MATH. SCAND. 48 (1981), 241-252

GEOMETRIC ASPECTS OF THE TOMITA–TAKESAKI THEORY II

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Introduction.

In the present paper we will study some problems, which grew out of the second author's work reported in the paper [6]. As in [6] we consider a σ-finite von Neumann algebra M on standard form $(M, H, J, P^\natural)$ in the sense of [3]. To each cyclic and separating vector $\xi \in P$ are associated two cones

$$P_\xi^\sharp = (M_+\xi)^- \quad \text{and} \quad P_\xi^\flat = (M'_+\xi)^- .$$

Since $P_\xi^\flat = J(P_\xi^\sharp)$, $P_\xi^\flat$ is the reflected image of $P_\xi^\sharp$ with respect to the "selfadjoint" part $H^\natural$ of the Hilbert Space H,

$$H^\natural = \{\zeta \in H \mid J\zeta = \zeta\} = P^\natural - P^\natural .$$

We shall study the orthogonal projected image Q_ξ of $P_\xi^\sharp$ onto the real subspace $H^\natural$:

$$Q_\xi = \tfrac{1}{2}(1+J)P_\xi^\sharp = \tfrac{1}{2}(1+J)P_\xi^\flat .$$

It turns out that Q_ξ is a closed cone in $H^\natural$, and that $P^\natural \subseteqq Q_\xi$ for any choice of ξ. Moreover $Q_\xi = P^\natural$ if and only if ξ is a trace vector for M. Our main result is:

If M is a factor not a type III$_1$, *and $\xi, \eta \in P^\natural$ are cyclic and separating for M, then $Q_\xi = Q_\eta$ if and only if*

1) $\eta = \lambda\xi$, $\lambda \in \mathbf{R}_+$ *or*
2) *M is finite, and $\eta = \lambda\xi^{-1}$ for a $\lambda \in \mathbf{R}_+$.*

Case 2) should be understood in the following way: When M is a finite factor, we may identify H with $L^2(M,\tau)$ and $P^\natural$ with $L^2(M,\tau)_+$, where τ is the normalized trace on M. Doing this ξ, η become positive, injective, selfadjoint operators affiliated with M, and the equation $\eta = \lambda\xi^{-1}$ makes sense.

Using [6, § 3] the above statement may also be expressed:

If M is a factor not of type III$_1$, and $\xi, \eta \in P^\natural$ are cyclic and separating for M then $Q_\xi = Q_\eta$ if and only if 1) $P_\xi^\sharp = P_\eta^\sharp$ or 2) $P_\xi^\sharp = P_\eta^\flat$.

Received May 21, 1980.

A crucial step in the proof is to show, that $Q_\xi = Q_\eta$ implies that the centralizers M_φ and M_ψ for the vectorfunctionals $\varphi = \omega_\xi$ and $\psi = \omega_\eta$ are equal. For factors of type III_1, the centralizer M_φ gives little information about the functional φ, and that is the reason why our proof fails in this case. However, we are strongly convinced that the above statement is also valid for factors of type III_1.

1. The cone Q_ξ.

Let $(M, H, P^\natural)$ be a σ-finite von Neumann algebra on standard form. For each cyclic and separating vector $\xi \in P$, the natural cone can be recovered from ξ, by the formula

$$P^\natural = (\Delta_\xi^{\frac{1}{4}} M_+ \xi)^- ,$$

where Δ_ξ is the modular operator associated with ξ. The cone $P^\natural$ induces a partial ordering $\leqq$ of the real Hilbert space $H^\natural = \{\zeta \mid J\zeta = \zeta\}$. When $\xi_1, \xi_2 \in H^\natural$ and $\xi_1 \leqq \xi_2$, we let $[\xi_1, \xi_2]$ denote the set

$$[\xi_1, \xi_2] = \{\eta \in H^\natural \mid \xi_1 \leqq \eta \leqq \xi_2\} .$$

Since J coincides with the unitary involution J_ξ obtained from ξ, we have

$$J\Delta_\xi^{\frac{1}{2}} a\xi = a^*\xi, \qquad a \in M .$$

Consider now the cone $Q_\xi = \frac{1}{2}(1+J)P_\xi^\sharp$. Clearly $Q_\xi \subseteqq H^\natural$. Since $\Delta_\xi^{\frac{1}{2}}$ and J coincide on $P_\xi^\sharp = \{a\xi \mid a \in M_+\}^-$, the map $\frac{1}{2}(1+J)$ of $P_\xi^\sharp$ onto Q_ξ, is bounded and has bounded inverse. Since $P_\xi^\sharp$ is closed, it follows that Q_ξ is complete, and hence closed in $H^\natural$. Using that $J\xi = \xi$, one has $(1+J)a\xi = (a+JaJ)\xi$, $a \in M_+$. Therefore

