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ON h-BASES FOR n

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1. Introduction.

The sum $A+B$ of two non-empty integer sequences A, B is defined to be the sequence of all distinct integers of the form

$$a+b; \quad a \in A,\ b \in B .$$

The sum of more than two sequences is defined similarly. In particular, for a positive integer h, we write hA for the h-fold sum $A+A+\ldots+A$.

We shall be concerned with finite integer sequences

$$B_k : 0 = b_0 < b_1 < \ldots < b_k ,$$

and their duals

$$B_k^* : 0 = b_0^* < b_1^* < \ldots < b_k^* ,$$

where

$$b_i^* = b_k - b_{k-i}, \quad i = 0, 1, \ldots, k .$$

Note that $\gcd B_k = \gcd B_k^*$ $(k \geqq 1)$.

If an integer M has an integral representation

$$M = b_1 x_1 + b_2 x_2 + \ldots + b_k x_k, \quad x_i \geqq 0 ,$$

we shall say that M is *dependent* on B_k. If $\gcd B_k = 1$, it is well known that every sufficiently large integer is dependent on B_k. In this case we denote the largest integer *not* dependent on B_k, the *Frobenius number* of B_k, by $g(B_k)$ or by $g(b_1, b_2, \ldots, b_k)$.

For integers a, b we use $[a, b]$ to denote the set of integers in the interval $a \leqq x \leqq b$. We also use $[x]$ to denote the integral part of a real number x.

An integer sequence

(1.1) $$A_k : \cdot 0 = a_0 < 1 = a_1 < a_2 < \ldots < a_k$$

is called an *h-basis* for a non-negative integer n if $[0, n] \subseteq hA_k$ (Rohrbach [15]). In this paper we consider the *h-range* $n(h, A_k)$ of A_k ("die Reichweite von A_k

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bezüglich h"), which is the largest n for which A_k is an h-basis. Thus A_k is an h-basis for n if and only if $0 \leqq n \leqq n(h, A_k)$.

In the literature, the h-range has been considered from two different points of view, named the *local* and the *global* case by Selmer [18]. In the local case, h and A_k are considered as given, and the problem consists in determining $n(h, A_k)$. In the global case, h and k are considered as given, and the problem consists in determining the *extremal* h-range

$$n(h,k) = \max_{A_k} n(h, A_k) ,$$

and also the corresponding *extremal bases*, i.e. the bases A_k for which $n(h,k) = n(h, A_k)$. In this paper we shall mainly be concerned with the local problem.

Given the sequence (1.1), $k \geqq 1$, we write A_{k-1} for the sequence

$$A_{k-1} : 0 = a_0 < a_1 < \ldots < a_{k-1} .$$

We define $h_0 = h_0(A_k)$ to be the smallest positive h for which

$$a_k \leqq n(h, A_k) ,$$

or, equivalently, as the smallest positive h for which

$$a_k \leqq n(h, A_{k-1}) + 1 .$$

Putting $n(0, A_k) = 0$, we then have $n(h, A_k) = n(h, A_{k-1})$ if $0 \leqq h < h_0$.

We trivially have $n(h, A_1) = h$. Thus $h_0(A_2) = a_2 - 1$. Stöhr [20] showed that

$$n(h, A_2) = a_2(h+3-a_2)-2, \quad h \geqq h_0 - 1 , \tag{1.2}$$

from which it also follows that

$$h_0(A_3) = a_2 + \left[\frac{a_3}{a_2}\right] - 2 . \tag{1.3}$$

Meures [11] was the first to discover that there is a connection between the h-range and the Frobenius number: Given A_k, if h is sufficiently large, then

$$n(h, A_k) = a_k h - g(A_k^*) - 1 . \tag{1.4}$$

Let $h_1 = h_1(A_k)$ be the smallest $h \geqq h_0 - 1$ for which (1.4) is valid. Then (1.4) is true for all $h \geqq h_1$. (For details, see Section 2.) In particular we have $h_1(A_1) = h_0(A_1) - 1 = 0$, and since

$$g(A_2^*) = a_2^2 - 3a_2 + 1 ,$$

we also have, by (1.2), that $h_1(A_2) = h_0(A_2) - 1$.

Hofmeister [5] introduced a special type of h-range called *regular*. An integral representation

(1.5) $$M = a_1r_1 + a_2r_2 + \ldots + a_kr_k, \quad r_i \geqq 0\,,$$

is *regular* if

$$\sum_{i=1}^{j} a_ir_i < a_{j+1}, \quad j = 1, 2, \ldots, k-1\,.$$

Thus, to represent M regularly, a_k is used a maximal number of times, then a_{k-1} is used a maximal number of times, and so on.

Now, the regular h-range of A_k is defined as the largest n for which all integers M, $0 \leqq M \leqq n$, have a regular representation (1.5) with $r_1 + r_2 + \ldots + r_k \leqq h$. The regular h-range of A_k was completely determined by Hofmeister [5, Satz 1].

For each positive integer M, let $\Lambda(M)$ denote the least number of elements of A_k with sum M. Also, put $\Lambda(0) = 0$. Then $M \in hA_k$ if and only if $\Lambda(M) \leqq h$.

If, for each $M \geqq 0$, we have

$$\Lambda(M) = \sum_{i=1}^{k} r_i\,,$$

where the r_i are those appearing in the regular representation (1.5) of M, then the basis A_k is called *pleasant* ("angenehm").

