MATH. SCAND. 48 (1981), 153–164

NIELSEN METHODS IN GROUPS WITH A LENGTH FUNCTION

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Many theorems have been proved using cancellation arguments in groups for which a normal form theorem holds. Here we prove a general theorem on groups with an integer valued length function satisfying three of the axioms given by Lyndon ([2]) and show that a large number of cancellation theorems are special cases or immediate corollaries of this theorem.

In section 1 we give definitions and preliminary lemmas. In section 2 we prove the main theorem and two corollaries and in section 3 we give some applications. Further applications will appear in a later paper.

1.

Let G be a group, with identity e, which has a normalized integer valued length function, that is a function $x \mapsto |x|$ satisfying

A1′. $|e| = 0$,
A2. $|x| = |x^{-1}|$,

and

A4. $d(x,y) > d(y,z)$ implies $d(x,z) = d(y,z)$,

where

$$2d(x,y) = |x| + |y| - |xy^{-1}| .$$

[Intuitively $d(x,y)$ is the length of the largest common terminal segment of x and y.]

As observed by Lyndon ([2, p. 210]) A4 is equivalent to

$$d(x,y) \geqq \min\{d(y,z), d(x,z)\}$$

and to

$$d(y,z), d(x,z) \geqq m \quad \text{implies } d(x,y) \geqq m .$$

Received September 21, 1979. In revised form May 27, 1980.

It can also be shown easily that A1′, A2, and A4 imply

$$|x| \geqq d(x,y) = d(x,y) \geqq 0 .$$

Let $X^{\pm 1}$ be a subset of G. A *word* in $X^{\pm 1}$ is a sequence $x_1 \ldots x_n$, $n \geqq 0$, with x_i in $X^{\pm 1}$. A *reduced word* is one in which $x_{i+1} \neq x_i^{-1}$. A *subword* is a subsequence, proper or not, of consecutive elements of the sequence. The *inverse* of $x_1 \ldots x_n$ is $x_n^{-1} x_{n-1}^{-1} \ldots x_1^{-1}$. We do not distinguish in notation between a word and the group element given by the corresponding product.

DEFINITION. A reduced word $x_0 x_1 \ldots x_{n+1}$ is a *sink* if

$$|x_0 x_1 \ldots x_n| > |x_0 x_1 \ldots x_{n+1}|$$

and no proper subword or its inverse satisfies the corresponding inequality. A reduced word is *sink-free* if no subword or its inverse is a sink.

The following extends the Lemma in [1].

LEMMA 1. *If every proper subword of the reduced word* $w = x_0 x_1 \ldots x_{n+1}$ *is sink free and if*

$$|w| \leqq \max \{|x_0|, |x_1|, \ldots, |x_{n+1}|\}$$

then

$$|x_i x_{i+1} \ldots x_j| = \max \{|x_i|, |x_{i+1}|, \ldots, |x_j|\}$$

for all proper subwords $x_i x_{i+1} \ldots x_j$. *If strict inequality holds then* $|x_0| \geqq |x_1|, \ldots, |x_n| \leqq |x_{n+1}|$.

PROOF. Let $p_i = x_0 x_1 \ldots x_i$ and $q_i = x_{i+1} \ldots x_{n+1}$ for $i = 0, 1, \ldots, n$. Since every proper subword is sink free we have, for $i = 1, 2, \ldots, n$,

$$(1) \qquad \begin{aligned} |p_i| &\geqq \max \{|p_{i-1}|, |x_i|\} , \\ |q_{i-1}| &\geqq \max \{|q_i|, |x_i|\} , \end{aligned}$$

and by induction

$$(2) \qquad \begin{aligned} |p_i| &\geqq \max \{|x_0|, |x_1|, \ldots, |x_i|\} , \\ |q_{i-1}| &\geqq \max \{|x_i|, |x_{i+1}|, \ldots, |x_{n+1}|\} , \end{aligned}$$

