$. But then for any x_2 in X_2 we have

$$1 < y(x_1) = (y_1+y_2)x_1 = y_2(x_1) = y_2(x_1+x_2) \leqq m\|x_1+x_2\| .$$

Doing the same for X_2 we see that any element in X_1+X_2 can be written in the form x_1+x_2 with x_1 in X_1, x_2 in X_2 and $\|x_1\|, \|x_2\| \leqq (m+1)\|x_1+x_2\|$. Letting F_3 and F_4 approach F_1 and F_2 and using 3) we obtain 2) with a decomposition bound $\leqq (m+1)$.

COROLLARY. *On a dual space the operators T_f are all* w**-continuous if and only if the spaces $J(F)$ are* w**-closed for each F.*

PROOF. If the operators T_f are all w*-continuous then $T_f=S'_f$ for some operator S_f on the pre-dual Y. $f \mapsto S_f$ is a continuous algebra homomorphism of $C(K)$ into $B(Y)$. Let J_{co} be the corresponding bounded co-capacity. It is easy to check that $J(F)=J_{co}(F)^{\circ}$ for all F. The other implication goes by setting

$J_{co}(F) = {}^\circ J(F)$, constructing a map $f \mapsto S_f$ and again checking that $T_f = S'_f$ for all f.

5. Proof of the main theorem.

The rest of the paper is devoted to a step-by-step proof of Theorem 1 of section 3. Until the final step everything is assumed to be real. We shall need the following concepts:

DEFINITION 4. An *optimal covering* of a compact subset D of the real line is a finite number $I_1, \ldots, I_n$ of open intervals which cover D and for which we have:

i) $I_i \cap I_{i+1} \neq \emptyset$ for all i.
ii) $I_i \cap I_j = \emptyset$ if $|i-j| > 1$.

Let x be some element of the Banach space X, supp the given bounded support representation of X on K, $G_1, \ldots, G_n$ an open covering of K and M a positive constant. The elements $x_1, \ldots, x_n$ are said to form a $\{G_i\}$*-decomposition* of x if

i) $x = x_1 + \ldots + x_n$.
ii) $\operatorname{supp} x_i \subseteqq G_i$ for each i.

This decomposition is said to be *M-bounded* if in addition

iii) $\|\sum \lambda_i x_i\| \leqq M \cdot \max |\lambda_i|$ for all real co-efficients λ_i.

Note that an optimal covering covers co D and that if $G_1, \ldots, G_n$ is any open covering of co D then there is an optimal covering $I_1, \ldots, I_m$ subordinate to it, i.e. each I_i is contained in some G_j.

LEMMA 1. *Let G be open in K and $G = G_1 \cup \ldots \cup G_n$ for a finite collection of disjoint open sets G_i. Then every element x in X with* $\operatorname{supp} x \subseteqq G$ *has an unique $\{G_i\}$-decomposition.*

PROOF. That x has at least one such decomposition follows by induction from iv) of the definition of support representation. Let $x_1 + \ldots + x_n$ and $y_1 + \ldots + y_n$ be two such decompositions. Then, for each i,

$$x_i - y_i = \Big(x - \sum_{j \neq i} x_j\Big) - \Big(x - \sum_{j \neq i} y_j\Big) = \sum_{j \neq i} y_j - x_j\,.$$

By iii) of the definition of support representation we have

$$\operatorname{supp}\Big(\sum_{j \neq i} y_j - x_j\Big) \subseteqq \bigcup_{j \neq i} G_j$$

and on the other hand $\operatorname{supp}(x_i - y_i) \subseteqq G_i$. Since $G_i \cap (\bigcup_{j \neq i} G_j) = \emptyset$ we have $\operatorname{supp}(x_i - y_i) = \emptyset$ and thus, by i) of the definition $x_i = y_i$.

LEMMA 2. *Let m be a decomposition bound for* supp *and $f_1, \ldots, f_n$ be functions in $C(K)$. For every x in X and every optimal covering $I_1, \ldots, I_N$ of $f_1(K) \cup \ldots \cup f_n(K)$ there is a $(2m)^{n+1}\|x\|$-bounded decomposition $\{x_{i_1 \ldots i_n}\}$ of x $(i_1, \ldots, i_n \in \{1, \ldots, N\})$ with*

$$f_j(\operatorname{supp} x_{i_1 \ldots i_n}) \subseteqq I_{i_j} \quad \text{for all } i_1, \ldots, i_n \text{ and all } j .$$

In particular, with one function f every x has a $(2m)^2\|x\|$-bounded $\{f^{-1}(I_j)\}$-decomposition for every optimal covering $I_1, \ldots, I_N$ of $f(K)$.

