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CLASSIFICATION OF DIMENSION GROUPS AND ITERATING SYSTEMS

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1. Introduction.

The notion of a dimension group has been introduced by Elliott in [4] in order to classify approximately finite dimensional C*-algebras. Recently some considerable progress has been done in the investigation of the dimension groups by Effros, Handelman, and Shen in [1], [2], [3], and [8]. For definition we refer to [3].

In [2], Effros and Shen conjectured that every finitely generated dimension group is isomorphic to the limit of a system

$$\mathbf{Z}^r \xrightarrow{A_1} \mathbf{Z}^r \xrightarrow{A_2} \cdots$$

where A_n is in the set $\mathrm{GL}(r,\mathbf{Z})^+$ of all unimodular matrices whose entries are non negative integers. Using the multidimensional continued fraction algorithm (the Jacobi–Perron algorithm, see [6] and [7]), Effros and Shen could show in [2] that the conjecture is true for finitely generated dimension groups which are simple and totally ordered. In the present paper our main purpose is to show that, more generally, the conjecture is also true for simple finitely generated dimension groups with a unique state. Since the Jacobi–Perron algorithm is no longer applicable in this situation we develop a new procedure to generate such a sequence $A_1, A_2, \ldots$ in $\mathrm{GL}(r,\mathbf{Z})^+$ (see Proposition 2.8).

2.

Let us first introduce some notations. Let J_r be the interior of $(\mathbf{R}^r)^+$. If A is any matrix with non negative entries then we denote by cone (A) the positive convex cone which is generated by the column vectors of A. Throughout in the remainder of this paper we only use the l^1-norm on $\mathbf{R}^r$ which we denote by $\|\cdot\|$. For any $x,y \in (\mathbf{R}^r)^+ \setminus \{0\}$ we define

$$d(x,y) = \big\| \|x\|^{-1}x - \|y\|^{-1}y \big\| .$$

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We note the following useful relation which will be needed below:

(2.1) $$d(x+y,y) = (1+\|y\|\,\|x\|^{-1})^{-1}d(x,y), \quad \text{if } x,y \in (\mathbf{R}^r)^+ \setminus \{0\} .$$

Observe that the norm $\|\cdot\|$ is additive on $(\mathbf{R}^r)^+$. For any matrix A with non negative entries we define

$$d(A) = \sup\{d(x,y) \mid x,y \in \text{cone}(A) \setminus \{0\}\} .$$

By [3], 3.5 and 4.1 any simple finitely generated dimension group with a unique state is isomorphic to one of the following forms. Fix a vector $x \in J_r$ such that $x \notin \mathbf{R}\mathbf{Z}^r$. Let $\mathbf{Z}^r$ be ordered by the cone

$$P_x = \{a \in \mathbf{Z}^r \mid \langle a,x\rangle > 0\} \cup \{0\} .$$

We want to prove the following.

2.2. THEOREM. *For every $x \in J_r \setminus \mathbf{R}\mathbf{Z}^r$ there exists a sequence $A_1, A_2, \ldots$ in $\mathrm{GL}(r,\mathbf{Z})^+$ such that*

$$P_x = \bigcup_{n=1}^{\infty} (A_n \ldots A_1)^{-1}(\mathbf{Z}^r)^+ .$$

Equivalently, for any $x \in J_r \setminus \mathbf{R}\mathbf{Z}^r$ we must find a sequence $A_1, A_2, \ldots$ in $\mathrm{GL}(r,\mathbf{Z})^+$ such that

(2.3) $$\mathbf{R}^+ x = \bigcap_{n=1}^{\infty} \text{cone}(A_1^t \ldots A_n^t) ,$$

where A_n^t denotes the transposed of the matrix A_n.

Instead of (2.3) we consider the following equivalent condition,

(2.4) $$(A_1^t \ldots A_n^t)^{-1}x \in J_r \quad \text{and} \quad \lim_{n\to\infty} d(A_1^t \ldots A_n^t) = 0 .$$

Thus, Theorem 2.2 is proved if for any $x \in J_r \setminus \mathbf{R}\mathbf{Z}^r$ we can find a sequence $\{A_n\}_{n\in\mathbf{N}}$ in $\mathrm{GL}(r,\mathbf{Z})^+$ such that (2.4) is satisfied. In order to determine such a sequence $\{A_n\}_{n\in\mathbf{N}}$ we only use matrices of the following kind.