If

$$(3) \qquad |p_k| + |q_{k-1}| > |p_{k-1}| + |q_k|$$

for some $k = 1, 2, \ldots, n$ then

$$\begin{aligned} 2d(p_k, x_k) &= |p_k| + |x_k| - |p_{k-1}| \\ &> |q_k| + |x_k| - |q_{k-1}| \\ &= 2d(q_k^{-1}, x_k)\,. \end{aligned}$$

Therefore, by A4, $2d(p_k, q_k^{-1}) = 2d(q_k^{-1}, x_k)$ that is, since $w = p_k q_k$,

$$|p_k| + |q_k| - |w| = |q_k| + |x_k| - |q_{k-1}|\,. \tag{4}$$

Suppose that one of the inequalities in (1) is strict for some k, then (3) and hence (4) holds for that k, and moreover

$$\begin{aligned} |p_k| + |q_{k-1}| - |x_k| &> \max\{|p_{k-1}|, |x_k|\} + \max\{|q_k|, |x_k|\} - |x_k|\,. \\ &\geqq |p_{k-1}|, |q_k|, |x_k|\,, \end{aligned}$$

so by (2) and (4)

$$|w| > \max\{|x_0|, |x_1|, \ldots, |x_{n+1}|\}$$

contradicting the hypothesis. Therefore equalities hold in (1) and hence in (2) for $i = 1, 2, \ldots, n$. Thus we have proved the first part of the lemma for the proper subwords p_j and q_{i-1}. However this means that the words $q_{i-1} = x_i x_{i+1} \ldots x_{n+1}$, $i \geqq 1$, also satisfy the hypotheses of the lemma and applying the corresponding equalities in (2) to these words we get

$$|x_i x_{i+1} \ldots x_j| = \max\{|x_i|, \ldots, |x_j|\}$$

for $1 \leqq i < j \leqq n$, which together with the result for p_j and q_{i-1} proves the first part of the lemma for all $i < j$.

Now by symmetry we may assume $|x_0| \geqq |x_{n+1}|$. Then strict inequality in the hypothesis gives

$$\begin{aligned} |w| &< \max\{|x_0|, |x_1|, \ldots, |x_{n+1}|\} \\ &= \max\{|x_0|, |x_1|, \ldots, |x_n|\} \\ &= |p_n|, \quad \text{by the first part}\,. \end{aligned}$$

Let k be the greatest integer, if any, such that either $|p_k| > |p_{k-1}|$ or $|q_{k-1}| > |q_k|$, then (3) and hence (4) holds for that k and moreover

$$|w| < |p_n| = |p_{n-1}| = \ldots = |p_k|$$

so

$$0 < |p_k| - |w| = |x_k| - |q_{k-1}| \qquad \text{by (4)}\,.$$

But $|q_{k-1}| \geqq |x_k|$ from (1), therefore $|p_i| = |p_{i-1}|$ and $|q_{i-1}| = |q_i|$ for all $i = 1, 2, \ldots, n$. Thus using the first part

$$|x_0| = |p_0| = |p_n| = \max\{|x_0|, |x_1|, \ldots, |x_n|\}$$

and

$$|x_{n+1}| = |q_n| = |q_0| = \max\{|x_1|, \ldots, |x_{n+1}|\} .$$

We now introduce Lyndon's abstract lexicographic ordering on ideals in G.

DEFINITION. An *ideal* is a non-empty subset Γ of G such that for any x and y in Γ, z is in Γ whenever

$$d(x,z) \geqq d(x,y) .$$

For any x in G and any integer i, $0 \leqq i \leqq |x|$, we put

$$\Gamma_i(x) = \{y \in G : 2d(x,y) \geqq i\} ,$$

and we abbreviate $\Gamma_{|x|}(x)$ to $\Gamma(x)$. Each $\Gamma_i(x)$ is an ideal and we have a chain

$$G = \Gamma_0(x) \supseteq \Gamma_1(x) \supseteq \ldots \supseteq \Gamma_{|x|}(x) = \Gamma(x) .$$

[Intuitively $\Gamma_i(x)$ represents the terminal segment of x of length $i/2$ and $\Gamma(x)$ the "right half" of x (where $|x|$ may be even or odd).]