PROOF. For reasons of readability we demonstrate explicitly only the case with one function f.

Set

$$G_o := \bigcup \{f^{-1}(I_i) : i \text{ odd}\}, \qquad G_e := \bigcup \{f^{-1}(I_i) : i \text{ even}\} .$$

Since $K \subseteqq G_o \cup G_e$, we can write x as $x_o + x_e$ with $\operatorname{supp} x_o \subseteqq G_o$ and similarly $\operatorname{supp} x_e \subseteqq G_e$ and $\|x_e\|, \|x_o\| \leqq m\|x\|$. As $I_1, \ldots, I_N$ is an optimal covering, the $f^{-1}(I_i)$'s in G_o and in G_e are pairwise disjoint. Thus, by Lemma 1, there are unique decompositions of x_o and x_e as $x_1 + x_3 + x_5 + \ldots$ and $x_2 + x_4 + x_6 + \ldots$ respectively with $f(\operatorname{supp} x_i) \subseteqq I_i$ for all i.

Consider now an element of the form $\sum \alpha_i x_i$ where $\alpha_i = \pm 1$ for all i. Then

$$\begin{aligned} \sum \alpha_i x_i &= \sum \{x_i : i \text{ odd}, \alpha_i = 1\} + \sum \{x_i : i \text{ even}, \alpha_i = 1\} - \\ &\quad - \sum \{x_i : i \text{ odd}, \alpha_i = -1\} - \sum \{x_i : i \text{ even}, \alpha_i = -1\} \\ &= y_1 + y_2 - y_3 - y_4 \end{aligned}$$

where y_1, y_2, y_3 and y_4 are the values of the four sums. Then

$$\|\sum \alpha_i x_i\| \leqq \|y_1\| + \|y_2\| + \|y_3\| + \|y_4\| .$$

But $y_1 + y_3 = x_o$ and $y_2 + y_4 = x_e$. Moreover, these decompositions of x_o and x_e are, by Lemma 1, unique. Thus they satisfy

$$\|y_1\|, \|y_3\| \leqq m\|x_o\| \qquad \text{and} \qquad \|y_2\|, \|y_4\| \leqq m\|x_e\|$$

since we know that decompositions with these properties exist. It follows that

$$\|\sum \alpha_i x_i\| \leqq 2m(\|x_o\| + \|x_e\|) \leqq (2m)^2\|x\| .$$

Now the points $(\pm 1, \pm 1, \ldots, \pm 1)$ are the extreme points of the unit ball of $\mathbf{R}^N$ with the maximum norm. It follows that the mapping $\mathbf{R}^N \to X$ defined by

$(\lambda_1,\ldots,\lambda_n)\mapsto \|\sum \lambda_i x_i\|$ has norm $\leqq (2m)^2\|x\|$, when $\mathbf{R}^N$ carries the maximum norm. This is the required result.

In the case of n functions we must decompose x first into 2^n elements corresponding to the choices of odd and even for each i in $1,\ldots,n$, e.g. for $n=3$ we have such elements as $x_{o,e,o}$ with

$$\operatorname{supp}(x_{o,e,o}) \subseteqq f_1^{-1}(G_o)\cap f_2^{-1}(G_e)\cap f_3^{-1}(G_o)\,.$$

The proof goes through exactly as before though the indices make it unreadable. For each odd-even choice we have a factor of $2m$ and also finally for the choice $\alpha_i=+1$ or -1. This gives $(2^m)^{n+1}\|x\|$ as the appropriate constant.

LEMMA 3. *Let f be a function in $C(K)$ and $I_1,\ldots,I_n$ an optimal covering of $f(K)$. Let $x_1+\ldots+x_n$ be an M_1-bounded and $y_1+\ldots+y_n$ an M_2-bounded $\{f^{-1}(I_i)\}$-decomposition of x. Then for any set of scalars λ_i, $i=1,\ldots,n$ we have*

$$\|\sum \lambda_i(x_i-y_i)\| \leqq 2m\cdot(M_1+M_2)\cdot\max|\lambda_{i+1}-\lambda_i|$$

where m is, of course, a decomposition bound for supp.

