MATH. SCAND. 48 (1981), 184–188

THE CROSSING NUMBERS OF SOME GENERALIZED PETERSEN GRAPHS

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Abstract.

The generalized Petersen graphs $P(2n+1,2)$ are shown to have crossing number 3 when $n \geqq 3$.

1. Introduction.

The generalized Petersen graphs $P(m,k)$ were first studied by Coxeter [3], and Bannai [1]. Their line chromatic number χ' was investigated twice: Watkins conjectured in [7], and Castagna and Prins proved in [2] that of these only the Petersen graph P itself is in class 2, that is, $\chi' = \Delta + 1 = 4$. All other $P(m,k)$ are in class 1 so that $\chi' = 3$. Thus this gives still another characterization of P; see [6].

We write uAv to indicate that points u and v are adjacent in a graph. For $m \geqq 3$ and $1 \leqq k \leqq m$ we define the *generalized Petersen graph* $P(m,k)$ as follows: its point set is $U \cup W$ where $U = \{u_1, \ldots, u_m\}$, $W = \{w_1, \ldots, w_m\}$; its lines are given by (1) $u_i A w_i$, (2) $u_i A u_{i+1}$, and (3) $w_i A w_{i+k}$, all for $i \in [1,m] = \{1, \ldots, m\}$, with addition of subscripts modulo m.

It will be useful to call the subgraph induced by U the *u-cycle*, and that induced by W the *w-cycle*. We also refer to u-lines, w-lines and uw-lines, with the expected meanings. Let the line $u_i u_{i+1}$ be denoted by e_i, $w_i w_{i+k}$ by f_i, and $u_i w_i$ by g_i. Otherwise the notation and terminology of [5] is used.

Our object is to determine the crossing numbers for the subfamily $P(m,2)$ of these graphs.

2. Crossing numbers.

As the graphs $P(m,1)$ are prisms $C_m \times K_2$, they are planar. One can easily verify that for even $m = 2n$, $P(2n,2)$ is also planar for each n. It is well known that the graph $P(3,2) = K_3 \times K_2$ is planar, and that the Petersen graph $P = P(5,2)$ has crossing number $\nu = 2$. Thus we can concentrate on the graphs $P(2n+1,2)$, $n \geqq 3$.

Received April 23, 1979; in revised form January 28, 1980.

As usual for crossing numbers we obtain an upper bound by producing a drawing in the plane. Figure 1 shows $P(7,2)$ with three crossings, a construction which clearly generalizes to $P(2n+1,2)$.

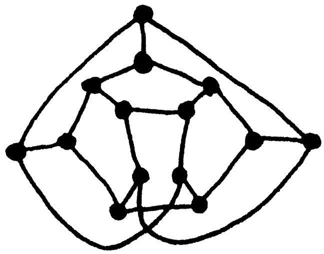

Fig. 1. $\nu(P(2n+1,2)) \leqq 3$.

LEMMA 1. *Every generalized Petersen graph* $P(2n+1,2)$ *has crossing number* $\nu \leqq 3$.

A second result is obtained by noticing that for $n \geqq 3$, $P(2n+1,2)$ contains a subdivision of $P(7,2)$ as a subgraph.

LEMMA 2. *Every generalized Petersen graph* $(P(2n+1,2)$ *has crossing number* $\nu \geqq \nu(P(7,2))$.

We can now prove the main result. Recall that a *good drawing* of a graph G is one with precisely $\nu(G)$ crossings.

THEOREM. *The crossing number of the generalized Petersen graph* $P(m,2)$ *is*

0 *when* $m=3$ *or* m *is even,*
2 *for* $m=5$, *and*
3 *if* m *is odd and* $m \geqq 7$.

PROOF. In view of the lemmas, all that remains to be shown is that $\nu=\nu(P(7,2)) \geqq 3$. Of course $\nu \geqq 2$ since $P(7,2)$ contains a subdivision of the Petersen graph. So we assume that $P(7,2)$ can be drawn with just two crossings.

Clearly any uw-line can be removed from $P(7,2)$ leaving a graph containing a subdivision of the Petersen graph, and thereby one with $\nu=2$. It follows that no uw-line is involved in a crossing if $\nu=2$.

