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THE MAXIMUM DISTANCE BETWEEN TWO-DIMENSIONAL BANACH SPACES

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Abstract.

The maximum Banach–Mazur distance between two two-dimensional Banach spaces is shown to be $\frac{3}{2}$. This is done by defining a plane figure R which is closed, bounded, convex, symmetric, and has a non-empty interior — i.e., which can be the unit ball of a Banach space structure in the plane. It is shown that any plane figure C with the same properties has an affine image C' such that $R \subseteq C' \sqrt{3/2} R$.

The author wishes to thank Professor William Davis for his help and encouragement, and the referee for his constructive suggestions. Most of this work was done while the author was with the Applied Mathematics Department of Brookhaven National Laboratory.

Let Γ be the set of subsets C of the plane R^2 which are closed, bounded, convex, and symmetric, and which have non-empty interiors. By *symmetric* we mean that $x \in C$ implies $-x \in C$; that is, C is centered on the origin and is invariant under 180° rotation.

Two elements of Γ are *equivalent* if one is the image of the other under an affine (linear) transformation. This is an equivalence relation, and we let γ be the set of equivalence classes. Following Edgar Asplund [1] we let $i\colon \Gamma \to \gamma$ be the canonocal map.

The sets in Γ are exactly those which can be the unit balls of Banach space structures on R^2, and the elements of γ are in $1-1$ correspondence with the isomorphism classes of two-dimensional Banach spaces. The results in this paper draw their significance from their Banach-space interpretation, but no knowledge of Banach spaces is necessary for their understanding.

We define a "distance" function in γ by

$$d(a,b) = \inf\{h \geqq 1 \mid \exists A \in i^{-1}(a),\ B \in i^{-1}(b), \text{ such that } A \subseteqq B \subseteqq hA\}\ .$$

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In other words, to determine the distance between a and b, we choose representatives A and B such that B contains A but fits as closely as possible. The relation $A \subseteqq B \subseteqq hA$ expresses the closeness of fit, and the minimized value of h is the "distance". This is not a conventional metric because it has the property that $d(a,b) \geqq 1$ with equality only when $a=b$. The "triangle inequality", which is easy to verify, is multiplicative:

$$d(a,c) \leqq d(a,b)d(b,c) .$$

One could obtain a conventional distance function be replacing this one by its logarithm.

In the Banach-space context this distance function is called the Banach–Mazur distance, and is usually defined by

$$d(X,Y) = \inf_T \|T\| \, \|T^{-1}\|$$

where the infimum is taken over all linear transformations $T: X \to Y$.

As examples we may take the class $p \in \gamma$ consisting of all parallelograms in Γ, and the class $h \in \gamma$ which includes regular hexagons. It is known that $d(p,h) = \frac{3}{2}$; this is illustrated in Figure 1. Asplund conjectured that $d(a,b) < \frac{3}{2}$ in all other cases, hence that $\frac{3}{2}$ is the maximum distance between elements of γ. He proved that $d(p,c) < \frac{3}{2}$ when $c \neq h$, and that $d(c,h) < \frac{3}{2}$ when $c \neq p$. Here we will derive Asplund's conjecture as a consequence of a stronger result.

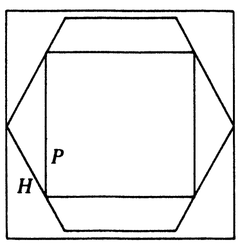

Fig. 1. $P \subseteqq H \subseteqq \frac{3}{2}P$.

THEOREM. *There exists a class $r \in \gamma$ such that for any $c \in \gamma$, $d(c,r) \leqq \sqrt{3/2}$, with equality only when $c=p$ or $c=h$.*

We define $r = i(R)$, where R is the closed convex set bounded by the lines $y=+1$ and $y=-1$, the circle $x^2+y^2=2$, and the ellipse $\frac{1}{2}x^2+y^2=\frac{4}{3}$. (See Figure 2.)

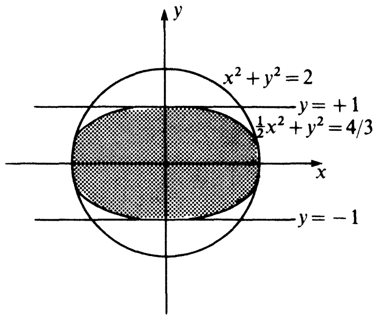

Fig. 2. Definition of R.

The relationships $d(p,r)=\sqrt{3/2}$ and $d(h,r)=\sqrt{3/2}$ are illustrated in Figure 3. The class r is a "center" for the metric space γ, but it is not the only one. From its existence we may conclude that an infinity of shapes exist that could play the role of r in the theorem. Some can be obtained by making small modifications in R or by passing to the dual.

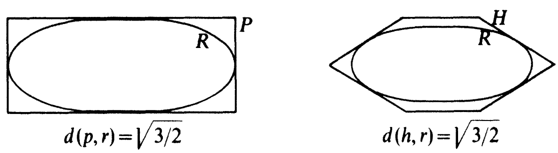

Fig. 3.

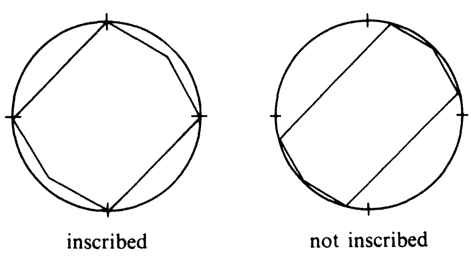

Fig. 3a. Definition of "inscribed".

COROLLARY. *For any $a,b\in\gamma$, $d(a,b)\leqq 3/2$, with equality only when $a,b=p,h$.*

This establishes that the diameter of the metric space of two-dimensional Banach spaces is $\frac{3}{2}$. In higher dimensions, very little is known; even examples are hard to compute. (In three dimensions, even the distance between l_1 and

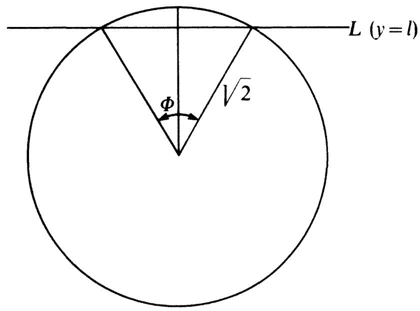

Fig. 4. $\Phi = 2 \arccos \frac{l}{\sqrt{2}}$.

l_∞ — in other words, between an icosahedron and a cube — is unknown.) W. J. Davis and others have obtained some partial results for very high dimensions, and for the limiting case as the dimension approaches infinity; see for example [2], [3], and [4].

