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MODELS FOR SYNTHETIC INTEGRATION THEORY*

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The present note gives an axiom for the treatment of integration in the context of syntetic differential geometry, and proves that the full, well adapted model considered by Dubuc [1] satisfies this integration axiom.

We shall, as was to be expected, define integration in terms of the notion "primitive of a function". But we want to have integrals for functions which have only been defined on *intervals*. This leads us to consider as basic object a ring object of line type equipped with a preorder relation $\leqq$.

We give the axiom in Section 1, and derive, by means of "standard synthetic calculus" some consequences. In Section 2, we prove that the axiom is valid for Dubuc's model, referred to above, essentially as a corollary of the embedding theorem of Porta and Reyes, [8].

In Section 1, we shall use set theoretic notation as shorthand for internal logic of the (unspecified) ambient topos in which we work. Thus, when we say: "for every $f\colon [0,1] \to R$, there exists a unique $g \ldots$", we mean that the sentence "$\forall f \in R^{[0,1]}\ \exists!\ g \ldots$" is internally valid. In Section 2, we shall have to unravel these internal sentences into statements about actual smooth manifolds with or without boundary.

Compared to the preprint version [7], the present version contains some simplifications.

1. Axioms for integration.

Let R be a ring object of line type (in the sense of Kock–Lawvere, [2]) in a topos $\mathscr{E}$, and let $\leqq$ be a pre-order relation on it, compatible with the ring structure. Thus, in particular, $\leqq$ is transitive and reflexive. It is not assumed to be antisymmetric; on the contrary, we shall assume that for any nilpotent $x \in R$ we have

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$$x \leqq 0 \quad \text{and} \quad 0 \leqq x .$$

If $a \leqq b$, the "set" $\{x \mid a \leqq x \leqq b\}$ will be denoted $[a, b]$ and called the closed interval determined by a and b. Note that a and b are *not*, in turn, determined by $[a, b]$ since, for instance, for any nilpotent $d \in A$

$$[a, b] = [a, b+d] .$$

In particular, $[0, 0]$ contains all nilpotent elements. We also note that any interval is "subeuclidean" or "etale" in the sense of [3] or [6], meaning that it is stable under the addition of nilpotents. In particular, for any $g\colon [a, b] \to R$, it makes sense to talk about $g'\colon [a, b] \to R$. Also, we note that $[a, b]$ is *convex*: if $x, y \in [a, b]$ and $t \in [0, 1]$, then

$$t \cdot x + (1-t) \cdot y \in [a, b] .$$

This is a standard consequence of the compatibility of $\leqq$ with the ring structure.

The axiom for integration says:

INTEGRATION AXIOM. For any $f\colon [0, 1] \to R$, there is a unique $g\colon [0, 1] \to R$ with $g' \equiv f$ and $g(0) = 0$.

We can then define

$$\int_0^1 f(t)\, dt := g(1);$$

several of the standard rules for integration then follow from the corresponding rules for differentiation (see [2, Theorem 8]) purely formally. In particular:

(1.1) $\int_0^1 f(t)\, dt$ depends in an R-linear way on f.

(1.2) Let $h\colon [0, 1] \to R$. Then $\int_0^1 h'(t)\, dt = h(1) - h(0)$.

From these two properties we can deduce:

PROPOSITION 1 (Hadamard's lemma). *Let $a \leqq b$ and $f\colon [a, b] \to R$. Then, for $x, y \in [a, b]$,*

$$f(y) - f(x) = (y-x) \cdot \int_0^1 f'(x + t \cdot (y-x))\, dt .$$

PROOF. (Note that the integrand makes sense because of convexity and étaleness of $[a, b]$.) By convexity, we have a map $\varphi\colon [0, 1] \to [a, b]$ given by

$$\varphi(t) = x + t \cdot (y-x) .$$

We have $\varphi' \equiv y - x$. So

$$\begin{aligned} f(y) - f(x) &= f(\varphi(1)) - f(\varphi(0)) \\ &= \int_0^1 (f \circ \varphi)'(t)\, dt && \text{by (1.2)} \\ &= \int_0^1 (y-x) \cdot (f' \circ \varphi)(t) dt && \text{(chain rule)} \\ &= (y-x) \cdot \int_0^1 (f' \circ \varphi)(t)\, dt && \text{by (1.1)}, \end{aligned}$$

which is the desired result.