We next show that no line can be in two crossings. As the automorphism group of $P(7,2)$ is transitive on u-lines, on w-lines, and on uw-lines, it will suffice to show that we can remove at least one line of each type and leave a nonplanar graph. In Figure 2 this is shown. The same figure shows that two consecutive u-lines are not in crossings if $v=2$. Hence if $v=2$ the u-cycle does not cross itself twice.

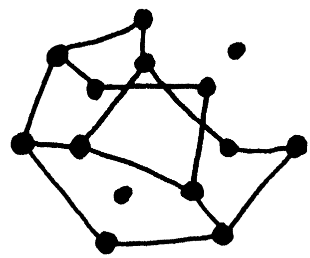

Fig. 2. A subdivision of $K_{3,3}$.

One can also remove three consective w-lines and leave a graph containg a subdivision of $K_{3,3}$, as shown in Figure 3. Thus no more than two w-lines are in crossings, and if two w-lines are in crossings, then they are at distance 3 in the line graph $L(P(7,2))$.

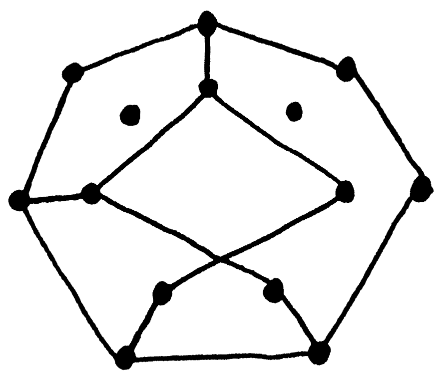

Fig. 3. A different subdivision of $K_{3,3}$.

There are now two possibilities: (1) both the u-cycle and the w-cycle cross themselves, or (2) the u-cycle crosses the w-cycle.

Case 1. Let P' be the planar graph obtained from $P(7,2)$ by including the crossing points. Then P' has 25 lines and 16 points, and is 2-connected. Euler's formula tells us that P' divides the plane into 11 regions. There are four regions bounded exclusively by u-lines or exclusively by w-lines. These four regions are bounded by a total of 18 lines. Every other region has at least 5 lines on its boundary since the girth of $P(7,2)$ is 5. We now count the pairs (e, R), where e is a line on the boundary of region R. Let C be the number of such pairs. Then each line of P' bounds two regions so $C=25\times 2=50$. But there are seven other regions since there are a total of 11 so $C\geqq 18+7\times 5=53$, from the above arguments. This contradiction dispenses with Case 1.

CASE 2. In this case we know that the u-cycle crosses the w-cycle and the w-lines in the crossings are at distance 3. So without loss of generality we can suppose that there are three w-points exterior to the u-cycle and 4-points in the interior. And if $\nu=2$ we can take the three exterior points to be w_0, w_2 and w_4.

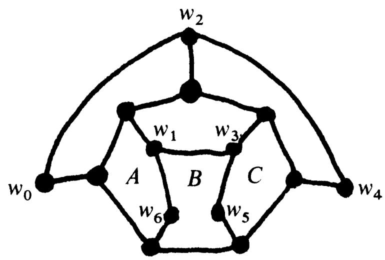

Fig. 4.

Thus we must have Figure 4, where we have labeled three regions A, B and C. Now f_5 must cross the boundary of $B \cup C$ since w_5 is in the interior of $B \cup C$. And because w_6 is interior to $A \cup B$, f_4 must cross its boundary. But as we have seen, $\nu=2$ implies that no two consecutive u-lines are crossed. So f_4 must cross e_6 and f_5 must cross e_4 in order to have $\nu=2$. But then f_4 must cross f_5 so $\nu\geqq 3$.

3. Comments and problems.

Since the graphs $P(2n+1,n)$, $P(2n+1,n+1)$ and $P(2n+1,2n-1)$ are all isomorphic to $P(2n+1,2)$, the crossing numbers of all are now established as 3 for all $n\geqq 3$. Thus the smallest unknown case is $P(8,3)$. Beyond this, $P(8,4)$ is easily seen to be a subdivision of the Möbius ladder M_8, and in fact $P(2n,n)$ is a subdivision of M_{2n}. Since Guy and Harary [4] have established the crossing number of all the Möbius ladders to be 1, we also have $\nu(P(2n,n))=1$ for $n\geqq 3$.

The determination of the crossing numbers of the remaining generalized Petersen graphs remains unsolved.

ACKNOWLEDGEMENT. We thank both referees for helpful comments which suggested the present improved proof of the theorem.

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