The proof of the two-dimensional theorem is long, mainly because of the great variety of objects in γ. It takes several elaborate constructions to cover all of them. Special cases, however, are generally easy, with one major exception: the case of nearly-regular hexagons. The reader may wish to approach the proof with that special case in mind. It falls within Case IIB, below.

To prove the theorem, we will let $c \in \gamma$ and find a representative $C' \in i^{-1}(c)$ such that $R \subseteqq C' \subseteqq \sqrt{3/2}R$. It will be clear from the proof that the constant can be improved when c is not p or h, but we will not check this explicitly.

A convex set C is *inscribed* in a circle S if no point of C is outside S, and every arc of S of more than 90° contains a point of C. (This definition differs slightly from the usual definition of "inscribed" polygons; see, for example, figure 3a.) Every class $c \in \gamma$ has a representative C which is inscribed in the circle $S_{\sqrt{2}}$ (that is, $x^2 + y^2 = 2$), and this representative is unique up to rotations and reflections. (This fact is due to Behrend [5].) We choose this representative C, and we will show how to modify if to obtain the desired C'.

CASE I. In the first case we assume that it is *not* possible to rotate C so that, while still inscribed in the circle $S_{\sqrt{2}}$, it also lies between the lines $y = +\sqrt{3/2}$ and $y = -\sqrt{3/2}$.

Let L be a chord of $S_{\sqrt{2}}$ of maximal length which does not intersect the interior of C. Rotate C so that L is horizontal and above the x axis. Now L coincides with a line $y = l$, with $\sqrt{3/2} < l \leqq \sqrt{2}$. It follows from the choice of l that the circle S_l is contained in C. Let C' be the image of C under the affine transformation $\Psi\colon (x, y) \to (\sqrt{3/2}x, (\sqrt{3/2}/l)y)$. We will show that $R \subseteqq C' \subseteqq \sqrt{3/2}R$.

First we show that $C' \subseteqq \sqrt{3/2}R$. The set $\sqrt{3/2}R$ is bounded by the lines $y = +\sqrt{3/2}$ and $y = -\sqrt{3/2}$, the circle $x^2+y^2=3$, and the ellipse $\frac{1}{2}x^2+y^2=2$. Let $(x,y) \in C$, and $(x',y') = (\sqrt{3/2}x, (\sqrt{3/2}/l)y) \in C'$. From $|y| \leqq l$, it follows that $|y'| \leqq \sqrt{3/2}$. Since C is inscribed in $S_{\sqrt{2}}$, we know $x^2+y^2 \leqq 2$; therefore

$$\begin{aligned}(x')^2+(y')^2 &= \tfrac{3}{2}x^2+\frac{3/2}{l^2}y^2 \\ &\leqq \tfrac{3}{2}(x^2+y^2) \leqq 3\,,\end{aligned}$$

and

$$\begin{aligned}\tfrac{1}{2}(x')^2+(y')^2 &= \tfrac{3}{4}x^2+\frac{3/2}{l^2}y^2 \\ &\leqq x^2+y^2 \leqq 2\,,\end{aligned}$$

so that $(x',y') \in \sqrt{3/2}R$.

To show that $R \subseteqq C'$ we must subdivide the case.

CASE IA. $\sqrt{8/5} < l \leqq \sqrt{2}$. Since C contains the circle S_l, C' must contain the image of S_l under affine transformation Ψ. That image is an ellipse with the equation

$$\left(\frac{x}{\sqrt{3/2}}\right)^2+\left(\frac{ly}{\sqrt{3/2}}\right)^2 = l^2\,,$$

or, more simply, $(1/l^2)x^2+y^3=\frac{3}{2}$. We will show that R is in the interior of that ellipse. Let $(x,y) \in R$. Then $x^2+y^2 \leqq 2$, which implies that

$$\begin{aligned}\frac{1}{l^2}x^2+y^2 &= \frac{1}{l^2}(x^2+y^2)+\left(1-\frac{1}{l^2}\right)y^2 \\ &\leqq \frac{2}{l^2}+\left(1-\frac{1}{l^2}\right)y^2 \\ &= y^2+\frac{1}{l^2}(2-y^2) \\ &< y^2+\tfrac{5}{8}(2-y^2) \\ &= \tfrac{5}{4}+\tfrac{3}{8}y^2 \text{ which is } \leqq \tfrac{3}{2} \text{ if } y^2 \leqq \tfrac{2}{3}\,.\end{aligned}$$

Also, we know that $\frac{1}{2}x^2+y^2 \leqq \frac{4}{3}$, which implies that

$$\frac{1}{l^2}x^2+y^2 = \frac{2}{l^2}(\tfrac{1}{2}x^2+y^2)-\left(\frac{2}{l^2}-1\right)y^2$$

$$\leq \frac{8}{3l^2} - \left(\frac{2}{l^2} - 1\right)y^2$$

$$= y^2 + \frac{1}{l^2}\left(\tfrac{8}{3} - 2y^2\right)$$

$$< y^2 + \tfrac{5}{8}\left(\tfrac{8}{3} - 2y^2\right)$$

$$= \tfrac{5}{3} - \tfrac{1}{4}y^2 \text{ which is } \leq \tfrac{3}{2} \text{ if } y^2 \geq \tfrac{2}{3}.$$

It follows that R is contained in the interior of C' as desired.

CASE IB. $\sqrt{3/2} < l \leq \sqrt{8/5}$. This case is difficult because R does not lie inside the ellipse used in case IA, so it is necessary to find some room between that ellipse and the boundary of C', or equivalently, between S_l and the boundary of C. But except for its difficulty, this case is not important. For example, in this case C cannot be a parallelogram, hexagon, or any regular polygon.

Let Φ be the angle subtended by the chord L, given by $\Phi = 2 \arccos l/\sqrt{2}$. See figure 4. Now Φ is a decreasing function of l and when $\sqrt{3/2} < l \leq \sqrt{8/5}$, we have $53° < \Phi \leq 60°$.

Let P be any point of contact between C and $S_{\sqrt{2}}$, as in figure 5. Since S_l is contained in C, so is the region bounded by S_l and the tangents to S_l through P. The arc of S_l which bounds this region has measure Φ.

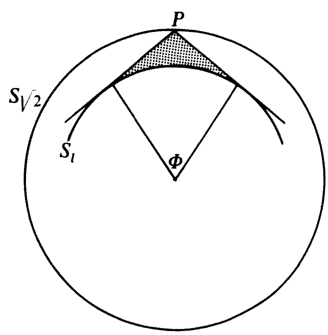

Fig. 5.