For $1 \leq p,q \leq r$, $p \neq q$ let $T^{(p,q)} = (t_{ij}^{(p,q)})_{1\leq i,j\leq r}$ be the matrix which is defined as follows

$$t_{ij}^{(p,q)} = \begin{cases} 1 & \text{if } i=j \text{ or } i=p,\ j=q \\ 0 & \text{elsewhere .} \end{cases}$$

The inverse $S^{(p,q)} = (s_{ij}^{(p,q)})_{1\leq i,j\leq r}$ of the matrix $T^{(p,q)}$ is given by

$$s_{ij}^{(p,q)} = \begin{cases} 1 & \text{if } i=j \\ -1 & \text{if } i=p,\ j=q \\ 0 & \text{elsewhere.} \end{cases}$$

Hence all the matrices $T^{(p,q)}$ belong to $\mathrm{GL}(r,\mathbf{Z})^+$.

For convenience we also include permutation matrices. Thus let Ω_r be the set of all matrices which are products of matrices of the form $T^{(p,q)}$ and of permutation matrices.

If $A=(a_{ij})_{1\leqq i,j\leqq r-1}$ is any matrix then we define the matrix $\hat{A}=(\hat{a}_{ij})_{1\leqq i,j\leqq r}$ as follows

$$\hat{a}_{ij} = \begin{cases} a_{ij} & \text{if } 1\leqq i,j\leqq r-1 \\ 1 & \text{if } i=j=r \\ 0 & \text{elsewhere.} \end{cases}$$

Clearly, if A is contained in Ω_{r-1} then $\hat{A}$ is contained in Ω_r. For any vector $x=(x_1,\ldots,x_r)^t\in \mathsf{R}^r$ $(r\geqq 2)$ we denote by $\check{x}$ the vector $(x_1,\ldots,x_{r-1})^t$.

Next we need some lemmas.

2.5. Lemma *Let $A=(a_{ij})_{1\leqq i,j\leqq r}$ be any matrix such that the entries a_{ij} are not negative and the column vectors of A are not zero. Then for any $\varepsilon>0$ there is some $\delta>0$ such that for any vectors $x,y\in(\mathsf{R}^r)^+\setminus\{0\}$ we have*

$$d(Ax,Ay)\leqq\varepsilon \quad \text{if } d(x,y)\leqq\delta\,.$$

Proof. Let $\lambda:=\inf\{\sum_{i=1}^r a_{ij} \mid 1\leqq j\leqq r\}$. By our assumption we have $\lambda\neq 0$. For any vector $x=(x_1,\ldots,x_r)^t\in(\mathsf{R}^r)^+$ whose norm is 1 we obtain

$$\|Ax\| = \sum_{i=1}^r\sum_{j=1}^r a_{ij}x_j \geqq \sum_{j=1}^r \lambda x_j = \lambda\|x\| = \lambda\,.$$

Now, let $x,y\in(\mathsf{R}^r)^+\setminus\{0\}$. If we set $\tilde{x}:=\|x\|^{-1}x$, $\tilde{y}:=\|y\|^{-1}y$ then we obtain

$$\begin{aligned} d(Ax,Ay) &= \|\,\|A\tilde{x}\|^{-1}A\tilde{x}-\|A\tilde{y}\|^{-1}A\tilde{y}\| \\ &\leqq \|A\tilde{x}\|^{-1}\|A\tilde{x}-A\tilde{y}\|+|\,\|A\tilde{x}\|^{-1}-\|A\tilde{y}\|^{-1}|\,\|A\tilde{y}\| \\ &\leqq \|A\|\lambda^{-1}\|\tilde{x}-\tilde{y}\|+\|A\|\lambda^{-2}|\,\|A\tilde{y}\|-\|A\tilde{x}\|\,| \\ &\leqq (\|A\|\lambda^{-1}+\|A\|^2\lambda^{-2})\|\tilde{x}-\tilde{y}\| = (\|A\|\lambda^{-1}+\|A\|^2\lambda^{-2})\,d(x,y)\,. \end{aligned}$$

Thus our assertion follows from this.