PROPOSITION 2. *Let $a \leqq b$ and $g\colon [a,b] \to R$. If $g' \equiv 0$ on $[a,b]$, then g is constant on $[a,b]$.*

PROOF. By Hadamard's lemma above, we have, for any $y \in [a,b]$,

$$g(y) - g(a) = (y-a) \cdot \int_0^1 g'(a + t \cdot (y-a))\, dt .$$

By assumption, the integrand is $\equiv 0$, so by (1.1) we conclude that the integral is 0, so $g(y) = g(a)$.

Our next proposition says that "one can differentiate under the integral sign":

PROPOSITION 3. *Let $f\colon [a,b] \times [0,1] \to R$. Let $h\colon [a,b] \to R$ be defined by*

$$h(s) = \int_0^1 f(s,t)\, dt .$$

Then

$$h'(s) = \int_0^1 \frac{\partial f}{\partial s}(s,t)\, dt .$$

PROOF. Let $d \in D$ (i.e., d is an element of square zero). Then

$$\begin{aligned} h(s+d) - h(s) &= \int_0^1 (f(s+d,t) - f(s,t))\, dt \\ &= \int_0^1 d \cdot \frac{\partial f}{\partial s}(s,t)\, dt \\ &= d \cdot \int_0^1 \frac{\partial f}{\partial s}(s,t)\, dt , \end{aligned}$$

where we have used (1.1) in the first and third equality sign, and Taylor's formula (see e.g. [2, Proposition 6]) in the second. On the other hand, by Taylor,

$$h(s+d)-h(s) = d\cdot h'(s) .$$

Since this holds for all $d \in D$, we get the conclusion by the uniqueness assertion in the "line type" axiom [2] (cf. the formulation of [4]).

We can now prove the following strengthening of the integration axiom. Again, the theorem should be understood "internally":

THEOREM 4. *For any $a \leqq b$ and any $f\colon [a,b] \to R$, there is a unique $g\colon [a,b] \to R$ with $g' \equiv f$ and $g(a)=0$.*

PROOF. The uniqueness assertion follows immediately from Proposition 2 and linearity of differentiation. To prove existence, we construct $g\colon [a,b] \to R$ as follows. Let $c \in [a,b]$. To define $g(c)$, consider the unique affine map $\varphi_c\colon [0,1] \to [a,b]$ with $\varphi_c(0)=a$ and $\varphi_c(1)=c$ (it maps into $[a,c] \subseteqq [a,b]$, by convexity of intervals). We put

$$\begin{aligned} g(c) &:= \int_0^1 f(\varphi_c(t))\cdot \varphi_c'(t)\,dt \\ &= \int_0^1 f(a+t\cdot(c-a))\cdot(c-a)\,dt . \end{aligned}$$

By (1.1), $g(a)=0$. To differentiate g with respect to c, we may, by Proposition 3, differentiate under the integral sign. We get, using also chain rule and Leibniz rule for differentiation,

$$\begin{aligned} g'(c) &= \int_0^1 f'(a+t\cdot(c-a))\cdot t\cdot(c-a)+f(a+t\cdot(c-a))\,dt \\ &= \int_0^1 F'(t)\,dt , \end{aligned}$$

where $F(t) := f(a+t\cdot(c-a))\cdot t$. By (1.2), this equals

$$F(1)-F(0) = f(a+(c-a))\cdot 1 = f(c) .$$

This proves $g'(c)=f(c)$. This proves existence of the desired g, and thus the theorem.