PROOF. Set $z_i:=x_i-y_i$ and $u_i:=\sum_{j=1}^i z_j$ for each i, further let $u_0:=0$. Then

$$\sum_1^n \lambda_i(x_i-y_i) = \sum_1^n \lambda_i z_i = \sum_1^n \lambda_i(u_i-u_{i-1})\,.$$

On the other hand

$$\sum_1^n \lambda_i(u_i-u_{i-1}) = \sum_1^n \lambda_i u_i - \sum_0^{n-1} \lambda_{i+1}u_i = \sum_1^{n-1} (\lambda_i-\lambda_{i+1})u_i$$

since $u_n=\sum_1^n z_j=\sum_1^n x_i-\sum_1^n y_i=x-x=0$.

Now, for each i, $u_i=\sum_1^i z_j=-\sum_{i+1}^n z_j$ since $\sum_1^n z_j=0$. Thus

$$\begin{aligned}\operatorname{supp} u_i &\subseteqq \left(\bigcup_1^i \operatorname{supp} z_j\right)\cap\left(\bigcup_{i+1}^n \operatorname{supp} z_j\right)\\ &\subseteqq f^{-1}\left(\bigcup_1^i I_j\right)\cap f^{-1}\left(\bigcup_{i+1}^n I_j\right) = f^{-1}(I_i\cap I_{i+1})\,.\end{aligned}$$

Set $G_i:=I_i\cap I_{i+1}$ for each $i=1,\ldots,n-1$. Then the G_i's are disjoint intervals. Consider the element $u:=\sum(-1)^i u_i$. Then

$$\operatorname{supp} u \subseteqq \bigcup \operatorname{supp} u_i \subseteqq f^{-1}(\bigcup G_i)\,.$$

By Lemma 1, $u=\sum(-1)^i u_i$ is the only $\{f^{-1}(G_i)\}$-decomposition of u and an argument as in the proof of the previous lemma shows that it is a $2m\|u\|$-bounded one.

Thus

$$\left\|\sum_1^n \lambda_i(x_i-y_i)\right\| = \left\|\sum_1^n (\lambda_i-\lambda_{i+1})u_i\right\| = \left\|\sum_1^n (-1)^i(\lambda_i-\lambda_{i+1})(-1)^i u_i\right\|$$
$$\leqq 2m\cdot\|u\|\cdot\max|\lambda_{i+1}-\lambda_i| .$$

On the other hand

$$\|u\| = \|\sum \{z_j : j \text{ even}\}\|$$
$$\leqq \|\sum \{x_j : j \text{ even}\}\| + \|\sum \{y_j : j \text{ even}\}\| \leqq M_1+M_2 .$$

This completes the proof.

LEMMA 4. *Let f be a function in $C(K)$ and $I_1,\ldots,I_n$ and $J_1,\ldots,J_N$ two optimal coverings of $f(K)$. Further let $x_1+\ldots+x_n$ be an M_1-bounded $\{f^{-1}(I_i)\}$-decomposition and $y_1+\ldots+y_N$ an M_2-bounded $\{f^{-1}(J_j)\}$-decomposition of an element x. Then*

$$\left\|\sum_1^n \lambda_i x_i-\sum_1^N \mu_j y_j\right\| \leqq \delta\cdot 4m(M_1+M_2+(8m^2+2m)\|x\|)$$

where δ is the length of the largest interval amongst the I_i's and J_j's, m is any decomposition bound for supp *and $\{\lambda_i\}$, $\{\mu_j\}$ are sets of scalars with $\lambda_i \in I_i$, $\mu_j \in J_j$ for all i, j.*

PROOF. The intersections of pairs $I_i \cap J_j$ is a collection of open intervals covering co $f(K)$. Let G_r, $r=1,\ldots,s$ be an optimal covering subordinate to this covering and $z_1+\ldots+z_s$ a $(2m)^2\|x\|$-bounded $\{f^{-1}(G_r)\}$-decomposition of x as in Lemma 2. Each G_r is contained in at least one intersection $I_i \cap J_j$ so that we can find two maps φ: $\{1,\ldots,s\} \to \{1,\ldots,n\}$ and ψ: $\{1,\ldots,s\} \to \{1,\ldots,N\}$ such that $G_r \subseteq I_{\varphi(r)} \cap J_{\psi(r)}$ for all r. Set

$$\hat{x}_i := \sum \{z_r : \varphi(r)=i\} \quad \text{and} \quad \hat{y}_j := \sum \{z_r : \psi(r)=j\} \quad \text{for each } i,j .$$

Clearly the $\hat{x}_i$'s and $\hat{y}_j$'s are $(2m)^2\|x\|$-bounded $\{f^{-1}(I_i)\}$- and $\{f^{-1}(J_j)\}$-decompositions of x respectively.