Given an arbitrary well-ordering of all the ideals of G, we have an induced lexicographic partial well-ordering on the ideals $\Gamma(x)$ defined as follows. If $|x| = |x'| = l$ then put

$$\Gamma(x) > \Gamma(x')$$

whenever, in the chains

$$G = \Gamma_0(x) \supseteq \Gamma_1(x) \supseteq \ldots \supseteq \Gamma_l(x) = \Gamma(x)$$
$$G = \Gamma_0(x') \supseteq \Gamma_1(x') \supseteq \ldots \supseteq \Gamma_l(x') = \Gamma(x') ,$$

$\Gamma_i(x)$ is greater than $\Gamma_i(x')$ (in the given well-order of all the ideals of G) for the first i for which they are not equal.

REMARK. $\Gamma(x) = \Gamma(x')$ if and only if

$$2d(x,x') \geqq |x| = |x'| .$$

Moreover if $2d(x,x') < |x| = |x'|$ then we must have

$$\Gamma(x) < \Gamma(x') \quad \text{or} \quad \Gamma(x) > \Gamma(x') .$$

Suppose that $|x| \geqq |y|$ and $2d(x,y) \geqq |y|$. If $r \leqq y$ then by A4

$$2d(x,z) \geqq r \quad \text{if and only if} \quad 2d(y,z) \geqq r .$$

That is

$$\Gamma_r(x) = \Gamma_r(y) \qquad \text{for all } r \leqq |y| .$$

Thus we have the following.

LEXICOGRAPHIC PROPERTY. *If* $|x| = |x'| \geqq |y| = |y'|$ *and if*

$$2d(x,y) \geqq |y| \quad \text{and} \quad 2d(x',y') \geqq |y'| ,$$

then $\Gamma(x) < \Gamma(x')$ *whenever* $\Gamma(y) < \Gamma(y')$.

[Intuitively this says that if the right halves of y and y' are segments of the right halves of x and x' respectively, then the right half of x is before the right half of x' whenever the right half of y is before the right half of y'.]

We now use this lexicographic partial well-order to define a partial well-order on the elements of G as follows.

Put $x < y$ if

(i) $|x| < |y|$, or
(ii) $|x| = |y|$ and $\{\Gamma(x), \Gamma(x^{-1})\} < \{\Gamma(y), \Gamma(y^{-1})\}$

where the partial order of pairs is defined by

$$\{\Gamma(x), \Gamma(x^{-1})\} < \{\Gamma(y), \Gamma(y^{-1})\}$$

if $\Gamma(x^{\varepsilon}) \leqq \Gamma(y^{\eta})$ and $\Gamma(x^{-\varepsilon}) < \Gamma(y^{-\eta})$ for some ε, $\eta = \pm 1$.

LEMMA 2. *If* $|xy| = |x| \geqq |y|$ *and* $\Gamma(y^{-1}) > \Gamma(y)$ *then* $x > xy$ *and* $x \neq y^{\pm 1}$.

PROOF.

$$2d(x, y^{-1}) = |x| + |y| - |xy| = |y| \tag{5}$$

and

$$2d(xy, y) = |xy| + |y| - |x| = |y| .$$

Therfore by the lexicographic property, $\Gamma(y^{-1}) > \Gamma(y)$ implies $\Gamma(x) > \Gamma(xy)$. Moreover

$$2d(x^{-1}, (xy)^{-1}) = |x| + |xy| - |y| \geqq |xy| = |x|$$

and so by the Remark above

$$\Gamma(x^{-1}) = \Gamma((xy)^{-1}) .$$

Thus by condition (ii) of the definition we have $x > xy$.

If $x = y^{-1}$, then $|xy| = 0 = |x| = |y|$ and $\Gamma(y) = \Gamma_0(y) = G = \Gamma(y^{-1})$ contradicting the hypotheses. If $x=y$, then $|x|=|y|$ and from (5) and the Remark above, $\Gamma(y^{-1})=\Gamma(x)$. So $\Gamma(y^{-1})=\Gamma(y)$ again contradicting the hypotheses.

2.

For a group with length function we now define a subset, denoted M, which plays a central role in what follows.