If f and g are as in the theorem, we define $\int_a^b f(t)\,dt$ as $g(b)$. This is consistent with our previous definition of $\int_0^1 f(t)\,dt$. Also, (1.1) and (1.2) and Proposition 3 generalize to the case where the limits of integration are a, b, and we furthermore have the following:

(1.3) $$\int_a^b f(t)\,dt + \int_b^c f(t)\,dt = \int_a^c f(t)\,dt \quad \text{for } a \leqq b \leqq c\,.$$

(1.4) Let h: $[a,b] \to R$ be defined by $h(s) = \int_a^s f(t)\,dt$. Then $h' \equiv f$.

(1.5) Let φ: $[a,b] \to [a_1, b_1]$ have $\varphi(a) = a_1$, $\varphi(b) = b_1$. Then

$$\int_{a_1}^{b_1} f(t)\,dt = \int_a^b f(\varphi(s)) \cdot \varphi'(s)\,ds\,.$$

To see (1.3), take any g: $[a,c] \to R$ with $g' \equiv f$ (such exists, by the theorem). By the generalized version of (1.2),

$$\int_a^b f = \int_a^b g' = g(b) - g(a)\,,$$

and similar $\int_b^c f = g(c) - g(b)$, from which (1.3) is immediate. To see (1.4), let again $g' \equiv f$. For any d of square zero, we then have, using Taylor's formula and (1.3),

$$\begin{aligned} d \cdot h'(s) &= h(s+d) - h(s) = \int_s^{s+d} f(t)\,dt = g(s+d) - g(s) \\ &= d \cdot g'(s) = d \cdot f(s)\,, \end{aligned}$$

and since this holds for all such d, we conclude $h'(s) = f(s)$ by the uniqueness assertion in the line type axiom. Finally, (1.5) is a formal consequence of the chain rule ([2, Theorem 8]).

2. Integration in Dubuc's model.

"Dubuc's model" is a Grothendieck topos $\mathscr{E}$ with a full and faithful functor i: $Mf \to \mathscr{E}$, where Mf is the category of smooth manifolds. The pair $\mathscr{E}, i$ is a well adapted model in the sense of [5], from where we shall freely use notations, except that we write R instead of A for $i(\mathsf{R})$, the basic ring of line type in $\mathscr{E}$. In particular, j: $\mathscr{W}^{\mathrm{op}} \to \mathscr{E}$ embeds the dual of the category of real Weyl algebras.

We shall utilize a theorem of Porta and Reyes ([8, Proposition p. 43]—which in turn depends on Whitney's theorem on smooth even functions) according to which i: $Mf \to \mathscr{E}$ can be extended to a full embedding (also denoted i) from the category Mf' of smooth manifolds with boundary. The functor $i: Mf \to \mathscr{E}$ preserves finite products, and i: $Mf' \to \mathscr{E}$ preserves those finite products that exist.

The manifold-with-boundary,

$$\mathsf{R}_{\geqq 0} = \{r \in \mathsf{R} \mid r \geqq 0\}$$

embeds by i to a subobject $R_{\geqq 0}$ of R, which in turn defines a preorder relation $\leqq$ on R, compatible with the ring structure. All nilpotents are in $R_{\geqq 0}$ (which is in fact étale, or subeuclidean subobject: stable under addition of infinitesimals). With this preorder relation we shall prove

THEOREM. *The Integration Axiom holds in Dubuc's model.*

We shall need the following specific feature of Dubuc's topos $\mathscr{E}$, namely that objects of the form

$$iM \times jX \qquad (\text{with } M \in Mf,\ X \in \mathscr{W})$$

form a site of definition for the topos $\mathscr{E}$. In particular, the preorder relation $\leqq$ on R may be defined in an alternative way, using i and j, as follows. It suffices to define when an element a in R, defined at stage $iM \times jX$, has $0 \leqq a$. Consider such an element

$$iM \times jX \xrightarrow{a} R\ .$$

There is a unique map $1 \to jX$, and therefore a canonical map $iM \to iM \times jX$. The composite, denoted a_0,

$$a_0 = iM \to iM \times jX \to R = i\mathsf{R}$$

comes, by i being full and faithful, from a unique smooth map $\bar{a}$: $M \to \mathsf{R}$, and we let $a \geqq 0$ provided $\bar{a}(m) \geqq 0$, $\forall\, m \in M$. (The fact that the order relation thus described agrees with the one defined by $R_{\geqq 0}$ essentially follows from [9, Proposition p. 15].)