2.6 Lemma. *Let $x=(x_1,\ldots,x_r)^t\in J_r\setminus\mathsf{R}\mathsf{Z}^r$ and let $\{A_n\}_{n\in\mathsf{N}}$ be a sequence of invertible matrices whose entries are non negative integers. Furthermore assume that $A_n^{-1}x\in J_r$ holds for any $n\in\mathsf{N}$ and $\lim_{n\to\infty}d(A_n)=0$. Let $A_n=(a_{ij}^{(n)})_{1\leqq i,j\leqq r}$. Then the following conditions are satisfied*

(2.6.1) $$\lim_{n\to\infty} a_{ij}^{(n)} = \infty \quad \textit{for } 1 \leqq i, j \leqq r ,$$

(2.6.2) $$\lim_{n\to\infty} \|A_n^{-1}x\| = 0 .$$

PROOF. Let us show that (2.6.1) is true. Let $a_1^{(n)}, \ldots, a_r^{(n)}$ be the column vectors of A_n. Since x lies in cone $(\{a_1^{(n)}, \ldots, a_r^{(n)}\})$ we have $d(x, a_i^{(n)}) \leqq d(A_n)$ if $1 \leqq i \leqq r$. Hence

$$\lim_{n\to\infty} d(x, a_i^{(n)}) = 0$$

holds. By our assumption, at least one of the numbers $\|x\|^{-1}x_1, \ldots, \|x\|^{-1}x_r$ is not rational, say $\|x\|^{-1}x_k$ for some k.

Since

$$|\|a_j^{(n)}\|^{-1}a_{kj}^{(n)} - \|x\|^{-1}x_k| \leqq d(a_j^{(n)}, x)$$

holds, we must have $\lim_{n\to\infty} \|a_j^{(n)}\| = \infty$ for any j, $1 \leqq j \leqq r$.

Since x_i is non zero and since

$$\lim_{n\to\infty} |\|a_j^{(n)}\|^{-1}a_{ij}^{(n)} - \|x\|^{-1}x_i| = 0$$

holds for $1 \leqq i \leqq r$, we obtain from this that $\lim_{n\to\infty} a_{ij}^{(n)} = \infty$ is satisfied.

Next let us prove (2.6.2). Since $A_n^{-1}x$ is contained in J_r we obtain for any $n \in \mathbf{N}$,

$$\left(\inf_{1\leqq i,j\leqq r} a_{ij}^{(n)}\right)\|A_n^{-1}x\| \leqq \|A_nA_n^{-1}x\| = \|x\| .$$

By (2.6.1) we know that $\lim_{n\to\infty} \inf_{1\leqq i,j\leqq r} a_{ij}^{(n)} = \infty$. Hence we conclude from this that $\lim_{n\to\infty} \|A_n^{-1}x\| = 0$ holds.

We need another notation. For any subset $M \subseteqq \mathbf{R}$ we denote by $\langle M \rangle$ the vector space over the rational field Q generated by M.

2.7 LEMMA. *Let $r \geqq 3$ and let $x = (x_1, \ldots, x_r)^t \in J_r$ such that $\check{x}$ is not contained in $\mathbf{R}\mathbf{Z}^{r-1}$. Furthermore let $A = (a_{ij})_{1\leqq i,j\leqq r-1}$ be a matrix such that a_{ij} is a positive rational number. Then at least one of the vectors*

$$x \quad \text{or} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_r \end{bmatrix} = \hat{A}x$$

satisfies the following condition on some vector $z = (z_1, \ldots, z_r)^t$.

(2.7.1) There is some k, $1 \leqq k \leqq r-1$ such that $\dim \langle \{z_k, z_r\} \rangle = 2$ and $\dim \langle \{z_i \mid 1 \leqq i \leqq r,\ i \neq k\} \rangle \geqq 2$ holds.

PROOF. Suppose that the vector x does not satisfy the condition (2.7.1). Since $\dim \langle \{x_i \mid 1 \leqq i \leqq r-1\} \rangle \geqq 2$ holds, there is some k such that $\dim \langle \{x_k, x_r\} \rangle = 2$. Hence, since x does not satisfy (2.7.1), x_i is the product of x_r by a positive rational number for $i \neq k$. By our assumption on A we infer from this that y_i has the form $px_k + qx_r$ for some positive rational numbers p and q. Since $y_r = x_r$ holds we obtain from this

$$\dim \langle \{y_i \mid 1 \leqq i \leqq r,\ i \neq k\} \rangle = \dim \langle \{y_k, y_r\} \rangle = 2 ,$$

i.e. (2.7.1) holds for y.