For certain sequences A_k, the h-range equals the regular h-range. In particular, this is so if A_k is pleasant. In this case we have $h_1(A_k) = h_0(A_k) - 1$ (Meures [11]). Some sufficient, but very restrictive, conditions for A_k to be pleasant have been given by Zöllner [23], Hofmeister [7], [8], and Djawadi [3].

In particular, put $a_3 = qa_2 - s$, $0 \leqq s < a_2$. Then A_3 is pleasant if and only if $s < q$ (Djawadi [3]), and Hofmeister's result on the regular h-range gives us

(1.6) $$n(h, A_3) = a_3(h + 1 - h_0) + a_2\left[\frac{a_3}{a_2}\right] - 2, \quad h \geqq h_0 - 1\,.$$

This result also contains some of the special results on $n(h_0, A_3)$ given by Salié [16].

In the case where Djawadi's condition $s < q$ is *not* satisfied, and algorithm for the computation of $n(h, A_3)$ has been given by Windecker [22]. From these results it follows that

$$n(h+1, A_3) = a_3 + n(h, A_3) \quad \text{for all } h \geqq h_0\,,$$

which is equivalent to $h_1(A_3) \leqq h_0(A_3)$. Using (1.4) and the result on the Frobenius number given in [13] (and also by Siering [19]), we get other algorithms for $n(h, A_3)$, which are simpler to apply than that of Windecker.

Unfortunately, in spite of several missing details, Windecker's proof of his algorithm is very long; it is also rather difficult to follow. In this paper we give a shorter and simpler deduction of the main facts about $n(h, A_3)$.

Most of the previous authors on this subject have been concerned with the global problem. Apart from some tabulated values of $n(h, k)$ for small h and k (see Mossige [12] for some results and further references), the exact value of $n(h, k)$, however, is known only for $k=1$ (trivial), $k=2$, $k=3$, and for $h=1$ (trivial).

Stöhr [20] showed that

$$n(h, 2) = \left[\frac{h^2+6h+1}{4}\right],$$

and that the corresponding extremal bases are given by $0, 1, (h+3)/2$ if h is odd, and by $0, 1, (h+3\pm 1)/2$ if h is even. (This is an easy consequence of (1.2).)

Hofmeister [6] solved the global problem for $k=3$ and h greater than some effectively computable constant. Hertsch [4] showed that Hofmeister's results are valid for all $h \geqq 500$. Recently, Hofmeister [9] showed that his results are valid for all $h \geqq 200$, and using the Univac 1110 at the University of Bergen, Mossige [12] showed that Hofmeister's results are also valid if $23 \leqq h < 200$.

Using Theorem 1′ of this paper, we can show that Hofmeister's results are true for all $h \geqq 96$. In our proof, the v defined in Section 3 plays the role of Hofmeister's "s-Stelle". But apart from this difference and some simplifications, our proof and that of Hofmeister [6], [9] (giving $h \geqq 200$) are rather similar. In this paper, therefore we are content with giving a lower bound for $n(h, 3)$, which is an easy consequence of Theorem 1′.

2. The connection with the Frobenius number.

Let $N_l = N_l(h, A_k)$ be the smallest non-negative integer which is $\equiv l \pmod{a_k}$ and does *not* belong to hA_k. Then

$$n(h, A_k) = \min_{l \in L} N_l - 1, \tag{2.1}$$

where L is some complete residue system modulo a_k.

Recalling the definition of Λ given in Section 1, we have $\Lambda(N_l) \geqq h+1$. On the other hand, if $N_l \geqq a_k$, then $-a_k + N_l \in hA_k$, so that $\Lambda(N_l) \leqq h+1$. If $N_l < a_k$, then $h < h_0$, and $\Lambda(N_l) \leqq h_0$. Thus

$$\Lambda(N_l) = h+1 \quad \text{if } h \geqq h_0 - 1. \tag{2.2}$$

Let $t_l^* = t_l(A_k^*)$ be the smallest integer which is $\equiv l \pmod{a_k}$ and dependent on A_k^*. Then t_l^* has an integral representation

$$t_l^* = \sum_{i=1}^{k-1} a_{k-i}^* x_i^{(l)}, \quad x_i^{(l)} \geqq 0 . \tag{2.3}$$

By a lemma of Brauer and Shockley [2], we also have

$$g(A_k^*) = -a_k + \max_{l \in L} t_l^* . \tag{2.4}$$

Now, let x_i be non-negative integers such that

$$N_l = \sum_{i=1}^{k} a_i x_i, \quad \Lambda(N_l) = \sum_{i=1}^{k} x_i .$$

Then, using (2.2), we get

$$N_l = a_k(h+1) - \sum_{i=1}^{k-1} a_{k-i}^* x_i, \quad h \geqq h_0 - 1 . \tag{2.5}$$

Since

$$\sum_{i=1}^{k-1} a_{k-i}^* x_i \equiv -N_l \equiv -l \pmod{a_k} ,$$

we have, by the definition of t_l^*,

$$\sum_{i=1}^{k-1} a_{k-i}^* x_i = a_k w + t_{-l}^*, \quad w \geqq 0 .$$

Hence, by (2.5),

$$N_l = a_k(h+1-w) - t_{-l}^*, \quad w \geqq 0 ,$$

and, by (2.1) and (2.4),

$$n(h, A_k) \leqq a_k h - g(A_k^*) - 1, \quad h \geqq h_0 - 1 . \tag{2.6}$$

On the other hand, for some integer u we have

$$N_l = a_k u - t_{-l}^* ,$$

so that, by (2.3),

$$N_l = a_k\left(u - \sum_{i=1}^{k-1} x_i^{(-l)}\right) + \sum_{i=1}^{k-1} a_i x_i^{(-l)} .$$