We remark that $[0, 1]$ (also denoted I) denotes both a certain manifold-with-boundary, and a certain subobject of R. However, they correspond under i: $Mf' \to \mathscr{E}$.

PROOF OF THE THEOREM. Let f denote an "element" of $R^{[0,1]}$ defined at stage $iM \times jX$, say (letting I denote $[0, 1]$),

$$f\colon iM \times jX \to R^I\ .$$

We shall produce an element g of R^I (defined at the same stage!), g: $iM \times jX \to R^I$ such that

$$\hat{f} = \frac{\partial \hat{g}}{\partial t} \colon iM \times jX \times I \to R \,, \tag{2.2}$$

where $\hat{}$ denotes exponential adjoint, and $\partial/\partial t$ means deriving in the direction of the I-factor.

By twisted exponential adjointness, we get from f a map $\tilde{f}\colon iM \times I \to R^{jX} \cong iX$, which (by the fact that $i\colon Mf' \to \mathscr{E}$ is full and faithful and preserves those products that exist) comes from a unique map in Mf'

$$\bar{f}\colon M \times I \to X \,.$$

We form a map $\bar{g}\colon M \times I \to X$ by putting

$$\bar{g}(m,t) = \int_0^t \bar{f}(m,s)\,ds \,. \tag{2.3}$$

Note that the integration makes sense, X being a finite dimensional real vector space. Since $\bar{f}$ is smooth, it is classical that $\bar{g}$ is also smooth. Now the bijection by which we produced $\bar{f}$ out of f,

$$\hom_{\mathscr{E}}(iM \times jX,\, R^I) \cong \hom_{Mf'}(M \times I, X) \,,$$

yields when applied (backwards) to $\bar{g}$ a map

$$g\colon iM \times jX \to R^I \,.$$

We claim that g satisfies (2.2). This fact must of course be derived from the evident equality

$$\frac{\partial \bar{g}}{\partial t} = \bar{f}\colon M \times I \to X \tag{2.4}$$

coming from (2.3) and the fundamental theorem of calculus. From (2.4) we get by applying the functor i (which transforms the "analytic" differentiation in Mf to the "synthetic" one in $\mathscr{E}$, essentially by [5, Theorem 1]) the equality

$$\frac{\partial \tilde{g}}{\partial t} = \tilde{f}\colon iM \times I \to iX = R^{jX} \tag{2.5}$$

which expresses that the directional derivative of $\tilde{g}$ along the vector field $\partial/\partial t$ of $iM \times I$ is $\tilde{f}$ (in the terminology of [4], say). Now we have a completely general fact about derivatives with values in Euclidean modules of form R^S (S arbitrary).

LEMMA. *Let Y be a vector field on object N, and $h\colon N \to R^S$ a function. Then for the directional derivative*

$$Y(h)\colon N \to R^S$$

we have

$$Y(h)\hat{\ } = \bar{Y}(\hat{h}) \colon N \times S \to R$$

where $\hat{\ }$ *denotes exponential adjointness, and* $\bar{Y}$ *is the vector field on* $N \times S$ *given by* $(\langle n, s\rangle, d) \mapsto \langle Y(n, d), s\rangle$.

The proof is straightforward in the style of [4].

We deduce in particular from the lemma and (2.5) that

$$\frac{\partial \hat{g}}{\partial t} = \hat{f} \colon iM \times I \times jX \to R\ .$$

Since clearly $\hat{g}(0) = 0$, this means that the element g of $R^{[0,1]}$ defined at stage $iM \times jX$ has $g' = f$ and $g(0) = 0$. This proves the existence.

The uniqueness comes similarly from the uniqueness of a $\bar{g} \colon M \times I \to X$ with $\partial \bar{g}/\partial t = \bar{f}$ and $\bar{g}(m, 0) = 0$, $\forall m \in M$.

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