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SOME REMARKS ON THE C*-ALGEBRAS ASSOCIATED WITH CERTAIN TOPOLOGICAL MARKOV CHAINS

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In this note we show how certain algebras $\mathcal{O}_A$, of the type constructed in [5], and associated to certain topological Markov chains, can be constructed as crossed products and fixed point algebras of certain natural group actions on $\mathcal{O}_n$, the algebra of the full shift. This helps to clarify their structure somewhat, including their K-theory. For simplicity, we work in detail with actions of $\mathbf{Z}_2$ only, and merely indicate what happens with more general group actions.

Let $\mathcal{O}_n$ be generated by n isometries $S_1,\ldots,S_n$ on a Hilbert space H, with $\sum S_iS_i^*=1$. If $r_0,r_1 \in \mathbf{N}$, with $r_0+r_1=n$, then by [2], there is an action $\alpha=\alpha(r_0,r_1)$ of $\mathbf{Z}_2$ on $\mathcal{O}_n$ given by $S_i \to S_i$, $i\leqq r_0$, and $S_i \to -S_i$ if $i>r_0$.

THEOREM $\quad \mathbf{Z}_2 \times \mathcal{O}_n \cong \mathcal{O}_A$, *where* $A = \begin{pmatrix} r_0 & r_1 \\ r_1 & r_0 \end{pmatrix}$.

PROOF. The crossed product acts on $H\oplus H$, and is generated by

$$\left\{\pi(X) = \begin{pmatrix} X & 0 \\ 0 & \alpha(X) \end{pmatrix} : X \in \mathcal{O}_n\right\}, \quad \text{and the unitary } U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\pi(\alpha(X))=U\pi(X)U^*$, $X \in \mathcal{O}_n$. If $U=R_0-R_1$ is the spectral decomposition of U, then

$$\begin{aligned}\mathbf{Z}_2 \times \mathcal{O}_n &= C^*(\pi(S_i),\ R_j :\ 1\leqq i\leqq n,\ 0\leqq j\leqq 1)\\ &= C^*(\pi(S_i)R_j :\ 1\leqq i\leqq n,\ 0\leqq j\leqq 1)\end{aligned}$$

where $\pi(S_i)R_j \neq 0$, $\forall\, i,j$. Now $\alpha(S_iS_i^*)=S_iS_i^*$, $\forall\, i$, hence if P_i denotes the range projection $S_iS_i^*$, then $\pi(P_i)$ commutes with each spectral projection R_j. Hence we can decompose $H\oplus H$ into $2n$ orthogonal subspaces by

$$1 = \sum_{j=0}^{1} \sum_{i=1}^{n} \pi(P_i)R_j\,.$$

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Then $\mathbf{Z}_2 \times \mathcal{O}_n$ is generated by certain transitions, given by the operators $\pi(S_i)R_j$, relative to this decomposition. This enables $\mathbf{Z}_2 \times \mathcal{O}_n$ to be expressed as a C*-algebra of the form $\mathcal{O}_B$, where B is a $2n \times 2n$ matrix, with entries consisting entirely of zeros and ones. The admissable transitions can be obtained from

$$(R_0 - R_1)\pi(S_i)(R_0 - R_1) = \begin{cases} S_i & i \leqq r_0 , \\ -S_i & i > r_0 , \end{cases}$$

so that certainly

$$R_0\pi(S_i)R_1 = R_1\pi(S_i)R_0 = R_0\pi(S_j)R_0 = R_1\pi(S_j)R_1 = 0$$

for $i \leqq r_0 < j$. Moreover, $R_0\pi(S_i)R_0$, $R_1\pi(S_i)R_1$, $R_0\pi(S_j)R_1$, $R_1\pi(S_j)R_0$ are all non zero for $i \leqq r_0 < j$. Thus we see

$$\mathbf{Z}_2 \times \mathcal{O}_n = \mathcal{O}_B ,$$

where $B^t = [b_{\alpha\beta}]_{\alpha,\beta \in \Lambda}$, $\Lambda = \{1,\ldots,n\} \times \{0,1\}$ is given by

$$\begin{aligned} b_{\alpha\beta} = 0 \quad \text{if } \alpha &= (j,1),\ \beta = (i,0), && i \leqq r_0 \\ \alpha &= (j,0),\ \beta = (i,1), && i \leqq r_0 \\ \alpha &= (i,0),\ \beta = (j,0), && j > r_0 \\ \alpha &= (i,1),\ \beta = (j,1), && j > r_0 \end{aligned}$$

and $b_{\alpha\beta} = 1$ otherwise. This matrix can be reduced to an equivalent 2×2 matrix A as follows: First define

$$A = \begin{pmatrix} r_0 & r_1 \\ r_1 & r_0 \end{pmatrix} = [a_{ij}]$$

and set

$$\Lambda' = \{(i, \varkappa, j) : 1 \leqq i,j \leqq 2,\ \varkappa \leqq a_{ij}\} .$$