Hence, if

$$N_l = \sum_{i=1}^{k-1} a_i x_i^{(-l)} ,$$

then

$$h+1 = \Lambda(N_l) \leqq u ,$$

and

$$N_l \geqq a_k(h+1)-t^*_{-l} .$$

By (2.1), (2.4), and (2.6), we now get

LEMMA 1. *Given A_k and $h \geqq h_0 - 1$. For each non-negative integer M, satisfying*

$$M \leqq -a_k + \sum_{i=1}^{k-1} a_i x_i^{(-M)} ,$$

suppose that $M \in hA_k$. Then

$$n(h, A_k) = a_k h - g(A_k^*) - 1 . \tag{2.7}$$

The $x_i^{(-M)}$ depend only on the residue of M modulo a_k. Hence, if h is sufficiently large, then Meure's formula (2.7) is valid. Thus there is a smallest $h \geqq h_0 - 1$ for which (2.7) is true. We denote this smallest h by $h_1 = h_1(A_k)$.

By (2.2), we have

$$N_l(h+i, A_k) \geqq a_k i + N_l(h, A_k), \quad i \geqq 0,\ h \geqq h_0 - 1 .$$

Hence, by (2.1),

$$n(h+i, A_k) \geqq a_k i + n(h, A_k), \quad i \geqq 0,\ h \geqq h_0 - 1 . \tag{2.8}$$

Now, if $h = h_1 + i$, $i \geqq 0$, then

$$n(h, A_k) \geqq a_k i + n(h_1, A_k) = a_k h - g(A_k^*) - 1 .$$

In combination with (2.6) this gives us

PROPOSITION 1 (Meures [11]). *Given A_k, then* (2.7) *is valid for all $h \geqq h_1$.*

As mentioned in the Introduction, if $k \geqq 3$, then $h_1 \leqq h_0$. However, by an example we now show that if $k > 3$, then the situation is rather different.

We may alternatively describe h_1 as the smallest $h \geqq h_0 - 1$ for which

$$[0, a_k h - g(A_k^*) - 1] \subseteqq hA_k .$$

Since $M \in hA_k$ if and only if $a_k h - M \in hA_k^*$ (Meures [11]), we may dually characterize h_1 as the smallest $h \geqq h_0 - 1$ for which

$$[g(A_k^*)+1, a_k h] \subseteqq hA_k^* .$$

In particular, if $a_1^* = 1$, then $g(A_k^*) = -1$, so that

$$h_1 \geqq a_2^* - 1 = a_k - a_{k-2} - 1 \quad \text{if } h_0 \geqq 2 . \tag{2.9}$$

Now, given $h_0 \geqq 2$, $k \geqq 3$, take A_k to be the sequence

$$0, 1, 2, \ldots, k-2, (k-2)h_0+1, (k-2)h_0+2 .$$

Then $h_0 = h_0(A_k)$, and by (2.9),

$$h_1 - h_0 \ \geqq \ (k-3)(h_0-1) .$$

(This relation is, in fact, valid with equality.)

Hence, *for each $k \geqq 4$ there exist sequences A_k for which the difference $h_1 - h_0$ is greater than any given integer.*

We shall, however, give some upper bounds for h_1 in terms of h_0 and A_k. For this purpose, we require the lemma below.

LEMMA 2. *Given A_k and $h' \geqq h_0 - 1$. Then the following three statements are equivalent*:

(i) $n(h+1, A_k) = a_k + n(h, A_k)$ *for all* $h \geqq h'$;
(ii) $\Lambda(a_k + n(h, A_k) + 1) \ = \ h+2$ *for all* $h \geqq h'$;
(iii) $h' \geqq h_1$.

PROOF. Since

$$\Lambda(n(h+1, A_k)+1) \ = \ h+2 ,$$

(i) implies (ii).

Assuming (ii), we get

$$a_k i + n(h', A_k) \ \geqq \ n(h'+i, A_k) \qquad \text{for all } i \geqq 0 .$$

Hence, if i is sufficiently large, then, by Lemma 1,

$$a_k i + n(h', A_k) \ \geqq \ a_k(h'+i) - g(A_k^*) - 1 ,$$

and, by (2.6), (2.7) is satisfied for $h = h'$. Thus (iii) is true.

Finally, (i) is an obvious consequence of (iii).

In particular, if A_k is pleasant, then

$$\Lambda(a_k + M) \ = \ 1 + \Lambda(M) \qquad \text{for all } M \geqq 0 .$$

Hence the statement (ii) of Lemma 2 is satisfied for $h' = h_0 - 1$. Thus $h_1 = h_0 - 1$, as mentioned in the Introduction.

Since A_2 is always pleasant, this gives us

$$n(h, A_2) \ = \ a_2(h+1-h_0) + n(h_0-1, A_1), \qquad h \geqq h_0 - 1;$$

that is (1.2). (It is, of course, possible to give a much simpler direct proof of (1.2).)