DEFINITION.

$$M = \{xy \in G : d(x,y^{-1})+d(y,x^{-1})>|x|=|y|\}$$

[Lyndon ([2, p. 213–214]) showed that a free product with the usual length function satisfies his axiom A5, that is $M=\{e\}$. More generally if G is a free product of G_1 and G_2 with amalgamated subgroup A, endowed with the usual length function, then M consists of all conjugates of A in G (see section 3).]

If X is a subset of G and $w=x_0x_1 \ldots x_{n+1}$ is a word in $X^{\pm 1}$ we define a Nielsen transformation of X *attached* to w to be a replacement of an element of X occurring in w, say x_k, by $x_ix_{i+1} \ldots x_j$, where $0 \leqq i \leqq k \leqq j \leqq n+1$ and $x_k^{\pm 1} \neq x_i, \ldots, x_{k-1}, x_{k+1}, \ldots, x_j$, leaving all other elements of X fixed. We denote this by

$$x_k \mapsto x_ix_{i+1} \ldots x_j .$$

Clearly the resulting set generates the same subgroup as X. If $x_k > x_ix_{i+1} \ldots x_j$ then we say that the Nielsen transformation *reduces* x_k.

THEOREM. *Let X be a subset of G which is minimal under Nielsen transformations attached to a word $w=x_0x_1 \ldots x_{n+1}$, $x_i \in X^{\pm 1}$. Suppose w satisfies the hypotheses of Lemma* 1, *and is not xx^{-1}, for $x \in X^{\pm 1}$, then*

(i) $|x_{i-1}|>|x_i|= \ldots =|x_j|<|x_{j+1}|$ *implies*

$$\Gamma(x_i^{-1}) = \Gamma(x_i) = \ldots = \Gamma(x_j^{-1}) = \Gamma(x_j) .$$

(ii) $|x_0|=|x_1|= \ldots =|x_{n+1}|$ *implies*

$$\Gamma(x_0^{-1}) \geqq \Gamma(x_0) = \Gamma(x_1^{-1}) = \Gamma(x_1) = \ldots = \Gamma(x_{n+1}^{-1}) \leqq \Gamma(x_{n+1}) ,$$

and, if $|w|<|x_0|$, then $w \in M$ and either $x_0=x_{n+1}^{-1}$, or both $\Gamma(x_0^{-1})=\Gamma(x_0)$ and $\Gamma(x_{n+1}^{-1})=\Gamma(x_{n+1})$.

(iii) $|x_0x_1 \ldots x_{n+1}|<\max\{|x_0|,|x_1|,\ldots,|x_{n+1}|\}$ *implies $|x_0|=|x_{n+1}|$, $x_0=x_i^{\pm 1}$ for some $i=1,2,\ldots,n+1$, and $x_{n+1}=x_j^{\pm 1}$ for some $j=0,1,\ldots,n$.*

(iv) $|x_{i-1}|>|x_i|, |x_{i+1}|,\ldots,|x_j|<|x_{j+1}|$ *implies* $x_ix_{i+1}\ldots x_j$ *is in* M, *and*

$$2d(x_{i-1},x_{j+1}) \geqq \min\{|x_{i-1}|,|x_{j+1}|\}\ .$$

PROOF. Let $n=0$ and $|w|<\max\{|x_0|,|x_1|\}$, then $x_0=x_1^{\pm 1}$, for otherwise $x_k \mapsto x_0x_1$ is an attached Nielsen transformation reducing x_k for $k=0$ or 1. By assumption $x_0 \neq x_1^{-1}$, so $x_0=x_1$ and both (ii) and (iii) hold. Conclusions (i) and (iv) do not apply in this case. It remains to consider the cases $n>0$, and $n=0$ with $w=\max\{|x_0|,|x_1|\}$.