Then Λ' can be matched with Λ by

$$\begin{aligned} (1,i,1) &\leftrightarrow (i,0), && 1 \leqq i \leqq r_0 \\ (1,j,2) &\leftrightarrow (j+r_0,0), && 1 \leqq j \leqq r_1 \\ (2,i,1) &\leftrightarrow (i,1), && 1 \leqq i \leqq r_0 \\ (2,j,2) &\leftrightarrow (j+r_0,2), && 1 \leqq j \leqq r_1 \end{aligned}$$

Then in the notation of [5, Remark 2.16], it is easily checked that $A' = B$, so that $\mathcal{O}_A = \mathcal{O}_B$. Hence $\mathbf{Z}_2 \times \mathcal{O}_n = \mathcal{O}_A$.

REMARK 1. Note that the fixed point algebra

$$\mathcal{O}_n^\alpha \cong R_0(\mathbf{Z}_2 \times \mathcal{O}_n)R_0 \cong \mathcal{O}_C ,$$

where

$$C = \begin{pmatrix} r_0 & r_1 \\ r_1 & r_0 \end{pmatrix} \text{ if } r_1 \neq 0, \quad \text{and} \quad C = n \text{ if } r_1 = 0 .$$

In fact this can be seen directly as follows. If $r_0 = 0$, then

$$\mathcal{O}_n^\alpha = C^*(S_i S_j : 1 \leqq i, j \leqq n) \cong \mathcal{O}_n^2 ,$$

and if $r_0 \neq 0$, then $\mathcal{O}_n^\alpha$ is generated by $\mathcal{F}_n^\alpha S_1^r$, $(S_1^*)^r \mathcal{F}_n^\alpha$, $r = 0, 1, \ldots$, where $\mathcal{F}_n^\alpha$ is the fixed point algebra of the UHF algebra $\mathcal{F}_n$ under the restricted action of α. This is an AF algebra with homogeneous embeddings

$$\begin{pmatrix} r_0 & r_1 \\ r_1 & r_0 \end{pmatrix}$$

by [7]. Hence

$$\mathcal{O}_n^\alpha \cong \mathcal{O}_C, \quad \text{where } C = \begin{pmatrix} r_0 & r_1 \\ r_1 & r_0 \end{pmatrix}.$$

COROLLARY. *If $r_0, r_1, s_0, s_1 \in \mathbf{Z}$ with $r_0 + r_1 = n = s_0 + s_1$, then $\alpha(r_0, r_1)$ is not conjugate to $\alpha(s_0, s_1)$ if $r_0 \neq 1 + s_1, s_0$.*

PROOF. If $\theta = \alpha(r_0, r_1)$ is conjugate to $\varphi = \alpha(s_0, s_1)$, then $\mathbf{Z}_2 \times_\theta \mathcal{O}_n$ is isomorphic to $\mathbf{Z}_2 \times_\varphi \mathcal{O}_n$, that is, $\mathcal{O}_{A_1}$ is isomorphic to $\mathcal{O}_{A_2}$ where

$$A_1 = \begin{pmatrix} r_0 & r_1 \\ r_1 & r_0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} s_0 & s_1 \\ s_1 & s_0 \end{pmatrix}.$$

Hence by considering weak extensions [5],

$$|\det(1 - A_1')| = |\det(1 - A_2')| .$$

Straightforward computation shows that

$$|\det(1 - A_1')| = |\det(1 - A_1)| = |(1 - r_0 + r_1)|(n-1) .$$

Thus $|1 - r_0 + r_1| = |1 - s_0 + s_1|$.

Let φ_n denote the endomorphism $X \to \sum S_i X S_i^*$ on $\mathcal{O}_n$, which commutes with the action α of $\mathbf{Z}_2$, and hence extends to an endomorphism $\tilde{\varphi}_n$ of $\mathbf{Z}_2 \times \mathcal{O}_n$, by

$$\tilde{\varphi}_n(X_0 + X_1 U) = \varphi_n(X_0) + \varphi_n(X_1)U, \qquad X_i \in \mathcal{O}_n .$$

PROPOSITION. *$\tilde{\varphi}_n$ is homotopic to the identity on $\mathbf{Z}_2 \times \mathcal{O}_n$.*