Then

$$\left\|\sum_1^n \lambda_i x_i-\sum_1^N \mu_j y_j\right\|$$
$$\leqq \left\|\sum_1^n \lambda_i x_i-\sum_1^s \lambda_{\varphi(r)}z_r\right\| + \left\|\sum_1^s (\lambda_{\varphi(r)}-\mu_{\psi(r)})z_r\right\| + \left\|\sum_1^s \mu_{\psi(r)}z_r-\sum_1^N \mu_j y_j\right\|$$
$$= \|\sum \lambda_i(x_i-\hat{x}_i)\| + \|\sum (\lambda_{\varphi(r)}-\mu_{\psi(r)})z_r\| + \|\sum \mu_j(\hat{y}_j-y_j)\| .$$

Set $\alpha_r := \lambda_{\varphi(r)} - \mu_{\psi(r)}$ if $z_r \neq 0$, and $\alpha_r := 0$ if $z_r = 0$. Then the middle term is

$$\left\| \sum_1^s \alpha_r z_r \right\| \leqq (2m)^2 \|x\| \cdot \max |\alpha_r| .$$

But $\alpha_r \neq 0$ implies $z_r \neq 0$ which implies that $I_{\varphi(r)} \cap J_{\psi(r)} \neq \emptyset$ and thus that

$$|\lambda_{\varphi(r)} - \mu_{\psi(r)}| \leqq \operatorname{length} I_{\varphi(r)} + \operatorname{length} J_{\psi(r)} \leqq 2\delta .$$

Thus the middle term is $\leqq \delta \cdot 8m^2 \|x\|$. Applying Lemma 3 to the outer terms we obtain the required result, Note that

$$|\lambda_{i+1} - \lambda_i| \leqq \operatorname{length} I_{i+1} + \operatorname{length} I_i \leqq 2\delta$$

and similarly for the J_j's.

LEMMA 5. *Let f be a function in $C(K)$ and x an element in X. Then there is an unique element in X, which we denote by x_f, with the property:*

Whenever $I_1, \ldots, I_n$ is an optimal covering of $f(K)$, $x_1 + \ldots + x_n$ an M-bounded $\{f^{-1}(I_i)\}$-decomposition of x and $\{\lambda_i\}$ a set of scalars with $\lambda_i \in I_i$ for each i, then

$$\left\| x_f - \sum_1^n \lambda_i x_i \right\| \leqq \delta \cdot 4m(M + (12m^2 + 2m)\|x\|)$$

where δ is the maximal length of the I_i's and m any decomposition bound for supp.

PROOF. By Lemma 2 we can find an appropriate $(2m)^2\|x\|$-bounded decomposition of x however fine we choose an optimal covering of $f(K)$. By Lemma 4, the distance between the sums for two such decompositions is at most $\delta \cdot 4m(16m^2 + 2m)\|x\|$, where δ is the interval length, so that, taking finer and finer decompositions, the sums converge to an element in X which we call x_f. The above inequality is merely the limit form of that in Lemma 4 with $M_1 = M$ and $M_2 = (2m)^2\|x\|$, the latter sums converging to x_f. Clearly x_f is unique.