Since C is inscribed in $S_{\sqrt{2}}$, there must be a point of contact with an argument somewhere in the range from $\frac{1}{2}\Phi$ to $90° + \frac{1}{2}\Phi$. Since there is none above the chord L, there must be one whose argument is from $\frac{1}{2}\Phi$ to $90° - \frac{1}{2}\Phi$, inclusive. See figure 6. It follows that every point in the shaded region must be in C. The lines L_1 and L_2, which bound the shaded region, are tangent to S_l at the points $P_1 = (l \sin \Phi,\ l \cos \Phi)$ and $P_2 = (l \cos \Phi,\ l \sin \Phi)$ respectively.

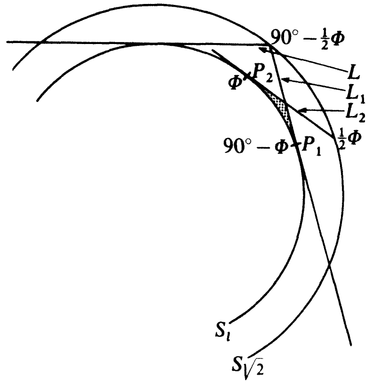

Fig. 6.

Let L_1' and L_2' be the images of L_1 and L_2 under Ψ. We will show that every point in R (in the first quadrant) is either inside the ellipse $(1/l^2)x^2+y^2=\frac{3}{2}$, or it is in the interior of the region bounded by the ellipse, L_1', and L_2'. (The other quadrants are symmetrical.) It will follow that R is contained in the interior of C'. The following four lemmas will fulfill the program and complete case IB.

LEMMA 1. *If* $(x,y)\in R$ *and* $y\geqq\sqrt{3/2}\sin\Phi$, *then* $(1/l^2)x^2+y^2<\frac{3}{2}$. ($\sqrt{3/2}\sin\Phi$ *is the ordinate of* ΨP_2.)

LEMMA 2. *If* $(x,y)\in R$ *and* $0\leqq y\leqq\sqrt{3/2}\cos\Phi$, *then* $(1/l^2)x^2+y^2<\frac{3}{2}$. ($\sqrt{3/2}\cos\Phi$ *is the ordinate of* ΨP_1.)

LEMMA 3. *Every point of* R *is strictly to the left of* L_2'.

LEMMA 4. *Every point of* R *is strictly to the left of* L_1'.

The proofs of the lemmas are at the end of the paper.

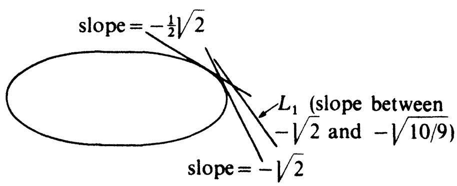

Fig. 7. Relationship between L_1' and R.

Case II. By rotating C if necessary we may assume that C is inscribed in $S_{\sqrt{2}}$ and lies between the lines $y=\sqrt{3/2}$ and $y=-\sqrt{3/2}$.

In this case we can find points $P_0, P_1, P_2 \in C \cap S_{\sqrt{2}}$ which satisfy all of these conditions: (1) these points together with their symmetric images P_0', P_1', P_2', respectively, divide the circle into arcs of no more than 90°; (2) $\arg P_2 \leqq \arg P_1 < \arg P_0$ (although P_1 may coincide with P_2); and (3) $30° \leqq \arg P_0 \leqq 60°$ and $-60° \leqq \arg P_2 \leqq -30°$. By a reflection and reindexing if necessary we can force arc P_1P_2 to be the shorter of P_0P_1 and P_1P_2 and hence no longer than 60°. We now choose a reflection and a rotation of C so that it lies between $y=\sqrt{3/2}$ and $y=-\sqrt{3/2}$ and so that $\arg P_0$ is as close as possible to 45° and we keep C in that position throughout case II.

Now let C' be the image of C under the linear transformation $\Psi\colon (x,y) \mapsto (\psi x, y)$ where ψ is chosen as large as possible such that no part of C' is outside the circle $x^2+y^2=3$. That is,

$$\psi = \min\left\{\frac{\sqrt{3-y^2}}{|x|} \;\middle|\; (x,y) \in C\right\}.$$

Since C includes no point outside $S_{\sqrt{2}}$, but at least one point on the arc of $S_{\sqrt{2}}$ between $(1,-1)$ and $(1,1)$, it is necessary that $\sqrt{3/2} \leqq \psi \leqq \sqrt{2}$.

We will show that $R \subseteqq C' \subseteqq \sqrt{3/2}R$. The proof that $C' \subseteqq \sqrt{3/2}R$ is easiest. Our construction guarantees that if $(x,y) \in C'$, then $|y| \leqq \sqrt{3/2}$ and $x^2+y^2 \leqq 3$. The other part of the boundary of $\sqrt{3/2}R$ is the ellipse $\frac{1}{2}x^2+y^2=2$. But if $(x,y) \in C'$ then $(x/\psi, y) \in C$ and therefore $(x/\psi)^2+y^2 \leqq 2$, and since $\psi \leqq \sqrt{2}$, this implies that $\frac{1}{2}x^2+y^2 \leqq 2$. Therefore $C' \leqq \sqrt{3/2}R$.

The proof that $R \subseteqq C'$ is more difficult. First we will require some lemmas which establish that certain line segments are outside R; then we will subdivide Case II according to the value of $\arg P_0$. The proofs of the lemmas are saved for the end.

Lemma 5. *If L is a chord of $S_{\sqrt{2}}$ which spans an arc of at most* 60° *and lies entirely in the region $-\sqrt{3/2}<y<+\sqrt{3/2}$, then ΨL is outside R. (If an endpoint of L is at $y=\pm\sqrt{3/2}$, ΨL may intersect the boundary of R.)*

Lemma 6. *If L is a chord of $S_{\sqrt{2}}$ whose endpoints are in the region $-1<y\leqq +1$, and if some point of ΨL is on (or outside) $S_{\sqrt{3}}$, then ΨL is outside R. (If the endpoints of L are at $(1,-1)$ and $(1,1)$, then ΨL may intersect the boundary of R.)*

Lemma 7. *If L is a chord of $S_{\sqrt{2}}$ whose endpoints have arguments α and β satisfying*

$$45^\circ \leqq \alpha < 60^\circ$$

$$-30^\circ \leqq \beta \leqq 0^\circ$$

$$\textit{and } \alpha - \tfrac{1}{2}\beta \leqq 60^\circ ,$$

and if some point ΨL is on (or outside) $S_{\sqrt{3}}$, then ΨL is outside R. (If $\alpha = 60^\circ$, $\beta = 0^\circ$, then ΨL may intersect the boundary of R.)