Suppose $|x_{k-1}|\geqq|x_k|$ then, using Lemma 1 in the case $n>0$, we have

$$|x_{k-1}x_k| = |x_{k-1}| \geqq |x_k|\ . \tag{6}$$

If $\Gamma(x_k^{-1})>\Gamma(x_k)$, then by Lemma 2, $x_{k-1}>x_{k-1}x_k$ and $x_{k-1}\neq x_k^{\pm 1}$, so $x_{k-1} \mapsto x_{k-1}x_k$ is an attached Nielsen transformation reducing x_{k-1}, which contradicts minimality. Thus $|x_{k-1}|\geqq|x_k|$ implies $\Gamma(x_k^{-1})\leqq\Gamma(x_k)$. Similarly, if $|x_k|\leqq|x_{k+1}|$, then $\Gamma(x_k^{-1})\geqq\Gamma(x_k)$. Moreover if $|x_{k-1}|=|x_k|$, then we have equality in (6), and hence by the Remark in section 1, $\Gamma(x_{k-1})=\Gamma(x_k^{-1})$. These three facts prove (i) and the first part of (ii).

Suppose $|w|<|x_0|$ and $\Gamma(x_0^{-1})>\Gamma(x_0)=\ldots=\Gamma(x_{n+1}^{-1})$ then $x_0\neq x_1^{\pm 1},\ldots, x_n^{\pm 1}, x_{n+1}$. If $x_0\neq x_{n+1}^{-1}$, then $x_0 \mapsto x_0\ldots x_{n+1}$ is an attached Nielsen transformation which reduces the length of x_0 contradicting minimality, so $x_0=x_{n+1}^{-1}$. Similarly if $\Gamma(x_{n+1}^{-1})<\Gamma(x_{n+1})$, then $x_0=x_{n+1}^{-1}$.

In either case, $\Gamma(x_0^{-1})=\Gamma(x_{n+1})$, and by the Remark in section 1, $2d(x_0^{-1},x_{n+1})\geqq|x_0|$. Put $x=x_0x_1\ldots x_n$ and $y=x_{n+1}$, then $w=xy$ and

$$2d(x^{-1},x_0^{-1}) = |x|+|x_0|-|x_0^{-1}x| \geqq |x_0|\ ,$$

so by A4, $2d(x^{-1},y)\geqq|x_0|$. Moreover

$$2d(x,y^{-1}) = |x|+|y|-|xy| > |x_0|$$

so w is in M. This completes the proof of (ii).

If $|x_0x_1\ldots x_{n+1}|<\max\{|x_0|,|x_1|,\ldots,|x_{n+1}|\}$, then by Lemma 1 $|x_0| \geqq|x_1|,\ldots,|x_n|\leqq|x_{n+1}|$. Suppose $|x_0|>|x_{n+1}|$, then $x_0 \mapsto x_0x_1\ldots x_{n+1}$ is a Nielsen transformation reducing x_0, so $|x_0|\leqq|x_{n+1}|$. Similarly $|x_{n+1}|\leqq|x_0|$. Therefore $|x_0x_1\ldots x_{n+1}|<|x_0|=|x_{n+1}|$, and because of the minimality neither $x_0 \mapsto x_0x_1\ldots x_{n+1}$ nor $x_{n+1} \mapsto x_0x_1\ldots x_{n+1}$ can be Nielsen transformations. The result (iii) follows.

It remains to prove (iv). Let $x=x_{i-1}$, $y=x_ix_{i+1}\ldots x_j$, and $z=x_{j+1}$, then by Lemma 1, $|x|=|xy|>|y|<|yz|=|z|$ and $|xyz|\leqq\max\{|x|,|z|\}$. Therefore

$$2d(x^{-1},(xy)^{-1}) = |x|+|xy|-|y| > |x| = |xy|\ . \tag{7}$$

By the Remark in section 1, $\Gamma(x^{-1})=\Gamma((xy)^{-1})$, and hence

$$\Gamma(x) \leqq \Gamma(xy)\,, \tag{8}$$

otherwise $x>xy$ and $x_{i-1} \mapsto x_{i-1}x_i \ldots x_j$ is an attached Nielsen transformation reducing x_{i-1}. Similarly