Next, suppose that $h_0 - 1 \leqq h < h_1$. By (2.6) and the definition of h_1, we have

$$a_k + n(h_1 - 1, A_k) + 1 \in h_1 A_k \ . \tag{2.10}$$

Since $n(h_1 - 1, A_k) + 1 \notin (h_1 - 1)A_k$, the left hand side of (2.10) does, in fact, belong to $h_1 A_{k-1}$, so that

$$a_k + n(h_1 - 1, A_k) + 1 \ \leqq \ a_{k-1} h_1 \ .$$

Now, using (2.8), we get

$$a_k(h_1 - h) + n(h, A_k) + 1 \ \leqq \ a_{k-1} h_1 \ ,$$

so that

$$n(h, A_k) \ \leqq \ a_{k-1}(h+1) - a_k - 1 \ .$$

Thus we have, as discovered independently by Selmer [18],

PROPOSITION 2. *Given A_k; if $h \geqq h_0 - 1$, and*

$$n(h, A_k) \ \geqq \ a_{k-1}(h+1) - a_k \ ,$$

then $h \geqq h_1$.

Putting $n_0 = n(h_0 - 1, A_{k-1})$, we have the

COROLLARY 1 (Meures [11]). *Given A_k, then $h_1 = h_0 - 1$, or*

$$h_1 \ \leqq \ \left\lceil \frac{a_k(h_0 - 1) - n_0 - 1}{a_k - a_{k-1}} \right\rceil .$$

PROOF. If $h \geqq h_0 - 1$, then, by (2.8),

$$n(h, A_k) \ \geqq \ a_k(h - h_0 + 1) + n_0 \ .$$

Hence, if

$$a_k(h - h_0 + 1) + n_0 \ \geqq \ a_{k-1}(h+1) - a_k \ , \tag{2.11}$$

then, by Proposition 2, we have $h \geqq h_1$.

Since, (2.11) is equivalent to

$$h \ \geqq \ \left\lceil \frac{a_k(h_0 - 1) - n_0 - 1}{a_k - a_{k-1}} \right\rceil ,$$

the result follows.

Using the trivial bound $n_0 \geqq h_0 - 1$, Corollary 1 gives us

$$h_1 \leqq \max\left\{h_0-1, \left[\frac{(a_k-1)(h_0-1)-1}{a_k-a_{k-1}}\right]\right\}.$$

It is easily seen that the second argument dominates for $h_0 \geqq 2$, $k \geqq 3$. Hence

$$h_1 \leqq \left[\frac{(a_k-1)(h_0-1)-1}{a_k-a_{k-1}}\right], \quad h_0 \geqq 2,\ k \geqq 3 . \tag{2.12}$$

From the trivial bound

$$h_0 \leqq \max_{1 \leqq i < k} \{a_{i+1}-a_i\}, \quad k \geqq 2 , \tag{2.13}$$

it follows that $h_0 \leqq a_k - k + 1$. Thus, by (2.12), we also have

$$h_1 \leqq \left[\frac{(a_k-1)(a_k-k)-1}{a_k-a_{k-1}}\right], \quad a_k > k > 2 .$$

This bound is usually far too large. However, it does not depend on any h-range, neither directly (the n_0 above) nor indirectly (through h_0).

Next, put

$$d_i = gcd(a_i, \ldots, a_k), \quad i = 1, 2, \ldots, k-1;$$

$$d_k = a_k, \quad d_{k+1} = 0;$$

and let

$$\beta_k = \sum_{i=1}^{k} a_i\left(\frac{d_{i+1}}{d_i} - 1\right).$$

Then, if $M > \beta_k$, we have by the theorem of Weidner [21],

$$\Lambda(a_k + M) = 1 + \Lambda(M) .$$

Hence, by Lemma 2, we have the

PROPOSITION 3. *Given A_k; if $h \geqq h_0 - 1$ and $n(h, A_k) \geqq \beta_k$, then $h \geqq h_1$.*

Since $g(A_k) \leqq \beta_k$ (Brauer [1]), we have $-1 \leqq \beta_k$, so that

$$h_0 - 1 \leqq h_0 + \left[\frac{\beta_k - h_0}{a_k}\right], \quad k \geqq 2 .$$

In combination with (2.8), Proposition 3 thus gives us

COROLLARY 2. *Given A_k, $k \geqq 2$, then*

$$h_1 \leqq h_0 + \left[\frac{\beta_k - h_0}{a_k}\right].$$

Moreover, we have

$$\beta_k \leqq \sum_{i=1}^{k-1} a_{k-1}(d_{i+1}-d_i)-a_k \;=\; a_{k-1}a_k-a_{k-1}-a_k$$

(with equality if $d_{k-1}=1$), so that Corollary 2 gives us

COROLLARY 3. *Given* A_k, $k \geqq 2$, *then* $h_1 \leqq h_0+a_{k-1}-2$.

Finally, in addition to the trivial bound (2.13), we now prove the

PROPOSITION 4. *Given* A_k, $k \geqq 2$, *we have*

$$h_0 \leqq 1+\max_{1 \leqq i<k}\left\lceil \frac{a_{i+1}-2}{i} \right\rceil . \tag{2.14}$$

PROOF. Let $\alpha_h(M)$ denote the number of positive integers in hA_k not exceeding M. Then

$$\alpha_1(M) \;=\; i \quad \text{if } a_i \leqq M < a_{i+1}, \quad i=1,2,\ldots,k-1 .$$

Denote the right hand side of (2.14) by h'. Then

$$h' \cdot \alpha_1(M) \geqq M \quad \text{for } M=1,2,\ldots,a_k-1 ,$$

and as an easy consequence of "the counting number form" of the $(\alpha+\beta)$-theorem of H. B. Mann (or, directly from Dyson's theorem; see Mann [10, Chap. 3]), it follows that $\alpha_{h'}(a_k-1)=a_k-1$; that is $[0,a_k-1] \subseteqq h'A_k$. Hence $h_0 \leqq h'$.