LEMMA 8. *If L is a chord of $S_{\sqrt{2}}$ which spans an arc of at most 60° and lies entirely in the region $y \leqq -\sqrt{1/2}$, then ΨL is outside R.*

LEMMA 9. *If L is a chord of $S_{\sqrt{2}}$ whose endpoints have arguments α and β satisfying*

$$-45^\circ \leqq \alpha \leqq -30^\circ ,$$

$$-120^\circ \leqq \beta \leqq -90^\circ ,$$

$$\textit{and } \alpha - \tfrac{1}{2}\beta \leqq 15^\circ ,$$

then ΨL is outside R.

CASE IIA: $\arg P_0 = 45^\circ$. Let P_3 be a point on the boundary of C such that ΨP_3 is on the circle $x^2 + y^2 = 3$. The construction guarantees that:

$$-60^\circ \leqq \arg P_2 \leqq -45^\circ$$

$$\arg P_2 \leqq \arg P_1 \leqq 0$$

$$-45^\circ \leqq \arg P_1 \leqq \arg P_3 \leqq \arg P_0 = 45^\circ .$$

We know that $\arg P_2 \leqq -45^\circ$ because the arc P_2P_0' can't exceed 90°; and $\arg P_1 \geqq -45^\circ$ because the arc P_0P_1 can't exceed 90°. We know that $\arg P_1 \leqq 0$ because P_1P_2 can't be longer than P_0P_1. It is possible that P_1 and P_2 coincide, as may P_3 and P_1. See figure 8.

Now the boundary of C, in the region defined by $x > 0$ and $-1 \leqq y \leqq +1$, is on or to the right of the broken line $P_0P_3P_1P_2$. But $\Psi(P_0P_3)$ and $\Psi(P_3P_1)$ are outside R by lemma 6, and $\Psi(P_1P_2)$ is outside R by lemma 5. Therefore all parts of the boundary of C' are outside R, and $R \subseteqq C'$.

(If $\arg P_1 = -45^\circ$ and the boundary of C includes the segment P_0P_1, then the corresponding segment of the boundary of C' intersects the boundary of R at $(\sqrt{2}, 0)$. This occurs, for example, when C is a square.)

CASE IIB: $\arg P_0 > 45^\circ$. Let $\arg P_0 = 45^\circ + \Phi$, where $0 < \Phi \leqq 15^\circ$. Define two chords L_1 and L_2 of the circle $S_{\sqrt{2}}$ as follows: The endpoints of L_1 have

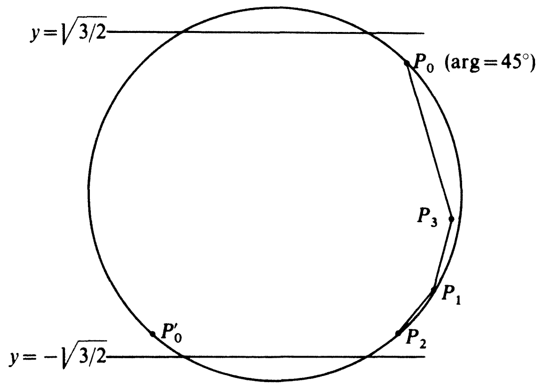

Fig. 8. Case IIA.

arguments $30°+2\Phi$ and $-30°+2\Phi$, and the endpoints of L_2 have arguments $30°$ and $-30°$. Each subtends an arc of 60°, and is therefore tangent to the circle $S_{\sqrt{3/2}}$. See figure 9.

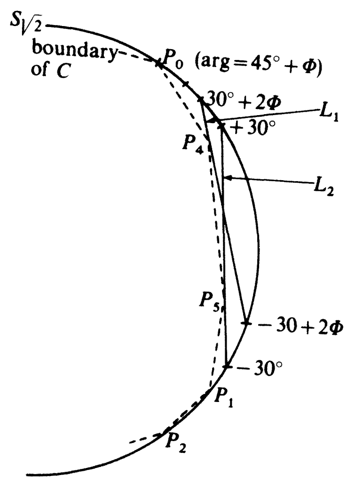

Fig. 9. Case IIB.

If L is any 60°-chord strictly between L_1 and L_2, then C must contain a point on or to the right of L. Otherwise, we could apply a rotation and reflection to make L coincident with the line $y=\sqrt{3/2}$; then C would still be in the region $-\sqrt{3/2}\leqq y\leqq\sqrt{3/2}$ but $\arg P_0$ would be closer to 45°.

Since L can be chosen arbitrarily close to L_1 and C is compact, there must be a point of C on L_1. Of all such points, let P_4 be the one with the largest y-

coordinate. (In figure 9, P_4 is shown on the upper part of L_1, but it could be as low as the lower endpoint of L_1.) Similarly there must be a point of C on L_2; let P_5 be the one with the lowest y-coordinate. If P_1 is to the roght of L_2, instead let $P_5 = P_1$.

We will now examine the parts of the boundary of C in the region $x > 0$, and show that the image of each part under Ψ is outside R. (In one case, the boundary of C' may intersect the boundary of R.) It will follow that $R \subseteqq C'$.

(a) Above P_0: This part of ∂C is above the line $y = 1$, and so is its image under Ψ, so both are outside R.

(b) From P_0 to P_4: If $\arg P_0 = 60°$, then $P_4 = P_0$ and there is nothing to prove; so we may assume that $\arg P_0 < 60°$. Let P be the lower endpoint of the chord of $S_{\sqrt{2}}$ which goes through P_0 and P_4. If $\arg P > 0$, then $\Psi(P_0 P)$ is outside of R by lemma 5; therefore, we may assume that $\arg P \leqq 0$. In this case no point of ∂C can be to the right of P. Since Ψ must map some point of ∂C onto the circle $S_{\sqrt{3}}$, it must also map some point of $P_0 P$ onto $S_{\sqrt{3}}$. Now $\arg P \geqq -30° + 2\Phi$, which marks the lower endpoint of L_1; and therefore

$$(\arg P_0) - \tfrac{1}{2}(\arg P) \leqq (45° + \Phi) - \tfrac{1}{2}(-30° + 2\Phi) = 60° .$$

Thus lemma 7 applies to the chord $P_0 P$, and $\Psi(P_0 P)$ is outside R.

(c) From P_4 to P_5: Every point on this segment of ∂C is on, or to the right of, some 60°-chord that lies between L_1 and L_2. Therefore, by lemma 5, Ψ moves this segment of ∂C outside R.