$$\Gamma(z^{-1}) \leqq \Gamma((yz)^{-1})\,. \tag{9}$$

We can suppose by symmetry that $|x| \leqq |z|$, so

$$2d(xy, z^{-1}) = |xy|+|z|-|xyz| \geqq |xy|\,, \tag{10}$$

and

$$2d(x, (yz)^{-1}) = |x|+|yz|-|xyz| \geqq |x|\,.$$

If $\Gamma(x)<\Gamma(xy)$ then by the Lexicographic Property, $\Gamma((yz)^{-1})<\Gamma(z^{-1})$, contradicting (9). Therefore from (8), $\Gamma(x)=\Gamma(xy)$, that is by the Remark in section 1,

$$2d(x, xy) \geqq |x| = |xy|\,.$$

Thus by A4 using (10)

$$2d(x, z^{-1}) \geqq |xy| = |x| = \min\{|x|, |z|\}\,.$$

Moreover from (7), $2d(x^{-1}, (xy)^{-1})>|x|=|xy|$, so $y=x^{-1}xy$ is in M.

Following Lyndon [2] we put

$$N = \{x \in G :\ \Gamma(x)=\Gamma(x^{-1})\}$$

and if x and y are in N we put $x \sim y$ if $2d(x, y) \geqq |x|=|y|$. This is easily shown to be an equivalence relation.

Definition. A subset X of G is *minimal* if there is no Nielsen transformation of X reducing one element of X and leaving the others fixed.

Corollary 1. *Let X be minimal and let H be the subgroup generated by X. If $H \cap M=\{e\}$ then $X \setminus N$ is a basis for a free subgroup F of H, and H is the free product of F and the subgroups generated by equivalent elements of $X \cap N$.*

Proof. Suppose w is a reduced word in $X^{\pm 1}$ which gives the identity in G and which has no proper subword giving the identity. Then either all the letters in w are of length zero and are thus equivalent elements of N or some subword or its inverse say $x_0x_1 \ldots x_{n+1}$ forms a sink. By Lemma 1 and part (iii) of the Theorem

$$|x_{n+1}| = |x_0| \geqq |x_1|, \ldots, |x_n| .$$

Since $H \cap M = e$, part (iv) of the Theorem cannot occur so

$$|x_0| = |x_1| = \ldots = |x_{n+1}|$$

and, by part (ii) and the definitions of N and $\Gamma(x)$, $x_0, x_1, \ldots, x_{n+1}$ are equivalent elements of N with $x_0 x_1 \ldots x_{n+1} = e$. Thus every relation between elements of X is a consequence of relations between equivalent elements of $X \cap N$. The result follows from the definition of a free product.

Corollary 2. *Suppose G can be generated by elements of length zero or one. If $M = \{e\}$ then every minimal set of generators of G has elements of length zero or one only.*

Proof. Let X be a minimal set of generators of G. Suppose there is some x in X of length greater than one. Consider all words in $X^{\pm 1}$ which give elements of length zero or one in G and which have no subword giving the identity. Since G is generated by these elements either x is redundant in X or it appears in one of these words. In the first case there is a Nielsen transformation taking x to the identity, contradicting the minimality of X, and in the second case we have a sink $x_0 x_1 \ldots x_{n+1}$ with no subword giving the identity. By the theorem this is impossible if $M = \{e\}$.

Note that the same proof shows that if $M = \{e\}$ and if G is generated by elements of length less than r, then the elements of every minimal set of generators have length less than r.

3.

Although it is possible to prove general results about the sets M and N and about the equivalence relation on N, we will here apply the Theorem and Corollaries only to special cases.

I. *Free products with amalgamation.*

Suppose G is a free product of groups G_λ, called the factors, with a common proper amalgamated subgroup A. Then for every element g of G, not in A, there is a smallest integer l such that g is a product of l elements $g_1 g_2 \ldots g_l$ with successive g_i from different factors and not in A (see [4, section 4.2]). Call $g_1 g_2 \ldots g_l$ a reduced form for g and define $|g| = l$ and $|a| = 0$ for all $a \in A$. Then $x \mapsto |x|$ satisfies the axioms of Section 1. If $x_1 \ldots x_l$ and $y_1 \ldots y_m$ are reduced forms for x and y respectively, then $d(x, y^{-1}) > 0$ if and only if x_l and y_1 are from the same factor. Moreover