REMARK. Some of the results given in this section may also be extended to sequences $A_k : 0=a_0<a_1<\ldots<a_k$, where a_1 is an *arbitrary* positive integer, and $\gcd A_k=1$.

In this case we assume h to be so large that $g(A_k)+1 \in hA_k$. We then define $n(h,A_k)$ as the largest n for which

$$[g(A_k)+1,n] \subseteqq hA_k .$$

If h is sufficiently large, then (2.7) is valid also in this case. It is also possible to prove that

$$|\overline{hA_k}| \;=\; n(A_k)+n(A_k^*), \qquad h \text{ large} ,$$

where $|\overline{hA_k}|$ denotes the number of integers in the relative complement of hA_k in $[0,a_kh]$, and $n(A_k)$ (not to be confused with the h-range of A_k) is the number of non-negative integers not dependent on A_k. (For some results on $n(A_k)$ and further references, see Selmer [17] and Rödseth [13], [14].)

In particular, it is a simple matter to show that

$$|\overline{hA_2}| = n(A_2)+n(A_2^*) = \tfrac{1}{2}(a_2-1)(a_2-2), \qquad h \geqq a_2-2\,,$$

where the abscence of a_1 is easily explained by considering the mapping $a_1x+a_2y \to x+a_2y$, $0 \leqq x < a_2$.

Let us again assume that $a_1=1$, and let $h_2=h_2(A_k)$ be the smallest $h \geqq h_0-1$ satisfying

$$|\overline{hA_k}| = n(A_k^*)\,. \tag{2.15}$$

Then (2.15) is true for all $h \geqq h_2$. We have $h_2 \geqq h_1$. However, Propositions 2 and 3 with their consequences remain valid when h_1 is replaced by h_2. It is also possible to prove, using the results of the following two sections, that $h_2(A_3) \leqq h_0(A_3)$. (For the value of $n(A_3^*)$, see Rödseth [13, Th. 2].)

3. Preliminaries on $k=3$.

We now consider the sequence A_3: $0=a_0<1=a_1<a_2<a_3$. Putting $a_3=s_{-1}$, $a_2=s_0$, we shall use the Euclidean algorithm in the form (cf. [13])

$$\begin{aligned}
s_{-1} &= q_1s_0-s_1, && 0 \leqq s_1 < s_0\\
s_0 &= q_2s_1-s_2, && 0 \leqq s_2 < s_1\\
s_1 &= q_3s_2-s_3, && 0 \leqq s_3 < s_2\\
&\dots\\
s_{m-2} &= q_ms_{m-1}-s_m, && 0 \leqq s_m < s_{m-1}\\
s_{m-1} &= q_{m+1}s_m, && 0 = s_{m+1} < s_m\,.
\end{aligned}$$

We also recursively define integers P_i, Q_i, R_i for $i=-1,\dots,m+1$, by

$$P_{i+1} = q_{i+1}P_i-P_{i-1}, \quad P_0 = 1,\ P_{-1} = 0 \tag{3.1}$$

$$Q_{i+1} = q_{i+1}Q_i-Q_{i-1}, \quad Q_0 = 0,\ Q_{-1} = -1 \tag{3.2}$$

$$R_{i+1} = q_{i+1}R_i-R_{i-1}, \quad R_0 = a_2-1,\ R_{-1} = a_3-1\,.$$

Now,

$$\frac{a_3}{a_2} = q_1 + \cfrac{-1}{q_2 + \cfrac{-1}{q_3 + \cfrac{\cdot}{\cdot \atop {\cdot \atop + \frac{-1}{q_{m+1}}}}}} = q_1 + \frac{-1}{q_2} + \ldots + \frac{-1}{q_{m+1}},$$

where the ith convergent is given by

$$q_1 + \frac{-1}{q_2} + \ldots + \frac{-1}{q_i} = \frac{P_i}{Q_i}, \quad i = 1, \ldots, m+1,$$

and where $\gcd(P_i, Q_i) = 1$, because of the relation

$$P_i Q_{i+1} - P_{i+1} Q_i = 1, \quad i = -1, \ldots, m.$$

In particular, we thus have $P_{m+1} = a_3/s_m$, $Q_{m+1} = a_2/s_m$.

For later references, we also list the following easily proved formulae:

(3.3) $$a_3 Q_i = a_2 P_i - s_i$$

(3.4) $$a_3 R_i = (a_3 - 1)s_i - (a_3 - a_2)P_i$$

(3.5) $$s_i Q_{i+1} - s_{i+1} Q_i = a_2 .$$

Since $q_i \geqq 2$, we have $P_i < P_{i+1}$, $Q_i < Q_{i+1}$, and $R_{i+1} < R_i$. We also have

(3.6) $$R_i = Q_i - P_i + s_i ,$$

and

(3.7) $$-\frac{1}{s_m}(a_3 - a_2) = R_{m+1} < \ldots < R_0 = a_2 - 1 .$$

Hence there is a unique integer $v = v(A_3)$, $0 \leqq v \leqq m$, satisfying

(3.8) $$R_{v+1} \leqq 0 < R_v .$$

For $-1 \leqq i \leqq m$, we define subsets X_i, Y_i of the fundamental point lattice by

$$X_i = \{(x,y) \mid 0 \leqq x < s_i - s_{i+1}, \quad 0 \leqq y < P_{i+1}\}$$

$$Y_i = \{(x,y) \mid 0 \leqq x < s_i, \quad 0 \leqq y < P_{i+1} - P_i\} .$$

We shall say that two lattice points (x,y) and (x',y') are *congruent* if

$$x + a_2 y \equiv x' + a_2 y' \pmod{a_3} .$$

It is easily seen that $X_i \cup Y_i$ contains just a_3 elements. We continue to show that these a_3 elements are incongruent.