There is an exception in the case $\arg P_0 = 60°$, when $P_4 = P_0$. In that case part of ∂C may coincide with L_1, and ΨL_1 may intersect the boundary of R. This occurs if C is a regular hexagon.

(d) From P_5 to P_1: If $P_5 = P_1$ there is nothing to prove. Otherwise let L be the chord through P_5 and P_1; then ΨL is outside R by either lemma 5 or lemma 6.

(e) From P_1 to P_2: This chord spans an arc of at most 60°, so by lemma 5, $\Psi(P_1 P_2)$ is outside R, unless $\arg P_2 = -60°$ and $\arg P_1 = 0°$, in which case $\Psi(P_1 P_2)$ may intersect the boundary of R. The latter also occurs if C is a regular hexagon.

(f) Below P_2: If P_2 is below the line $y = -1$, there is no problem; but that is not necessary. See figure 10. We have $\arg P'_0 = -135° + \Phi$, so because the arc $P'_0 P_2$ cannot exceed 90°, we have $\arg P_2 \leqq -45° + \Phi$. Let L_3 be the chord of $S_{\sqrt{2}}$ with endpoints at $-120° + 2\Phi$ and $-60° + 2\Phi$, and let L_4 be the chord with endpoints at $-120°$ and $-60°$. If L is any 60°-chord between L_3 and L_4, then C must intersect L, or else C could be rotated to make L coincide with the line $y = -\sqrt{3/2}$, and then $\arg P_0$ would be closer to 45°. Therefore C also intersects

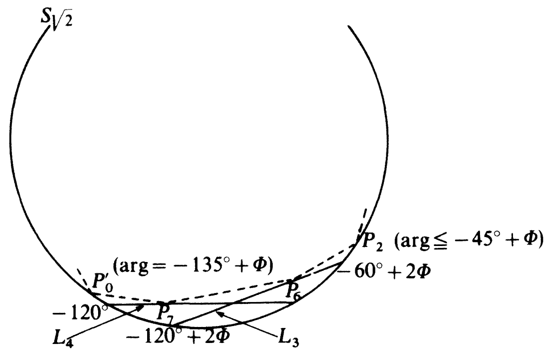

Fig. 10.

L_3 and L_4. Let P_6 be the point of $C \cap L_3$ with the largest x-coordinate, and let P_7 be the point of $C \cap L_4$ with the smallest x-coordinate.

Now Ψ moves the chord through P_2 and P_6 outside R, by Lemma 9. Any point of ∂C between P_6 and P_7 is below some 60° chord, and is therefore moved outside R by lemma 8. Any point of ∂C between P_7 and P'_0 is below the line $y = -1$, so Ψ moves this segment outside of R as well.

This completes the proof that $R \subseteqq C'$.

Case IIC: $\arg P_0 < 45°$. The problems of this case are symmetrical to those of case IIB and can be resolved in the same way.

This completes the proof of the theorem, except for the proofs of the lemmas.

Proof of Lemma 1. Since $\Phi = 2 \arccos l/\sqrt{2}$, we have

$$\begin{aligned}\sin \Phi &= \sin\left(2 \arccos \frac{l}{\sqrt{2}}\right) \\ &= 2\left(\sin \arccos \frac{l}{\sqrt{2}}\right)\left(\cos \arccos \frac{l}{\sqrt{2}}\right) \\ &= 2\left(\sqrt{1-\frac{l^2}{2}}\right)\left(\frac{l}{\sqrt{2}}\right) \\ &= l\sqrt{2-l^2}\,.\end{aligned}$$

Therefore

$$y = \sqrt{3/2}\sin \Phi \geqq \sqrt{3/2}\, l\sqrt{2-l^2}\,.$$

This function is decreasing for $1<l<\sqrt{2}$ so it achieves its lowest value in the range $\sqrt{3/2}<l\leqq\sqrt{8/5}$ when $l=\sqrt{8/5}$. Thus

$$y \geqq \sqrt{3/2}\sqrt{8/5}\sqrt{2-(8/5)} = \sqrt{24/25}\,.$$

Now following Case IA, we have

$$\begin{aligned}
\frac{1}{l^2}x^2+y^2 &= \frac{2}{l^2}\left(\tfrac{1}{2}x^2+y^2\right)-\left(\frac{2}{l^2}-1\right)y^2\\
&\leqq \frac{8}{3l^2}-\left(\frac{2}{l^2}-1\right)y^2\\
&= y^2+\frac{1}{l^2}\left(\tfrac{8}{3}-2y^2\right)\\
&< y^2+\tfrac{2}{3}\left(\tfrac{8}{3}-2y^2\right)\\
&= \tfrac{16}{9}-\tfrac{1}{3}y^2 \leqq \tfrac{16}{9}-\tfrac{1}{3}\left(\tfrac{24}{25}\right) = \tfrac{328}{225} < \tfrac{3}{2}\,.
\end{aligned}$$

PROOF OF LEMMA 2.

$$\begin{aligned}
\cos\Phi &= \cos\left(2\arccos\frac{l}{\sqrt{2}}\right)\\
&= 2\cos^2\left(\arccos\frac{l}{\sqrt{2}}\right)-1\\
&= (l^2-1)\,.
\end{aligned}$$

Therefore $y\leqq\sqrt{3/2}\cos\Phi=\sqrt{3/2}\,(l^2-1)$. Again following case IA,

$$\begin{aligned}
\frac{1}{l^2}x^2+y^2 &= \frac{1}{l^2}(x^2+y^2)+\left(1-\frac{1}{l^2}\right)y^2\\
&\leqq \frac{2}{l^2}+\left(1-\frac{1}{l^2}\right)y^2\\
&\leqq \frac{2}{l^2}+\left(1-\frac{1}{l^2}\right)\tfrac{3}{2}(l^2-1)^2\\
&= \frac{1}{l^2}\left(2+\tfrac{3}{2}(l^2-1)^3\right)\,.
\end{aligned}$$

It is routine to check that this is always less than 3/2 when $\sqrt{3/2}<l\leqq\sqrt{8/5}$.