(11) $$x_{l-r+1} \ldots x_l y_1 \ldots y_r \in A \quad \text{for } r \leqq d(x, y^{-1}) .$$

Let a be in A and let x be an element of G of length $l \geqq 1$. Put $y = ax^{-1}$, then $yx = a$, $xy = xax^{-1}$ and

$$0 = |yx| \leqq |xy| \leqq 2l - 1 .$$

Hence

$$d(x, y^{-1}) + d(y, x^{-1}) \geqq \tfrac{1}{2} + l > l = |x| = |y| ,$$

so xax^{-1} and a are in M. Thus every conjugate of an element of A is in M.

Conversely suppose

$$d(x, y^{-1}) + d(y, x^{-1}) > |x| = |y| = l .$$

Let s and t be the integer parts of $d(x, y^{-1})$ and $d(y, x^{-1})$, respectively. Since $2d(x, y^{-1})$ and $2d(y, x^{-1})$ are integers, we have $s + t \geqq l$. Put $r = l - s$, then from (11) we have that $x_{r+1} \ldots x_l y_1 \ldots y_s$ and $y_{s+1} \ldots y_l x_1 \ldots x_r$ are in A, say a and a', respectively. Since $xy = x_1 \ldots x_r a a' (x_1 \ldots x_r)^{-1}$, every element of M is a conjugate of an element of A.

Similar methods will show that N consists of conjugates of the factors of G, each conjugate of each factor being an equivalence class. Thus we have the following.

H. NEUMANN'S THEOREM ([3]). *If G is a free product with amalgamated subgroup A and if H is a subgroup which intersects all conjugates of A trivially, then H is a free product of a free group and conjugates of subgroups of the factors of G.*

If A is the identity, that is G is a free product, then this reduces to the following.

KUROS SUBGROUP THEOREM. *Every subgroup of G is a free product of a free group and conjugates of subgroups of the free factors.*

If G is a free product with factors G_λ and if $g(G_\lambda)$ is the minimum number of generators of G_λ, then by Corollary 2 any minimal set X of generators of G consists of elements of the factors G_λ. Moreover, in order to generate G_λ, X must have at least $g(G_\lambda)$ elements in G_λ.

GRUSHKO-NEUMANN THEOREM. *The cardinality of X is not less than $\sum_\lambda g(G_\lambda)$. Moreover if $\sum_\lambda g(G_\lambda)$ is finite and φ is an epimorphism of a free group F with finite basis B onto G, then there is an automorphism α of F such that $\varphi(\alpha(B)) \subset \bigcup_\lambda G_\lambda$.*

PROOF. The first part follows from the immediately preceeding remarks. Since $\varphi(B)$ is finite it can be minimised in the partial well order by a finite number of Nielsen transformation. The result follows since each Nielsen transformation of $\varphi(B)$ can be obtained by an automorphism of F.

We now show that Theorem 1 of Zieschang ([6, p. 11]) is a special case of the Theorem above. Let G be the free product of G_λ with amalgamated subgroup A, and with the length function described above. Let X be a minimal set in G. Suppose that $w = x_0 x_1 \dots x_{n+1}$, $x_i \in X^{\pm 1}$, is a sink, with no subword equal to the identity. Then by Lemma 1 and part (iii) of the Theorem, $|x_0| = |x_{n+1}| \geqq |x_1|, \dots, |x_n|$. Let $l = \min\{|x_1|, \dots, |x_n|\}$ and let $x_i \dots x_j$ be a maximal subword of w such that $|x_i| = \dots = |x_j| = l$ if $l \geqq 1$, and $|x_i|, \dots, |x_j| \leqq 1$ if $l = 0$. Then exactly one of the following holds:

(a) $l = 0$, $|x_i| = \dots = |x_j| = 0$ and $|x_{i-1}|, |x_{j+1}| \geqq 2$,

(b) $l = 0$ and $|x_i|, \dots, |x_j|$ are all zero or one, with at least one of each; moreover $|x_{i-1}|, |x_{j+1}| \geqq 2$, unless $i = 0$ and $j = n+1$.