The next proposition contains the crucial part of the proof of Theorem 2.2

2.8 PROPOSITION. *For any $x \in J_r \setminus \mathbf{R}\mathbf{Z}^r$ and for any $\varepsilon > 0$ there is some $A \in \Omega_r$ such that*

$$A^{-1}x \in J_r \quad \text{and} \quad d(A) \leqq \varepsilon$$

holds.

PROOF. We proceed by induction on r. Observe that our statement is trivial for $r=1$. Let us consider the case $r=2$. Let $x=(x_1, x_2)^t \in J_2$ such that $x_1 x_2^{-1}$ is not rational and let $k_0, k_1, k_2, \ldots$ be the sequence of positive integers which determines the continued fraction expansion of the number $x_1 x_2^{-1}$ (see [5, section 10.9]). We define a sequence $\{A_n\}_{n \in \mathbf{N}}$ of matrices in Ω_2 by

$$A_n^t = \begin{cases} (T^{(1,2)})^{k_{n-1}} & \text{for } n = 1, 3, 5, \ldots \\ (T^{(2,1)})^{k_{n-1}} & \text{for } n = 2, 4, 6, \ldots \end{cases}$$

if $x_1 > x_2$ holds and

$$A_n^t = \begin{cases} (T^{(2,1)})^{k_n} & \text{for } n = 1, 3, 5, \ldots \\ (T^{(1,2)})^{k_n} & \text{for } n = 2, 4, 6, \ldots \end{cases}$$

if $x_1 < x_2$ holds. By continued fraction theory and by the results of [1] we know that the sequence $\{A_n\}_{n \in \mathbf{N}}$ satisfies the condition (2.3). We conclude that our statement is true for $r=2$.

Suppose that our assertion has been proved already for some $r-1$, where $r > 2$. First we want to show that the following is true:

(2.8.1) For any $x \in J_r \setminus \mathbf{R}\mathbf{Z}^r$ and for any $A \in \Omega_r$ satisfying $A^{-1}x \in J_r$ there is some $B \in \Omega_r$ such that $B^{-1}A^{-1}x \in J_r$ and $d(AB) \leqq \frac{3}{4} d(A)$ holds.

Let $x=(x_1,\ldots,x_r)^t \in J_r \setminus \mathbf{R}\mathbf{Z}^r$ be given and let $A \in \Omega_r$ be chosen such that

$$x^{(1)} = \begin{bmatrix} x_1^{(1)} \\ \vdots \\ x_r^{(1)} \end{bmatrix} = A^{-1}x$$

is contained in J_r. We assume that $\dim\langle\{x_1^{(1)},\ldots,x_{r-1}^{(1)}\}\rangle \geqq 2$ holds. For, if this were not satisfied then we could apply some permutation matrix in order to achieve this situation. We denote by a the last column vector of A. Since our assertion is true for $r-1$, by 2.5 and 2.6 we can find some $C_1 \in \Omega_{r-1}$ such that the following conditions are satisfied:

(2.8.2) $\hat{C}_1^{-1}x^{(1)} \in J_r$.

(2.8.3) $d(y_i^{(1)}, y_j^{(1)}) \leqq \frac{1}{16}d(A)$ holds for $1 \leqq i, j \leqq r-1$, where $y_1^{(1)},\ldots,y_{r-1}^{(1)}$ are the first $r-1$ column vectors of the matrix $A\hat{C}_1$.

(2.8.4) $x_i^{(2)} < x_r^{(2)} = x_r^{(1)}$ holds for $1 \leqq i \leqq r-1$, where

$$x^{(2)} = \begin{bmatrix} x_1^{(2)} \\ \vdots \\ x_r^{(2)} \end{bmatrix} = \hat{C}_1^{-1}x^{(1)}.$$

(2.8.5) $\|y_i^{(1)}\| \geqq 4\|a\|$ holds for $1 \leqq i \leqq r-1$.