LEMMA 6. *For every f in $C(K)$, the mapping $x \mapsto x_f$ is a bounded linear operator on X which we denote by T_f. Also we have $\|T_f\| \leqq (2m)^2 \|f\|_\infty$.*

PROOF. If $I_1, \ldots, I_n$ is an optimal covering of $f(K)$ and $x_1 + \ldots + x_n$, $y_1 + \ldots + y_n$ are M-bounded $\{f^{-1}(I_i)\}$-decompositions of elements x and y respectively, then $(\alpha x_1 + \beta y_1) + \ldots + (\alpha x_n + \beta y_n)$ is an $\{f^{-1}(I_i)\}$-decomposition of $\alpha x + \beta y$ for all scalars α and β, since

$$\operatorname{supp}(\alpha x_i + \beta y_i) \subseteqq \operatorname{supp} x_i \cup \operatorname{supp} y_i \subseteqq f^{-1}(I_i) \quad \text{for all } i .$$

Further

$$\left\|\sum_1^n \lambda_i(\alpha x_i+\beta y_i)\right\| \leqq \left\|\sum_1^n \alpha\lambda_i x_i\right\| + \left\|\sum_1^n \beta\lambda_i y_i\right\| \leqq M(\max|\alpha\lambda_i| + \max|\beta\lambda_i|)$$
$$\leqq M(|\alpha|+|\beta|)\max|\lambda_i| .$$

Thus this decomposition is an $M(|\alpha|+|\beta|)$-bounded one. In particular, taking limits with $(2m)^2\|x\|$-bounded decompositions of x and y, Lemma 5 implies that

$$\sum \lambda_i(\alpha x_i+\beta y_i) \to (\alpha x+\beta y)_f$$

so that $(\alpha x+\beta y)_f = \alpha x_f + \beta y_f$, i.e. the mapping is linear.

Now for each x, x_f is the limit of sums $\sum \lambda_i x_i$ for any sequence of $(2m)^2\|x\|$-bounded $\{f^{-1}(I_i)\}$-decompositions where the maximal interval length in the optimal coverings tends to zero. We can clearly choose the optimal coverings $\{I_i\}$ so that $I_i \cap \operatorname{co}(f(K)) \neq \emptyset$ all i and thus choose the λ_i's in $\operatorname{co}(f(K))$. But then

$$\|\sum \lambda_i x_i\| \leqq (2m)^2\|x\|\cdot \max|\lambda_i| \leqq (2m)^2\|x\|\cdot\|f\|_\infty$$

and this inequality carries over to the limit x_f. This shows the mapping to be bounded with the given norm.

LEMMA 7. *The map $f \mapsto T_f$ from $C(K)$ into $B(X)$ is a continuous homomorphism of unital Banach algebras with norm $\leqq (2m)^2$.*

PROOF. That $T_{\lambda f} = \lambda T_f$ for any scalar λ is trivial.

Let f and g be two functions in $C(K)$ and x an element in X. Suppose δ given and take an optimal covering $I_1,\ldots,I_n$ of $f(K)\cup g(K)\cup(f+g)(K)$ with maximal interval length less than δ. By Lemma 2 we can find a $(2m)^4\|x\|$-bounded decomposition $\{x_{ijk}\}$ of x with

$$f(\operatorname{supp} x_{ijk}) \subseteqq I_i, \quad g(\operatorname{supp} x_{ijk}) \subseteqq I_j,$$
$$(f+g)(\operatorname{supp} x_{ijk}) \subseteqq I_k \quad \text{for all } i,j,k .$$

Set

$$u_i := \sum_{j,k} x_{ijk}, \quad v_j := \sum_{i,k} x_{ijk}, \quad w_k := \sum_{i,j} x_{ijk} \quad \text{for all } i,j,k .$$

Then $u_1+\ldots+u_n$ is a $(2m)^4\|x\|$-bounded $\{f^{-1}(I_i)\}$-decomposition of x and similarly for v_j and w_k.

Choose scalars $\lambda_i \in I_i$ for each i. Then

$$\|(T_f+T_g)x - T_{f+g}x\| = \|x_f+x_g-x_{f+g}\| \leqq \|x_f - \sum \lambda_i u_i\| +$$
$$+\|x_g - \sum \lambda_j v_j\| + \|x_{f+g} - \sum \lambda_k w_k\| + \|\sum \lambda_i u_i + \sum \lambda_j v_j - \sum \lambda_k w_k\| .$$