LEMMA 3. *If* $(x,y) \in X_{i-1} \cup Y_{i-1}$, $0 \leqq i \leqq m$, *then the lattice point*

$$(3.9) \qquad (x',y') = \left(x - s_i\left[\frac{x}{s_i}\right], y + P_i\left[\frac{x}{s_i}\right]\right)$$

belongs to $X_i \cup Y_i$ *and is congruent to* (x,y).

PROOF. By (3.3), we have

$$x' + a_2 y' = x + a_2 y + a_3 Q_i\left[\frac{x}{s_i}\right].$$

Hence (x',y') is congruent to (x,y).

We now show that $(x',y') \in X_i \cup Y_i$. Clearly, $0 \leqq x' < s_i$. If $(x,y) \in X_{i-1}$, then

$$y' < P_i + P_i\left[\frac{s_{i-1} - s_i - 1}{s_i}\right] = P_{i+1} - (P_i - P_{i-1}) < P_{i+1},$$

since $[(s_{i-1} - 1)/s_i] = q_{i+1} - 1$.

If $(x,y) \in Y_{i-1}$, then

$$y' < P_i - P_{i-1} + P_i\left[\frac{s_{i-1} - 1}{s_i}\right] = P_{i+1}.$$

Thus we have

$$(3.10) \qquad 0 \leqq x' < s_i \quad \text{and} \quad 0 \leqq y' < P_{i+1}.$$

Let us now assume that $x' \geqq s_i - s_{i+1}$ and $y' \geqq P_{i+1} - P_i$. If $(x,y) \in X_{i-1}$, then

$$P_{i+1} - P_i \leqq y + P_i\left[\frac{x}{s_i}\right] < P_i + P_i\left[\frac{x}{s_i}\right],$$

so that $q_{i+1} - 2 \leqq [x/s_i]$, which gives us

$$x = x' + s_i\left[\frac{x}{s_i}\right] \geqq s_i - s_{i+1} + s_i(q_{i+1} - 2) = s_{i-1} - s_i;$$

a contradiction.

If $(x,y) \in Y_{i-1}$, then

$$P_{i+1} - P_i \leqq y + P_i\left[\frac{x}{s_i}\right] < P_i - P_{i-1} + P_i\left[\frac{x}{s_i}\right],$$

so that $q_{i+1}-1\leqq[x/s_i]$, which gives us

$$x = x'+s_i\left[\frac{x}{s_i}\right] \geqq s_i-s_{i+1}+s_i(q_{i+1}-1) = s_{i-1},$$

and again we have reached a contradiction.

Hence we have $x'<s_i-s_{i+1}$ or $y'<P_{i+1}-P_i$, which, in combination with (3.10), shows that $(x',y')\in X_i\cup Y_i$.

Because of Lemma 3, for each $i=0,\ldots,m$, we may define a function

$$\varphi\colon X_{i-1}\cup Y_{i-1}\to X_i\cup Y_i$$

by putting $\varphi(x,y)=(x',y')$, for (x',y') given by (3.9).

If $(x,y)\in X_{i-1}\cup Y_{i-1}$, then $[x/s_i]=[y'/P_i]$. Hence φ has an inverse

$$\varphi^{-1}: X_i\cup Y_i\to X_{i-1}\cup Y_{i-1}$$

given by

$$\varphi^{-1}(x',y') = \left(x'+s_i\left[\frac{y'}{P_i}\right], y'-P_i\left[\frac{y'}{P_i}\right]\right).$$

Thus φ is a *bijection*, and, by Lemma 3, φ also has the property that if $(x,y)\in X_{i-1}\cup Y_{i-1}$, then (x,y) and $\varphi(x,y)$ are congruent lattice points.

Since

$$X_{-1}\cup Y_{-1} = \{(x,0)\mid 0\leqq x<a_3\},$$

it follows that $X_i\cup Y_i$ consists of a_3 incongruent lattice points. Thus the set

$$\{x+a_2y\mid (x,y)\in X_i\cup Y_i\}$$

forms a complete residue system modulo a_3 for each $i=-1,\ldots,m$.

Now fix $r, 0\leqq r<a_3$. Let (x_i,y_i) be the unique lattice point in $X_i\cup Y_i$ which is congruent to $(r,0)$, $i=-1,\ldots,m$. Then

$$x_i+a_2y_i = x_{i-1}+a_2y_{i-1}+a_3Q_i\left[\frac{x_{i-1}}{s_i}\right], \quad i\geqq 0. \tag{3.11}$$

Recalling the definition of $t_l^*=t_l(A_3^*)$ given in Section 2, we now prove

LEMMA 4. *We have* $t^*_{-r}=(a_3-1)x_v+(a_3-a_2)y_v$.

PROOF. A more general result is proved in [13]. However, for the convenience of the reader, we include a proof of this lemma.