PROOF. In the notation of [3], $\varphi_n = \lambda_V$, where $V = \sum_{i,j} S_i S_j S_i^* S_j^*$ is a self adjoint unitary with spectral decomposition $V = E_1 - E_2$ say. Then $\alpha(E_i) = E_i$, since $\alpha(V) = V$. Let V_t denote the unitary $E_1 - tE_2$, for $t \in \mathbf{T}$, so that λ_{V_t} gives a homotopy from the identity to φ_n. Now $\alpha = \lambda_D$, where

$$D = \sum_{i \leqq r_0} S_i S_i^* - \sum_{j > r_0} S_j S_j^* ,$$

and so $\lambda_X(D) = X^* D X$ for all unitaries X in $\mathcal{O}_n$. Then

$$\begin{aligned} \lambda_D \lambda_{V_t} &= \lambda_{D\lambda_D(V_t)} = \lambda_{DV_t} && \text{as } \alpha(V_t) = V_t \\ &= \lambda_{V_t} \lambda_{V_t}(D) && \text{as } \lambda_{V_t}(D) = V_t^* D V_t \\ &= \lambda_{V_t} \lambda_D , \end{aligned}$$

i.e. α commutes with λ_{V_t}, and so λ_{V_t} extends to an endomorphism $\tilde{\lambda}_{V_t}$ on $\mathbf{Z}_2 \times \mathcal{O}_n$, which gives a homotopy from the identity to $\tilde{\varphi}_n$ on the crossed product.

COROLLARY. *If $\hat{\alpha}$ denotes the dual action of $\hat{\mathbf{Z}}_2$ on $\mathbf{Z}_2 \times \mathcal{O}_n$, then $x = r_0 x + r_1 \hat{\alpha}_*(x)$, for all x in $K_i(\mathbf{Z}_2 \times \mathcal{O}_n)$ $(i = 0, 1)$.*

PROOF. Consider first the case $i = 0$. By [4] it is enough to consider classes $x = [F]_0$, where F is a projection majorized by R_0, thus $F = ER_0$, with $E \in \mathcal{O}_n^\alpha$. Then $F = \frac{1}{2}(E + EU)$, and so

$$\begin{aligned} \tilde{\varphi}_n(F) &= \tfrac{1}{2}(\varphi_n(E) + \varphi_n(E)U) = \varphi_n(E) R_0 \\ &= \sum_{i=1}^{n} S_i E S_i^* R_0 = \sum_{i \leqq r_0} S_i E R_0 S_i^* + \sum_{i > r_0} S_i E R_1 S_i^* \\ &= \sum_{i \leqq r_0} S_i F S_i^* + \sum_{i > r_0} S_i \hat{\alpha}(F) S_i^* \end{aligned}$$

since $S_i R_0 = R_0 S_i$, $i \leqq r_0$, $S_i R_0 = R_1 S_i$, $i > r_0$, and $\hat{\alpha}(ER_0) = ER_1$. Thus by the proposition,

$$[F]_0 = r_0 [F]_0 + r_1 [\hat{\alpha}(F)]_0 .$$

To get the result for $K_1(\mathbf{Z}_2 \times \mathcal{O}_n)$ it suffices, again by [4], to consider equivalence classes of unitaries U in $\mathbf{Z}_2 \times \mathcal{O}_n$ of the form $U = U' + R_1$ where U' is a unitary in $R_0(\mathbf{Z}_2 \times \mathcal{O}_n)R_0$. The argument then works just as for the case of K_0.

REMARK 2. This global condition is enough in a large number of interesting cases to deduce that $K_i(\mathcal{O}_A)$ are torsion groups. In this case one can proceed in the spirit of [4] to compute $K_0(\mathcal{O}_A)$. Consider the action $\alpha(r_0+1, r_1)$ of $\mathbf{Z}_2$ on $\mathcal{O}_{n+1}$ which leaves

$$\mathcal{E}_n = C^*(S_1, \ldots, S_{r_0}, S_{r_0+2}, \ldots S_{n+1})$$

globally invariant. If $\mathcal{J}$ is the ideal of $\mathcal{E}_n$ generated by $S_{r_0+1}, S^*_{r_0+1}$, which is also invariant and isomorphic to the compacts $\mathcal{K}$, then

$$\mathbf{Z}_2 \times \mathcal{E}_n/\mathbf{Z}_2 \times \mathcal{J} \cong \mathbf{Z}_2 \times \mathcal{O}_n, \quad \text{and} \quad \mathbf{Z}_2 \times \mathcal{J} \cong \mathcal{K} \oplus \mathcal{K} .$$

Then one can show using exact sequences as in [4], that

$$K_0(\mathcal{O}_A) = \mathbf{Z}^2/(1-A)\mathbf{Z}^2 = \mathbf{Z}_{n-1} \oplus \mathbf{Z}_{r_1-r_0-1} ,$$

where the components are generated by $[1]_0$, $[R_1]_0-[R_0]_0$, at least if $\mathbf{Z}^2/(1-A)\mathbf{Z}^2$ is 2-torsion free.