By Lemma 5 the first three terms are each smaller than $\delta\cdot 4m(16m^4+12m^2+2m)\|x\|$. The fourth term can be rewritten as $\|\sum_i\sum_j\sum_k(\lambda_i+\lambda_j-\lambda_k)x_{ijk}\|$. Since x_{ijk} is a $(2m)^4\|x\|$-bounded decomposition this implies that this term is smaller than $(2m)^4\|x\|\cdot\max|\alpha_{ijk}|$ where $\alpha_{ijk}:=\lambda_i+\lambda_j-\lambda_k$ if $x_{ijk}\neq 0$ and $\alpha_{ijk}:=0$ otherwise. However, when $x_{ijk}\neq 0$, we have

$$f^{-1}(I_i)\cap g^{-1}(I_j)\cap(f+g)^{-1}(I_k)\neq\varnothing\,.$$

Let t be a point in this intersection then

$$|\lambda_i-f(t)|<\delta,\quad |\lambda_j-g(t)|<\delta,\quad |(f+g)(t)-\lambda_k|<\delta\,.$$

Thus $|\alpha_{ijk}|<3\delta$.

So $\|(T_f+T_g)x-T_{f+g}x\|$ is bounded by a constant times δ. Since δ was arbitrary we have $(T_f+T_g)x=T_{f+g}x$. As x was arbitrary $T_f+T_g=T_{f+g}$.

An analogous argument with f, g and fg shows that $T_f\cdot T_g=T_{fg}$. Thus the map is an algebra homomorphism and continuous with norm $\leqq(2m)^2$ by Lemma 6. That $T_1=I$ is trivial.

LEMMA 8. *For all f in $C(K)$ and x in X, we have:*
i) *T_fx lies in $J_{co}(f^{-1}(0))$ and thus* $\operatorname{supp}T_fx\subseteq\operatorname{supp}f$.
ii) *$T_fx=\lambda x$ if and only if $f\equiv\lambda$ on* $\operatorname{supp}x$, *λ some scalar.*

PROOF. i) $T_fx=x_f=$limit of sums of the form $\sum\lambda_ix_i$. If $\operatorname{supp}x_i$ intersects $f^{-1}(0)$ for some i then, since $f(\operatorname{supp}x_i)\subseteq I_i$, we have $0\in I_i$. Thus we can choose $\lambda_i=0$. This shows that x_f is the limit of sums which can, without loss of generality, be assumed to have their supports disjoint from $f^{-1}(0)$. By the definition of J_{co} we have $x_f\in J_{co}(f^{-1}(0))$.

ii) If $f\equiv\lambda$ on $\operatorname{supp}x$ and $I_1,\dots,I_n$ is an optimal covering of $f(K)$ then λ lies in some I_i. Then setting $x_i=x$ and all others $=0$ we have a $\|x\|$-bounded $\{f^{-1}(I_i)\}$-decomposition of x. Choosing $\lambda_i=\lambda$ and the others arbitrarily we see that $x_f=\lambda x$.

On the other hand, suppose that $T_fx=\lambda x$. Let g be a function in $C(K)$ with $g\equiv 1$ on a neighbourhood of $f^{-1}(\lambda)$. Then there is a function h in $C(K)$ with $g-1=h(f-\lambda)$. This means $T_{g-1}x=T_hT_{f-\lambda}x=0$ so that $T_gx=x$. But then by i) we have

$$\operatorname{supp}x=\operatorname{supp}T_gx\subseteq\operatorname{supp}g\,.$$

Since g was arbitrary we have $\operatorname{supp}x\subseteq f^{-1}(\lambda)$ or in other words $f\equiv\lambda$ on $\operatorname{supp}x$.

LEMMA 9. *An operator A in $B(X)$ commutes with every operator T_f, f in $C(K)$, if and only if* $\operatorname{supp}Ax\subseteq\operatorname{supp}x$ *for all x in X.*

PROOF. For any M-bounded $\{G_i\}$-decomposition $x_1 + \ldots + x_n$ of an element x we have

$$\|\sum \lambda_i A x_i\| = \|A(\sum \lambda_i x_i)\| \leqq \|A\| \cdot \|\sum \lambda_i x_i\| \leqq \|A\| \cdot M \cdot \max |\lambda_i|$$

and $\operatorname{supp} Ax_i \subseteq \operatorname{supp} x_i \subseteq G_i$ for each i. Thus $Ax_1 + \ldots + Ax_n$ is a $\|A\| \cdot M$-bounded $\{G_i\}$-decomposition of Ax. Choosing suitable sums $\sum \lambda_i x_i$ converging to x_f we see that $A(\sum \lambda_i x_i) = \sum \lambda_i A x_i$ converges to $(Ax)_f$ and thus $(Ax)_f = Ax_f$ that is $T_f Ax = AT_f x$.