By the definition of t^*_{-r}, there are non-negative integers x, y such that

$$t^*_{-r} = (a_3-1)x+(a_3-a_2)y\ . \tag{3.12}$$

We choose such a pair (x,y) for which y is minimal.

By (3.4), we have

$$t^*_{-r}-a_3R_v = (a_3-1)(x-s_v)+(a_3-a_2)(y+P_v)\ .$$

Since R_v is a positive integer, and t^*_{-r} is the smallest integer $\equiv -r \pmod{a_3}$ with a representation (3.12), we have $x<s_v$.

Similarly,

$$t^*_{-r}+a_3R_{v+1} = (a_3-1)(x+s_{v+1})+(a_3-a_2)(y-P_{v+1})\ ,$$

and if $R_{v+1}<0$, then $y<P_{v+1}$. Also in the case $R_{v+1}=0$, we have $y<P_{v+1}$, because of the minimality of y.

Finally,

$$t^*_{-r}-a_3(R_v-R_{v+1}) = (a_3-1)(x-s_v+s_{v+1})+(a_3-a_2)(y+P_v-P_{v+1})\ ,$$

so that $x<s_v-s_{v+1}$ or $y<P_{v+1}-P_v$.

Hence $(x,y)\in X_v\cup Y_v$. Since $x+a_2y\equiv -t^*_{-r}\equiv r \pmod{a_3}$, we thus have $(x,y) = (x_v, y_v)$.

For $h_0=h_0(A_3)$ given by (1.3), we now prove the

LEMMA 5. *For* $1\leqq i\leqq v$, *we have*

$$x_{i-1}+y_{i-1}+Q_i-1 \leqq h_0 \quad \text{if } P_i\leqq s_i \tag{3.13}$$

$$x_i+y_i+R_i-1 \leqq h_0 \quad \text{if } P_i>s_i\ . \tag{3.14}$$

PROOF. For $m\geqq 1$, we have $s_1>0$. Hence, by (1.3), $h_0=a_2+q_1-3$.

Put

$$\gamma_i = \max_{(x,y)\in X_i}\{x+y\} = s_i-s_{i+1}+P_{i+1}-2$$

$$\delta_i = \max_{(x,y)\in Y_i}\{x+y\} = s_i+P_{i+1}-P_i-2\ .$$

We first prove (3.13) and therefore assume that $P_i\leqq s_i$. Then $\gamma_{i-1}<\delta_{i-1}$, and it is sufficient to show that

$$\delta_{i-1}+Q_i-1 \leqq h_0\ . \tag{3.15}$$

By (3.5), we have

$$h_0-\delta_{i-1}-Q_i+1 = (Q_i-1)s_{i-1}-Q_{i-1}s_i-P_i+P_{i-1}-Q_i+q_1\ ,$$

and, since $s_{i-1} \geqq s_i + 1$,

$$h_0 - \delta_{i-1} - Q_i + 1 \geqq (Q_i - Q_{i-1} - 1)s_i - P_i + P_{i-1} + q_1 - 1 .$$

Using the assumption $P_i \leqq s_i$, we further get

$$h_0 - \delta_{i-1} + Q_i + 1 \geqq (Q_i - Q_{i-1} - 2)P_i + P_{i-1} + q_1 - 1 . \tag{3.16}$$

If $Q_i - Q_{i-1} - 2 \geqq 0$, we thus have

$$\delta_{i-1} + Q_i - 1 \leqq h_0 - q_1 .$$

If $Q_i - Q_{i-1} - 2 \leqq -1$, then $i = 1$ or $q_2 = \ldots = q_i = 2$. Hence

$$Q_j = j, \quad P_j = (q_1 - 1)j + 1 \quad \text{for } 0 \leqq j \leqq i , \tag{3.17}$$

and the right hand side of (3.16) equals 0. This completes the proof of (3.15).

In the proof of (3.15) we did not explicitly use the assumption $i \leqq v$. However, it follows from (3.6) that the conditions $i \geqq 1$ and $P_i \leqq s_i$ imply $i \leqq v$.

Next we prove (3.14) and therefore assume that $P_i > s_i$. Then $\gamma_i > \delta_i$, and it is sufficient to show that

$$\gamma_i + R_i - 1 \leqq h_0 . \tag{3.18}$$

By (3.5) and (3.6), we have

$$h_0 - \gamma_i - R_i + 1 = (Q_{i+1} - 2)s_i - (Q_i - 1)s_{i+1} - P_{i+1} - Q_i + P_i + q_1 ,$$

and, since $s_{i+1} \leqq s_i - 1$, we get

$$h_0 - \gamma_i - R_i + 1 \geqq (Q_{i+1} - Q_i - 1)s_i - P_{i+1} + P_i + q_1 - 1 .$$

Since $i \leqq v$, we have $R_i \geqq 1$ by (3.7) and (3.8), so that, by (3.6), $s_i \geqq 1 + P_i - Q_i$. Hence, using (3.1) and (3.2), we get

$$\begin{aligned} h_0 - \gamma_i - R_i + 1 \geqq\ & (q_{i+1}(Q_i - 1) - Q_i - Q_{i-1})(P_i - Q_i) + \\ & + P_{i-1} - Q_{i-1} + q_1 - 2 . \end{aligned} \tag{3.19}$$

Thus, if $q_{i+1}(Q_i - 1) - Q_i - Q_{i-1} \geqq 0$, then (3.18) is true.