PROOF OF LEMMA 3. We will do this in reverse: we will let (x,y) be on the line L_2' and show that $\frac{1}{2}x^2+y^2>\frac{4}{3}$ and hence that (x,y) is not in R. The equation of L_2 is

$$(\cos\Phi)x+(\sin\Phi)y = l$$

and the equation of L_2' is

$$(\cos\Phi)\left(\frac{1}{\sqrt{3/2}}x\right)+(\sin\Phi)\left(\frac{l}{\sqrt{3/2}}y\right) = l\,.$$

If (x,y) satisfies this equation then

$$x = \frac{l-(\sin\Phi)\left(\frac{l}{\sqrt{3/2}}y\right)}{(\cos\Phi)\left(\frac{1}{\sqrt{3/2}}\right)} = \left(\frac{l\sqrt{3/2}}{\cos\Phi}\right)-y\left(\frac{l\sin\Phi}{\cos\Phi}\right).$$

Therefore

$$\frac{1}{2}x^2+y^2 = \frac{1}{2}\left(\frac{l\sqrt{3/2}}{\cos\Phi}-y\left(\frac{l\sin\Phi}{\cos\Phi}\right)\right)^2+y^2\,.$$

It is always true that $\frac{1}{2}(A-By)^2+y^2\geqq A^2/(2+B^2)$, and in this case that implies that

$$\frac{1}{2}x^2+y^2 \geqq \frac{\frac{\frac{3}{2}l^2}{\cos^2\Phi}}{2+\frac{l\sin^2\Phi}{\cos^2\Phi}} = \frac{\frac{3}{2}l^2}{2\cos^2\Phi+l^2\sin^2\Phi}$$

$$= \frac{\frac{3}{2}l^2}{(2-l^2)\cos^2\Phi+l^2} = \frac{\frac{3}{2}l^2}{(2-l^2)(l^2-1)^2+l^2}\,.$$

This function reaches its minimum value (for possible values of l) when $l^2=8/5$, and then $\frac{1}{2}x^2+y^2\geqq\frac{150}{109}>\frac{4}{3}$.

Proof of Lemma 4. The equation of L_1 is

$$(\sin\Phi)x+(\cos\Phi)y = l$$

and the equation of L_1' is

$$(\sin\Phi)\left(\frac{1}{\sqrt{3/2}}x\right)+(\cos\Phi)\left(\frac{l}{\sqrt{3/2}}y\right) = l\,.$$

The slope of L_1' is therefore

$$-\frac{(\sin\Phi)\left(\frac{1}{\sqrt{3/2}}\right)}{(\cos\Phi)\left(\frac{l}{\sqrt{3/2}}\right)} = -\frac{\sin\Phi}{l\cos\Phi}$$

This is an increasing function of l (in the relevant range) and has values from $-\sqrt{2}$ (when $l=\sqrt{3/2}$) to $-\sqrt{10/9}$ (when $l=\sqrt{8/5}$).

The slopes of the half-tangents to R at the corner point $(\sqrt{4/3}, \sqrt{2/3})$ are $-\sqrt{1/2}$ and $-\sqrt{2}$ (see figure 7). It is clear, therefore, that if $(\sqrt{4/3}, \sqrt{2/3})$ is to the left of L_1', then so is every point of R.

To see that $(\sqrt{4/3}, \sqrt{2/3})$ is to the left of R, it is necessary only to check that

$$(\sin \Phi)\left(\frac{1}{\sqrt{3/2}}\sqrt{4/3}\right)+(\cos \Phi)\left(\frac{l}{\sqrt{3/2}}\sqrt{2/3}\right) < l,$$

or equivalently that

$$l\sqrt{2-l^2}\,(\sqrt{\tfrac{8}{9}})+(l^2-1)(\tfrac{2}{3}l)-l < 0,$$

when $\sqrt{3/2}<l\leqq\sqrt{8/5}$. This is a routine calculation. There is equality if $l=\sqrt{3/2}$.

Proof of Lemma 5. Assume without loss of generality that L spans an arc of exactly 60°. We show that if (x,y) is on L, then $(\psi x)^2+y^2>2$, so that $(\psi x, y)\in \Psi L$ is outside of R. First assume $|y|<\sqrt{1/2}$. Since L is tangent to the circle $S_{\sqrt{3/2}}$, we know $x^2+y^2\geqq 3/2$; and therefore

$$\begin{aligned}(\psi x)^2+y^2 &\geqq \tfrac{3}{2}x^2+y^2 \qquad\qquad (\text{since } \Psi\geqq\sqrt{3/2})\\ &= \tfrac{3}{2}(x^2+y^2)-\tfrac{1}{2}y^2\\ &\geqq \tfrac{3}{2}(\tfrac{3}{2})-\tfrac{1}{2}y^2\\ &> \tfrac{3}{2}(\tfrac{3}{2})-\tfrac{1}{2}(\tfrac{1}{2}) = 2\end{aligned}$$

If $y\geqq\sqrt{1/2}$, then (x,y) must be to the right of the chord with endpoints at 0° and 60°. The equation of that chord is $\sqrt{3}\,x+y=\sqrt{6}$, so we have $x>(\sqrt{6}-y)/\sqrt{3}$ and

$$\begin{aligned}(\Psi x)^2+y^2 &\geqq \tfrac{3}{2}x^2+y^2\\ &> \tfrac{3}{2}\left(\frac{6-2\sqrt{6}\,y+y^2}{3}\right)+y^2\\ &= \tfrac{3}{2}(y-\sqrt{2/3})^2+2 \geqq 2.\end{aligned}$$

The case $y<-\sqrt{1/2}$ is symmetrical.

Equality occurs if $(x,y)=(\sqrt{8/9}, \sqrt{2/3})$, which is on the chord with endpoints at 0° and 60°, and $\psi=\sqrt{3/2}$. Of course, equality also occurs when $(x,y)=(\sqrt{8/9}, -\sqrt{2/3})$.

PROOF OF LEMMA 6. It suffices to consider the case of a chord L whose endpoints are at $(1,1)$ and $P=(\sqrt{2-b^2}, -b)$, with $0 \leqq b < 1$, and with ΨP on the circle $S_{\sqrt{3}}$. In this case $\psi = (\sqrt{3-b^2}/\sqrt{2-b^2})$ and the segment ΨL joins the points $((\sqrt{3-b^2}/\sqrt{2-b^2}), 1)$ and $(\sqrt{3-b^2}, -b)$.