On the other hand suppose that A commutes with all T_f. For each x in X and $k \notin \operatorname{supp} x$ there is a function f in $C(K)$ with $f \equiv 1$ on $\operatorname{supp} x$ but $k \notin \operatorname{supp} f$. Then by Lemma 8

$$\operatorname{supp} Ax = \operatorname{supp} AT_f x = \operatorname{supp} T_f Ax \subseteq \operatorname{supp} f .$$

Thus $k \notin \operatorname{supp} Ax$.

PROOF OF THEOREM 1. *The real case.* Lemma 7 shows the existence of the desired homomorphism and Lemma 9 and part ii) of Lemma 8 show it has the claimed properties. It remains to show that this is the only continuous homomorphism of unital Banach algebras $C(K) \to B(X)$ for which $T_f x = T_g x$ whenever $f \equiv g$ on $\operatorname{supp} x$.

Suppose $f \mapsto S_f$ were a second such homomorphism. Let f be a function in $C(K)$ and x an element in X. For given $\delta > 0$, let $I_1, \ldots, I_n$ be an optimal covering of $f(K)$ with maximal interval length less than δ. Let $x_1 + \ldots + x_n$ be a $(2m)^2\|x\|$-bounded $\{f^{-1}(I_i)\}$-decomposition of x constructed as in the proof of Lemma 2. Let $\lambda_i \in I_i$ be chosen scalars and f_o, f_e functions in $C(K)$ with $\|f_o - f\|_\infty$, $\|f_e - f\|_\infty < \delta$ and $f_o(k) = \lambda_i$ for k in $\operatorname{supp} x_i$, i odd, $f_e(k) = \lambda_i$ for k in $\operatorname{supp} x_i$, i even. Such functions exist since $|\lambda_i - f(k)| < \delta$ for k in $\operatorname{supp} x_i$. Then

$$S_f x = S_f x_e + S_f x_o = S_{f_e} x_e + S_{f_o} x_o + S_{f - f_e} x_e + S_{f - f_o} x_o$$

where x_o and x_e are as in Lemma 2. Now since f_e and f_o are constant on $\operatorname{supp} x_i$ for i odd and even, respectively and since the homomorphism $f \mapsto S_f$ also has the property that $S_f y = S_g y$ if $f \equiv g$ on $\operatorname{supp} y$, we have

$$S_{f_e} x_e + S_{f_o} x_o = \sum_1^n \lambda_i x_i .$$

If M is the norm of the map $f \mapsto S_f$, then the remaining terms can be estimated by means of

$$\|S_{f - f_e} x_e\| \leqq M \|f - f_e\|_\infty \cdot \|x_e\| \leqq M \cdot \delta \cdot m \|x\|$$

and similarly for $S_{f - f_0} x_o$. Thus

$$\|S_f x - \sum \lambda_i x_i\| \leqq 2M \cdot \delta \cdot m \|x\| .$$

Taking suitable limits we obtain $S_f x = x_f$ and thus $S_f x = T_f x$ which completes the proof.

The complex case. Consider X as a real Banach space of twice the dimension. By the theorem for the real case we obtain a map $C_{\mathbf{R}}(K) \to B(X_{\mathbf{R}})$. By setting $T_f := T_{\operatorname{Re} f} + iT_{\operatorname{Im} f}$ we can extend this to all of $C(K)$. It remains only to show that the operators T_f are in fact complex linear. However in $X_{\mathbf{R}}$ the fact that an operator is complex linear is equivalent to the fact that it commutes with the operator $x \mapsto ix$. By Lemma 9 this is equivalent to the fact that $\operatorname{supp} ix \subseteq \operatorname{supp} x$ for all x. Since ii) of the definition of support representation gives $\operatorname{supp} ix = \operatorname{supp} x$ for all x we have the theorem in the complex case.

6. Conclusion.

In this paper we have shown how to construct an operator T_f from a function f basically by 'integrating' f over a sort of measure whose values are closed subspaces, namely the capacity J. This is a generalisation of, for example, the construction of a continuous functional calculus for spectral operators of scalar type by integrating over a spectral measure. This allows one to extend results on spectral operators to suitable classes of decomposable operators and this is done in a separate paper of the author [6]. However because of its general nature it is to be hoped that this result will also have applications in many other fields.

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