We want to arrange the situation in such a manner that in addition (2.7.1) holds for the vector $x^{(2)}$. This can be achieved as follows. If $x^{(2)}$ satisfies the condition (2.7.1) then we change nothing. If $x^{(2)}$ does not satisfy this condition then, applying our assumption to the vector $\check{x}^{(2)}$ and using 2.5 as well as 2.6 we can find some matrix $C_0 \in \Omega_{r-1}$ such that the conditions (2.8.2), (2.8.3), (2.8.4), and (2.8.5) remain valid if we substitute C_1 by C_1C_0 and any entry of this matrix is positive. Now, by 2.7 the "new" vector $x^{(2)}$ satisfies the condition (2.7.1). For the remainder of the proof of (2.8.1) let us keep this situation. Thus let $k \in \{i \mid 1 \leqq i \leqq r-1\}$ be chosen such that

(2.8.6) $\dim\langle\{x_i^{(2)} \mid 1 \leqq i \leqq r,\ i \neq k\}\rangle \geqq 2$ and $\dim\langle\{x_k^{(2)}, x_r^{(2)}\}\rangle = 2$

holds.

We define a positive integer n_0 as follows

$$n_0 := \sup\{n \in \mathbf{N} \mid nx_k^{(2)} \leqq x_r^{(2)}\}.$$

By the second relation in (2.8.6) we must have $n_0x_k^{(2)} < x_r^{(2)}$. By (2.1) and (2.8.5) we can find some $n_1 \in \mathbf{N}$ such that

(2.8.7) $\frac{7}{16}d(a,y_k^{(1)}) \leqq d(ma+y_k^{(1)},y_k^{(1)}) \leqq \frac{9}{16}d(a,y_k^{(1)})$
for $m \in \{n_1, n_1+1\}$

holds. We will consider two different cases.

First, let $n_0 \leqq n_1$. We set $B := \hat{C}_1(T^{(r,k)})^{n_0}T^{(k,r)}$. Since $n_0 x_k^{(2)} < x_r^{(2)} < (n_0+1)x_k^{(2)}$ holds, we obtain that $B^{-1}x^{(1)}$ is contained in J_r.

Let $y_1^{(2)},\ldots,y_r^{(2)}$ be the column vectors of AB. Then the following identities are valid.

$$y_i^{(2)} = \begin{cases} y_k^{(1)} + n_0 a & \text{if } i=k \\ y_k^{(1)} + (n_0+1)a & \text{if } i=r \\ y_i^{(1)} & \text{if } i \neq k,\ i \neq r\,. \end{cases}$$

Since $n_0 \leqq n_1$ holds we obtain from (2.8.7) and (2.1)

$$d(y_k^{(2)},y_k^{(1)}) \leqq d(y_r^{(2)},y_k^{(1)}) \leqq \tfrac{9}{16}d(a,y_k^{(1)}) \leqq \tfrac{9}{16}d(A)\,.$$

Since

$$d(y_k^{(2)},y_r^{(2)}) = d(y_k^{(2)},y_k^{(2)}+a) \leqq \tfrac{1}{5}d(y_k^{(2)},a) \leqq \tfrac{1}{5}d(A)$$

holds, we obtain from (2.8.3) that

$$d(y_i^{(2)},y_j^{(2)}) \leqq \tfrac{3}{4}d(A)$$

is satisfied for $1 \leqq i, j \leqq r$. This implies that $d(AB) \leqq \frac{3}{4}d(A)$ holds, i.e. (2.8.1) is satisfied in this case.

Next, we assume that $n_0 > n_1$ holds. From (2.8.6) we deduce that the following is true

(2.8.8) $\dim\langle\{x_i^{(2)} \mid 1 \leqq i \leqq r,\ i \neq k,r\} \cup \{x_r^{(2)} - m x_k^{(2)}\}\rangle \geqq 2$
for some $m \in \{n_1, n_1+1\}$.

Let P be the permutation matrix which is obtained from the identity matrix, interchanging the kth and the rth column vector and let $C_2 := \hat{C}_1(T^{(r,k)})^m P$. Since $n_0 > n_1$ holds, the vector

$$x^{(3)} = \begin{bmatrix} x_1^{(3)} \\ \vdots \\ x_r^{(3)} \end{bmatrix} = C_2^{-1}A^{-1}x$$

is contained in J_r and the following identities are satisfied.

$$x_i^{(3)} = \begin{cases} x_r^{(2)} - m x_k^{(2)} & \text{if } i=k \\ x_k^{(2)} & \text{if } i=r \\ x_i^{(2)} & \text{if } i \neq k,r\,. \end{cases}$$