Alternatively, one can use the decomposition of $\mathcal{K} \otimes \mathcal{O}_A$ as a crossed product of an AF-algebra by a single automorphism and apply the exact sequence [9, 2.4] to show that

$$K_0(\mathcal{O}_A) = \mathbf{Z}^n/(1-A)\mathbf{Z}^n, \quad K_1(\mathcal{O}_A) = \operatorname{Ker}(1-A)$$

on $\mathbf{Z}^n$ for every $n \times n$-matrix A that satisfies condition (I) of [5].

REMARK 3. A similar analysis works for the actions of finite abelian groups, e.g. if $\omega = e^{2\pi i/3}$, and $r_0, r_1, r_2 \in \mathbf{Z}$ with $r_0+r_1+r_2=n$, let α denote the action of $\mathbf{Z}_3$ on $\mathcal{O}_n$ by

$$S_i \to \omega^r S_i \quad \text{if } \sum_0^{r-1} r_j \leqq i < \sum_0^{r} r_j .$$

Then $\mathbf{Z}_3 \times \mathcal{O}_n \cong \mathcal{O}_A$, where

$$A = \begin{bmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{bmatrix}.$$

In particular the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

for which $\operatorname{Ext}(\mathcal{O}_A) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ arises in this way.

REMARK 4. For a compact abelian group G, let $\gamma_1, \ldots, \gamma_n \in \hat{G}$ and let α be the corresponding action of G on $\mathcal{O}_n$ given by $\alpha(g)(S_i) = \gamma_i(g)S_i$, $g \in G$, $i = 1, \ldots, n$.

For example, if $G=\mathsf{T}$, and $\gamma_i(t)=t$, $\forall i$, then $\mathsf{T}\times\mathcal{O}_n$ is stably isomorphic to a Glimm algebra and so simple [8, Remark]. However, if $G=\mathsf{T}^2$, $n=2$, and if $\gamma_i(t_1,t_2)=t_i$, then $\mathsf{T}^2\times\mathcal{O}_2$ is not simple (see [6]), and cf. [1] for actions on UHF algebras. Some insight into the distinction may be found by considering the related objects

$$C^*(R_\eta\pi(S_i) : 1\leqq i\leqq n,\ \eta\in\hat{G}),$$

where (π,U) is the natural covariant representation of $(\mathcal{O}_n,G,\alpha)$ on $L^2(G,H)$ and where R_η are the projections with $U(g)=\sum_{\eta\in\hat{G}}\eta(g)R_\eta$. Then $C^*(R_\eta\pi(S_i))$ can be expressed, modulo the fact that B does not satisfy condition (I) in [5], as $\mathcal{O}_B$, where $B=[b_{\alpha\beta}]_{\alpha,\beta\in\Lambda}$ is the matrix with $\Lambda=\{1,\ldots,n\}\times\hat{G}$ and $\{0,1\}$ entries of admissable transitions $(i,\eta)\to(j,\eta-\gamma_j)$, $1\leqq i,j\leqq n$, $\eta\in\hat{G}$. In both examples cited above, the matrices obtained are not irreducible in the strict sense. However, in the first case, if $\alpha,\beta\in\Lambda$, then there exists a path of transitions from either α to β or β to α, but not both if $\alpha\neq\beta$. This is not the case in the second example. Further investigation of the ideal structure of these algebras, might shed more light on the question of simplicity of the crossed products themselves.

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REFERENCES

1. O. Bratteli, *Crossed products of* UHF *algebras by product type actions*, Duke Math. J. 46 (1979), 1–23.
2. J. Cuntz, *Simple* C*-*algebras generated by isometries*, Comm. Math. Phys. 57 (1977), 173–185.
3. J. Cuntz, *Automorphisms of certain simple* C*-*algebras*, in *Quantum Fields-Algebras, Processes*, ed. L. Streit, Springer-Verlag, Wien - New York, 1980.
4. J. Cuntz, *K-theory for certain* C*-*algebras*, Ann. of Math. (in press).
5. J. Cuntz and W. Krieger, *A class of* C*-*algebras and topological Markov chains*, Invent. Math. 56 (1980), 251–268.
6. D. E. Evans, *On* $\mathcal{O}_n$, Publ. Res. Inst. Math. Sci. (to appear).
7. Th. Fack and O. Marechal, *Sur la classification des symmetries de* C*-*algèbres* UHF, Canad. J. Math. 31 (1979), 496–523.
8. G. K. Pedersen and H. Takai, *Crossed products of* C*-*algebras by approximately uniformly continuous actions*, Preprint, Copenhagen, 1979.
9. M. Pimsner and D. Voiculescu, *Exact sequences for K-groups and* Ext-*groups of certain cross-products* C*-*algebras*, J. Operator Theory 4 (1980), 93–118.

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