Since $P_i > s_i$ and $v \geqq 1$, we have $i \geqq 2$. Hence, if $q_{i+1}(Q_i - 1) - Q_i - Q_{i-1} \leqq -1$, then $q_2 = \ldots = q_{i+1} = 2$. Thus, by (3.17), the right hand side of (3.19) equals 0, and (3.18) is true also in this case.

4. Determination of $n(h, A_3)$.

We now prove that if $k = 3$ and $h \geqq h_0$, then the hypotheses of Lemma 1 are satisfied. In the notation of the preceding section, by Lemma 4, we then have to show that for each r, $0 \leqq r \leqq a_3$, the sequence

$$r < r+a_3 < r+2a_3 < \dots < x_v+a_2y_v-a_3 \tag{4.1}$$

is a subsequence of h_0A_3.

If $v=0$, then the sequence (4.1) is empty, and $h_1=h_0-1$. We therefore assume that $v\geqq 1$.

By (3.11), we have

$$r = x_0+a_2y_0 \leqq x_1+a_2y_1 \leqq \dots \leqq x_v+a_2y_v .$$

Given i, $1\leqq i\leqq v$, suppose that $x_{i-1}\geqq s_i$. We then show that the integer

$$M = x_{i-1}+a_2y_{i-1}+a_3z, \quad 0 \leqq z < Q_i\left[\frac{x_{i-1}}{s_i}\right],$$

belongs to h_0A_3.

Put

$$z = Q_i\left[\frac{z}{Q_i}\right]+z' .$$

By (3.3), we then have $M=x'+a_2y'+a_3z'$, where

$$x' = x_{i-1}-s_i\left[\frac{z}{Q_i}\right] \geqq 0, \quad y' = y_{i-1}+P_i\left[\frac{z}{Q_i}\right], \quad 0 \leqq z' < Q_i ,$$

and

$$x'+y'+z' \leqq x_{i-1}+y_{i-1}+(P_i-s_i)\left[\frac{z}{Q_i}\right]+Q_i-1 .$$

If $P_i\leqq s_i$, then

$$x'+y'+z' \leqq x_{i-1}+y_{i-1}+Q_i-1 .$$

Since $z<Q_i[x_{i-1}/s_i]$ and $(x_i,y_i)=\varphi(x_{i-1},y_{i-1})$, we get, using (3.6),

$$x'+y'+z' \leqq x_i+y_i+R_i-1 \quad \text{if } P_i > s_i .$$

In both cases we have, by Lemma 5, that $x'+y'+z'\leqq h_0$, as required.

THEOREM 1. *We have*

$$n(h, A_3) = a_3h-g(A_3^*)-1 \quad \textit{for all } h\geqq h_0 ,$$

where h_0 is given by (1.3).

By (2.4) and Lemma 4, this theorem may be given the more explicit form:

THEOREM 1′. *In the notation of Section* 3, *we have*

$$n(h,A_3) = a_3(h+1)-(a_3-1)(s_v-1)-(a_3-a_2)(P_{v+1}-1) + \min\{(a_3-1)s_{v+1},(a_3-a_2)P_v\}-1$$

for $h \geqq h_0$, *where* v *is determined by* (3.8).

More briefly, Theorem 1 states that $h_1(A_3) \leqq h_0(A_3)$. If $v=0$, that is if Djawadi's condition $s_1 < q_1$ is satisfied, then we know that $h_1(A_3)=h_0(A_3)-1$, and Theorem 1′ coincides with (1.6). If $v \geqq 1$, it is not difficult to see that $h_1(A_3) \neq h_0(A_3)-1$. Thus $h_1(A_3)=h_0(A_3)$ if $v \geqq 1$.

For relatively prime positive integers a, b, c, an algorithmic formula for the Frobenius number $g(a,b,c)$ was given in [13]. Using Th. 1 in [13], we then get Theorem 1′ from Theorem 1, by putting $a=a_3$, $b=a_3-1$, $c=a_3-a_2$. Similar algorithmic formulae for $n(h,A_3)$ arise by pairing a_3, a_3-1, a_3-a_2 with a, b, c, in other ways. (See Selmer [18].)

Suppose that $h \geqq 2$, and let β, γ be integers satisfying $2 \leqq \gamma \leqq \beta$, $2\beta \leqq h+2$. Put

$$a_2 = 2\beta-\gamma+1, \quad a_3 = a_2\gamma-\beta .$$

Then $q_1=\gamma$, $s_1=\beta$, $q_2=2$, $s_2=\gamma-1$, $R_1=1-\gamma+\beta \geqq 1$, $R_2=2-\gamma \leqq 0$, so that $v=1$, and Theorem 1′ gives us

$$n(h,A_3) = a_3(h+5-\beta-\gamma)-2(\beta-\gamma+2) , \tag{4.2}$$

which shows that Hilfssatz 1 of Hofmeister [6] is valid with *equality*.

Hofmeister (Satz 2) made the following choice:

$$\beta = \left[\frac{4(h+1)}{9}\right]+2, \quad \gamma = \left[\frac{2h}{9}\right]+2 .$$

Now, for $h \geqq 18$, (4.2) gives us

$$n(h,A_3) = \tfrac{4}{81}h^3+\tfrac{2}{3}h^2+\varepsilon_1 h+\varepsilon_0 ,$$

where the coefficients $\varepsilon_1, \varepsilon_0$ depend on the residue of h modulo 9.

It is now known (cf. Section 1) that this choice of A_3 gives us the *unique* extremal basis for each $h \geqq 23$.

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