(c) $|x_i| = \dots = |x_j| = l \geqq 1$; moreover $|x_{i-1}|, |x_{j+1}| > l$ unless $i = 0$ and $j = n+1$.

If $i \neq 0$ and $j \neq n+1$, let $\xi_1 \xi_2 \dots \xi_n$ and $\eta_1 \dots \eta_m$ be reduced forms for x_{i-1} and x_{j+1}, respectively.

In case (a), $x_i, \dots, x_j$ are all in A and $m, n \geqq 2$. Moreover by part (iv) of the Theorem, $2d(x_{i-1}, x_{j+1}^{-1}) \geqq \min\{m, n\} \geqq 2$, and by Lemma 1, $|x_{i-1} x_i \dots x_j x_{j+1}| \leqq \max\{m, n\}$, that is $2d(x_{i-1}\alpha, x_{j+1}^{-1}) \geqq \min\{m, n\} \geqq 2$, where $e \neq \alpha = x_i \dots x_j \in A$. Therefore, by (11), we have $\xi_n \eta_1$ and $(\xi_n \alpha)\eta_1$ in A. Thus $\xi_n \alpha \xi_n^{-1} = \xi_n \alpha \eta_1 \eta_1^{-1} \xi_n^{-1} \in A$, where ξ_n lies in one of the factors and not in A, so (2.3) of Ziechang's theorem ([6, p. 11]) is satisfied.

In case (b), $x_i, \dots, x_j$ all lie in the same factor and have length $\leqq 1$. At least one of these has length 1, i.e. does not lie in A. If $i = 0$ and $j = n+1$, then $x_0 x_1 \dots x_{n+1}$ is in A and satisfies (2.4) of [6]. If $i \neq 0$ and $j \neq n+1$, then $x_i \dots x_j = \alpha$ lies in one of the factors but not in A and, as in case (a), $\xi_n \alpha \xi_n^{-1}$ lies in A; moreover ξ_n and α lie in the same factor, and again (2.4) of [6] is satisfied.

In case (c), if $i = 0$ and $j = n+1$, then $x_0, x_1, \dots, x_{n+1}$ are all of the same length and by part (ii) of the theorem either $x_0, x_1, \dots, x_{n+1}$ are equivalent elements of N (i.e. lie in a conjugate of one of the factors) and $x_0 x_1 \dots x_{n+1}$ is in M (i.e. is conjugate to a non-identity element of A), or $x_0 = x_{n+1}^{-1}$ and $x_1, \dots, x_n$ lie a conjugate of one of the factors and $x_0 x_1 \dots x_{n+1}$, and hence $x_1 \dots x_n$, is conjugate to a non-identity element of A. If $i \neq 0$ snd $j \neq n+1$, then $x_i, \dots, x_j$ lie in a conjugate of one of the factors and $x_i \dots x_j$ is conjugate to a non-identity element of A. Thus in every case (2.4) of [6] is satisfied. We have thus completed the proof that Theorem 1 of [6] is a special case of the Theorem above.

II. H.N.N. *extensions.*

Suppose that G is an H.N.N. group with base B and associated pair K_1, K_{-1}, then equivalent elements of N lie in the same conjugate of the base and elements of M are conjugates of the associated subgroups. Therefore Corollary 1 gives

H. NEUMANN'S THEOREM ([3]). *If G is an* H.N.N. *group and H is a subgroup of G which intersects all conjugates of the associated pair trivially then H is a free product of a free group and conjugates of subgroups of the base of G.*

If X is a minimal subset and $x_0 x_1 \ldots x_{n+1}$, $x_i \in X^{\pm 1}$, is a sink with no subword equal to the identity, then by applying the arguments above we get that there is a subword $x_i \ldots x_j$ such that $x_i, \ldots, x_j$ all lie in the same conjugate of the base B and $x_i \ldots x_j$ is conjugate to an element of K_1 or K_{-1}. Thus we have the main result of Peczynski and Reiwer [5].

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