Thus, by (2.8.6) and (2.8.8) our assumption is applicable to the vector $\check{x}^{(3)}$. Hence by 2.5 we can find some $C_3 \in \Omega_{r-1}$ such that

$$\hat{C}_3^{-1} x^{(3)} \in J_r \quad \text{and} \quad d(y_i^{(2)}, y_j^{(2)}) \leqq \tfrac{1}{16} d(A)$$

holds for $1 \leqq i, j \leqq r-1$, where $y_1^{(2)}, \ldots, y_{r-1}^{(2)}$ are the first $r-1$ column vectors of the matrix $AC_2\hat{C}_3$. The last column vector $y_r^{(2)}$ of $AC_2\hat{C}_3$ is $y_k^{(1)} + ma$. Since all vectors $y_1^{(2)}, \ldots, y_r^{(2)}$ are contained in cone $(A\hat{C}_1)$ there exists some non negative scalar λ and some $z \in$ cone $(y_1^{(1)}, \ldots, y_{r-1}^{(1)})$ such that

$$y_k^{(2)} = \lambda a + z$$

holds. Since $d(z, y_k^{(1)}) \leqq \frac{1}{16} d(A)$ is true we can easily deduce from (2.8.7) that the inequality $d(y_i^{(2)}, y_j^{(2)}) \leqq \frac{3}{4} d(A)$ is satisfied for $1 \leqq i, j \leqq r$ if $\lambda = 0$ or $\lambda = m$ holds. Now let us suppose that λ is not zero and $\lambda \neq m$. Then we obtain from the triangle inequality

$$\begin{aligned} d(y_k^{(2)}, y_r^{(2)}) &= d(z + \lambda a, y_k^{(1)} + ma) \\ &\leqq d(z + \lambda a, y_k^{(1)} + \lambda a) + d(y_k^{(1)} + \lambda a, y_k^{(1)} + ma) \\ &\leqq \tfrac{1}{16} d(A) + d(y_k^{(1)} + \lambda a, y_k^{(1)} + ma) . \end{aligned}$$

We consider two different cases. First let $\lambda > m$. Then we obtain from (2.1) and (2.8.7)

$$\begin{aligned} d(y_k^{(1)} + \lambda a, y_k^{(1)} + ma) &= d((y_k^{(1)} + ma) + (\lambda - m)a, y_k^{(1)} + ma) \\ &\leqq d(y_k^{(1)} + ma, a) \\ &= d(a, y_k^{(1)}) - d(y_k^{(1)} + ma, y_k^{(1)}) \\ &\leqq \tfrac{9}{16} d(a, y_k^{(1)}) . \end{aligned}$$

Next, let $\lambda < m$. Then we obtain from (2.1) and (2.8.7) again

$$\begin{aligned} & d(y_k^{(1)} + \lambda a, y_k^{(1)} + ma) \\ &= d((m^{-1} y_k^{(1)} + a) + (\lambda^{-1} - m^{-1}) y_k^{(1)}, m^{-1} y_k^{(1)} + a) \\ &\leqq d(m^{-1} y_k^{(1)} + a, (\lambda^{-1} - m^{-1}) y_k^{(1)}) = d(y_k^{(1)} + ma, y_k^{(1)}) \\ &\leqq \tfrac{9}{16} d(a, y_k^{(1)}) \leqq \tfrac{9}{16} d(A) . \end{aligned}$$

Hence, in any case we infer the validity of the following inequalities

$$d(y_i^{(2)}, y_j^{(2)}) \leqq \tfrac{11}{16} d(A) < \tfrac{3}{4} d(A)$$

for $1 \leqq i, j \leqq r$. If we define $B := C_2 \hat{C}_3$, then we obtain that our statement (2.8.1) is true also if $n_0 > n_1$.

Now we can easily deduce our main statement from (2.8.1). For, if

$x \in J_r \setminus \mathbf{R}\mathbf{Z}^r$ and $n \in \mathbf{N}$ are given, then by successive applications of (2.8.1) we can find some matrix $A \in \Omega_r$ such that $A^{-1}x \in J_r$ and $d(A) \leqq (3/4)^n$ holds. Thus our proposition follows from this.

Now the condition (2.4) and hence Theorem 2.2 is a trivial consequence of 2.8 and 2.5.

ADDED IN PROOF. Recently the author has shown that the conjecture of Effros–Shen is not true in general (to appear in Proc. Amer. Math. Soc.).

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