A general formula for the minimum distance from the origin to a line through (x,y) and (z,w) is

$$d = \frac{\begin{vmatrix} x & y \\ z & w \end{vmatrix}}{\sqrt{(x-z)^2+(y-w)^2}}.$$

In the case of ΨL, it is enough to show that $d > \sqrt{2}$, since that will imply that ΨL is outside $S_{\sqrt{2}}$, and hence outside R. It is easier to work with d^2, and prove that the numerator is more than twice the denominator.

$$d^2 = \frac{\begin{vmatrix} \sqrt{3-b^2} & -b \\ \dfrac{\sqrt{3-b^2}}{\sqrt{2-b^2}} & 1 \end{vmatrix}^2}{\left(\sqrt{3-b^2}-\dfrac{\sqrt{3-b^2}}{\sqrt{2-b^2}}\right)^2+(b+1)^2}.$$

$$\begin{vmatrix} \sqrt{3-b^2} & -b \\ \dfrac{\sqrt{3-b^2}}{\sqrt{2-b^2}} & 1 \end{vmatrix}^2 - 2\left(\sqrt{3-b^2}-\frac{\sqrt{3-b^2}}{\sqrt{2-b^2}}\right)^2 - 2(b+1)^2$$

$$= (3-b^2)\left(1+\frac{b}{\sqrt{2-b^2}}\right)^2 - 2(3-b^2)\left(1-\frac{1}{\sqrt{2-b^2}}\right)^2 - 2(b+1)^2$$

$$= (3-b^2)\left(1+\frac{2b}{\sqrt{2-b^2}}+\frac{b^2}{2-b^2}-2+\frac{4}{\sqrt{2-b^2}}-\frac{2}{2-b^2}\right) - 2(b+1)^2$$

$$= (3-b^2)\left(\frac{2b+4}{\sqrt{2-b^2}}-2\right) - 2(b+1)^2$$

$$= \left(\frac{3-b^2}{\sqrt{2-b^2}}\right)(2b+4) - 2(3-b^2) - 2(b+1)^2$$

$$= \left(\frac{1}{\sqrt{2-b^2}}+\sqrt{2-b^2}\right)(2b+4) - 2(2b+4) > 0$$

since $(1/A)+A>2$ whenever $A\neq 1$. Equality occurs when $\sqrt{2-b^2}=1$; that is, when $b=1$.

PROOF OF LEMMA 7. We need only consider the case in which $\alpha-\frac{1}{2}\beta=60°$, since any chord satisfying $\alpha-\frac{1}{2}\beta<60°$ must lie to the right of a chord satisfying $\alpha-\frac{1}{2}\beta=60°$. Similarly, we may assume that Ψ moves the lower endpoint of L onto the circle $S_{\sqrt{3}}$.

Let $\gamma=|\beta|$, so that the endpoints of L have arguments $(60°-\frac{1}{2}\gamma)$ and $(-\gamma)$. The chord L is bisected by the ray with argument $(30°-\frac{3}{4}\gamma)$, and includes the point $(\sqrt{2}\cos\gamma, -\sqrt{2}\sin\gamma)$. We can therefore write the equation of the line which includes L:

$$\begin{aligned}\cos(30°-\tfrac{3}{4}\gamma)x+\sin(30°-\tfrac{3}{4}\gamma)y \\ &= \sqrt{2}\cos(30°-\tfrac{3}{4}\gamma)\cos\gamma-\sqrt{2}\sin(30°-\tfrac{3}{4}\gamma)\sin\gamma \\ &= \sqrt{2}\cos(30°+\tfrac{1}{4}\gamma)\,.\end{aligned}$$

This line meets the positive y axis at the point $(0,\sqrt{2}c)$ where

$$c=\frac{\cos(30°+\frac{1}{4}\gamma)}{\sin(30°-\frac{3}{4}\gamma)}\,.$$

If the segment ΨL is extended, it meets the positive y axis at the same point. The image of $(\sqrt{2}\cos\gamma, -\sqrt{2}\sin\gamma)$ under Ψ is the point $(\sqrt{3-2\sin^2\gamma}, -\sqrt{2}\sin\gamma)$ on $S_{\sqrt{3}}$.

Now see Figure 11. The line PR is the extension of the chord ΨL, and R is the lower endpoint of that chord. The segment OS, whose length is given as $\sqrt{2}a$, is perpendicular to PR. We intend to show that $a>1$, and it will follow that PR is outside $S_{\sqrt{2}}$ and therefore that ΨL is outside R.

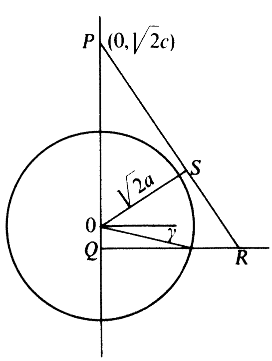

Fig. 11. $R=(\sqrt{3-2\sin^2\gamma}, -\sqrt{2}\sin\gamma)$.

By similarity of triangles, we have

$$\frac{a^2}{c^2} = \left(\frac{OS}{OP}\right)^2 = \left(\frac{QR}{PR}\right)^2 = \frac{(QR)^2}{(PQ)^2+(QR)^2}$$

$$= \frac{3-2\sin^2\gamma}{(\sqrt{2}c+\sqrt{2}\sin\gamma)^2+3-2\sin^2\gamma} = \frac{3-2\sin^2\gamma}{2c^2+4c\sin\gamma+3}\,.$$

Therefore

$$a^2 = \frac{c^2(3-2\sin^2\gamma)}{2c^2+4c\sin\gamma+3}$$

and in order to show that $a>1$, it suffices to show that

$$f(\gamma) = c^2(3-2\sin^2\gamma)-2c^2-4c\sin\gamma-3 > 0\,,$$

whenever $0<\gamma\leqq 30°$. Recall

$$c = \frac{\cos(30°+\frac{1}{4}\gamma)}{\sin(30°-\frac{3}{4}\gamma)}\,.$$

The verification that $f(\gamma)>0$ is tedious and mechanical, but crucial to the proof of the theorem, so an outline will be given here. The first step is to simplify f:

$$f(\gamma) = c^2(1-2\sin^2\gamma)-4c\sin\gamma-3$$
$$= c^2\cos 2\gamma-4c\sin\gamma-3\,.$$

If we treat f as a function of two variables, we find that $\partial f/\partial c>0$ for relevant values of γ and c, so that if we substitute a lower bound for c we obtain a lower bound for f.

Next we calculate the first four derivatives of c with respect to γ and show that they satisfy (writing μ for $30°-\frac{3}{4}$)

$$c' = c\frac{\cos\mu}{\sin\mu}-\frac{\cos\gamma}{4\sin^2\mu}$$

$$c'' = \tfrac{3}{2}c'\frac{\cos\mu}{\sin\mu}+\tfrac{1}{2}c$$

$$c^{(3)} = \tfrac{3}{2}c''\frac{\cos\mu}{\sin\mu}+\tfrac{9}{8}c'\frac{1}{\sin^2\mu}+\tfrac{1}{2}c'$$

$$c^{(4)} = \tfrac{3}{2}c^{(3)}\frac{\cos\mu}{\sin\mu}+\tfrac{9}{4}c''\frac{1}{\sin^2\mu}+\tfrac{1}{2}c''+\tfrac{27}{16}c'\frac{\cos\mu}{\sin\mu}\,.$$

It is clear that $c^{(5)}>0$ when $0 \leqq \gamma \leqq 30°$, and that when $\gamma=0$, we have $c=\sqrt{3}$, $c'=2$, $c''=\frac{7}{2}\sqrt{3}$, $c^{(3)}=\frac{103}{4}$, and $c^{(4)}=\frac{683}{8}\sqrt{3}$. It follows that when $0 \leqq \gamma \leqq 30°$,

$$c \geqq \sqrt{3}+2\gamma+\tfrac{7}{4}\sqrt{3}\gamma^2+\tfrac{103}{24}\gamma^3+\tfrac{683}{192}\sqrt{3}\gamma^4 .$$

We also have $\cos 2\gamma \geqq 1-2\gamma^2$, and $\sin\gamma \leqq \gamma$. (For these expressions, γ must be measured in radians.)

We next substitute these expressions for $c, \cos 2\gamma$, and $\sin\gamma$ into the equations for f. For convenience, the terms beyond γ^5 in the series for c^2 are deleted. Simplifying, we obtain

$$f(\gamma) \geqq \tfrac{1}{2}\gamma^2+\tfrac{7}{12}\sqrt{3}\gamma^3+\tfrac{49}{32}\gamma^4-\tfrac{775}{48}\sqrt{3}\gamma^5-\tfrac{4579}{48}\gamma^6-\tfrac{117}{2}\sqrt{3}\gamma^7 .$$

This is positive at least for $0<\gamma \leqq 0.2$ radians (about 11°).

For larger values of γ, we can divide the range $0.2 \leqq \gamma \leqq \frac{1}{6}\pi$ into smaller intervals, and use the same method to construct a power series for f centered around the lower endpoint of each interval. It is necessary to calculate $(\sin\gamma)$ to the first-power term and $(\cos 2\gamma)$ and c to the square term, and we can drop the fourth-power term from c^2. The result is a lower bound for f, and the lower bound is positive if we use the intervals $0.2 \leqq \gamma \leqq 0.3$, $0.3 \leqq \gamma \leqq 0.4$, and $0.4 \leqq \gamma \leqq 0.524 > \frac{1}{6}\pi$. We can conclude that $f(\gamma)>0$ whenever $0<\gamma<30°$, which proves the lemma.

PROOF OF LEMMA 8. Any point on one of these chords must be on or to the right of the chord L_0 through the points $(0, -\sqrt{2})$ and $(\sqrt{3/2}, -\sqrt{1/2})$, on or to the left of the corresponding chord in the third quadrant, or below the midpoints of these chords. The last possibility forces the point to be strictly below the line $y=-1$, so that both the point and its image under Ψ are outside of R. It therefore suffices to prove that ΨL_0 is outside of R. That is a special case of Lemma 9, whose proof follows.

PROOF OF LEMMA 9. We need to consider only chords L satisfying $\alpha-\frac{1}{2}\beta=15°$, since any chord with $\alpha-\frac{1}{2}\beta<15°$ lies below some chord for which equality holds. Also, we may assume that $\psi=\sqrt{3/2}$. These assumptions imply that the endpoints of L are at $(\sqrt{2}\cos\alpha, \sqrt{2}\sin\alpha)$ and $(\sqrt{2}\cos\beta, \sqrt{2}\sin\beta)$ and the endpoints of ΨL are at $(\sqrt{3}\cos\alpha, \sqrt{2}\sin\alpha)$ and $(\sqrt{3}\cos\beta, \sqrt{2}\sin\beta)$.

If $-34° \leqq \alpha \leqq -30°$, we can prove that this line lies outside of the ellipse $\frac{1}{2}x^2+y^2=\frac{4}{3}$, and therefore outside of R. It is equivalent to show that the line through $(\sqrt{3/2}\cos\alpha, \sqrt{2}\sin\alpha)$ and $(\sqrt{3/2}\cos\beta, \sqrt{2}\sin\beta)$ is outside the circle $x^2+y^2=\frac{4}{3}$. To do this, we let d be the perpendicular distance from the origin to the line, and apply the formula from lemma 6:

$$d^2 = \frac{\begin{vmatrix} \sqrt{3/2}\cos\beta & \sqrt{2}\sin\beta \\ \sqrt{3/2}\cos\alpha & \sqrt{2}\sin\alpha \end{vmatrix}^2}{(\sqrt{3/2}\cos\beta - \sqrt{3/2}\cos\alpha)^2 + (\sqrt{2}\sin\beta - \sqrt{2}\sin\alpha)^2}$$

$$= \frac{3(\cos\beta\sin\alpha - \cos\alpha\sin\beta)^2}{\frac{3}{2}[(\cos\beta-\cos\alpha)^2 + (\sin\beta-\sin\alpha)^2] + \frac{1}{2}(\sin\beta-\sin\alpha)^2}$$

$$= \frac{3\sin^2(a-\beta)}{3-3\cos(\alpha-\beta)+\frac{1}{2}(\sin\beta-\sin\alpha)^2}\,.$$

The last term in the denominator is clearly maximized by setting $\alpha = -30°$, $\beta = -90°$. Then its value is $\frac{1}{8}$, so we have

$$d^2 \geqq \frac{3\sin^2(\alpha-\beta)}{\frac{25}{8}-3\cos(\alpha-\beta)}\,.$$

The angle $(\alpha-\beta)$ must be from $60°$ to $64°$, and in this range, the last expression is a decreasing function of $(\alpha-\beta)$. Therefore

$$d^2 \geqq \frac{3°\sin^2 64°}{\frac{25}{8}-3\cos 64°} = 1.3390 > \tfrac{4}{3}\,.$$

Therefore in this case, ΨL must be outside R.

If $\alpha \leqq -34°$, a different approach is necessary. Note that the chord L is bisected by the ray in the direction $\frac{1}{2}(\alpha+\beta)$, which can vary from $-66°$ to $-82\frac{1}{2}°$. Therefore the slope of L is $-(\tan\frac{1}{2}(\alpha+\beta))^{-1}$, which must be a positive number not exceeding $(\tan 66°)^{-1}$. The slope of ΨL is therefore at most $\sqrt{2/3}/\tan 66°$. The slope of the upper half-tangent to R at the point $(\sqrt{2/3}, -1)$ is $\frac{1}{2}\sqrt{2/3}$, and since $\tan 66° = 2.246 > 2$, the latter slope is larger. It follows that if ΨL passes below $(\sqrt{2/3}, -1)$ it is entirely outside R. This verification is lengthy but elementary.

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