$$(*) \qquad Q_\xi = \{(a+JaJ)\xi \mid a \in M_+\}^- .$$

For any cone K in $H^\natural$ we put $K^\circ = \{\eta \in H^\natural \mid (\eta \mid \xi) \geqq 0, \ \forall \xi \in K\}$. From the Hahn–Banach Theorem one gets easily $K^{\circ\circ} = \bar{K}$. By (*) it follows that for $\eta \in H^\natural$

$$\begin{aligned} \eta \in Q_\xi^\circ &\Leftrightarrow ((a+JaJ)\xi \mid \eta) \geqq 0, \qquad \forall a \in M_+ \\ &\Leftrightarrow (a\xi \mid \eta) + (\eta \mid a\xi) \geqq 0, \qquad \forall a \in M_+ \\ &\Leftrightarrow \omega_{\xi\eta} + \omega_{\eta\xi} \geqq 0 . \end{aligned}$$

Hence we have the following characterization of Q_ξ°:

$$(**) \qquad Q_\xi^\circ = \{\eta \in H \mid \omega_{\xi\eta} + \omega_{\eta\xi} \geqq 0\} .$$

PROPOSITION 1.1.
1) *Q_ξ is the closed cone generated by* $(\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})[0,\xi]$.
2) *Q_ξ° is the closure of* $(\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})^{-1}P^\natural$.
3) $Q_\xi^\circ \subseteqq P^\natural \subseteqq Q_\xi$.

PROOF. 1) The map $a \to \Delta_\xi^{\frac14}(a\xi)$ is a bijection of $\{a \in M_+ \mid 0 \leqq a \leqq 1\}$ onto $[0,\xi]$ (cf. [2, § 3]). In particular $[0,\xi]$ is contained in $D(\Delta_\xi^{-\frac14})$, and since $J\Delta_\xi^{\frac14}J = \Delta_\xi^{-\frac14}$ we have also $[0,\xi] \subseteqq D(\Delta_\xi^{+\frac14})$.

For $a \in M_+$:

$$(a+JaJ)\xi = (1+\Delta_\xi^{\frac12})a\xi = (\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})\Delta_\xi^{\frac14}a\xi .$$

Hence

$$\{(a+JaJ)\xi \mid a \in M_+\} = \bigcup_{\lambda>0} \lambda(\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})[0,\xi] .$$

This proves 1).

2) Note first that $(\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})^{-1}$ is bounded and everywhere defined. Let $\eta \in (\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})^{-1}P^\natural$. We shall prove that $\eta \in Q_\xi^\circ$, i.e. that $(\eta \mid \zeta) \geqq 0$ for all $\zeta \in Q_\xi$. By 1) it is enough to consider $\zeta \in (\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})[0,\xi]$. However, in this case

$$(\eta \mid \zeta) = ((\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})\eta \mid (\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})^{-1}\zeta) \geqq 0 ,$$

because both sides in the last inner product belong to the selfdual cone $P^\natural$. Hence $((\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})^{-1}P^\natural)^- \subseteqq Q_\xi^\circ$. To prove the converse inclusion, put $K = (\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})^{-1}P^\natural$ and assume $\eta \in K^\circ$.

For any $\zeta \in P^\natural$:

$$((\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})^{-1}\eta \mid \zeta) = (\eta \mid (\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})^{-1}\zeta) \geqq 0 .$$

Hence $(\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})^{-1}\eta \in (P^\natural)^\circ = P^\natural$. Since $\Delta_\xi^{-\frac14}(P_\xi^\flat) \subseteqq P^\natural$ one has for every $\zeta \in P^\flat$, that

$$((1+\Delta_\xi^{\frac12})^{-1}\eta \mid \zeta) = ((\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})^{-1}\eta \mid \Delta_\xi^{-\frac14}\zeta) \geqq 0 .$$

Thus by [7, Lemma 15.2], $(1+\Delta_\xi^{\frac12})^{-1}\eta \in P_\xi^\sharp$ or equivalently $\eta \in (1+\Delta_\xi^{\frac12})P_\xi^\sharp = Q_\xi$. This proves that $K^\circ \subseteqq Q_\xi$, and hence $Q_\xi^\circ \subseteqq K^{\circ\circ} = \bar{K}$.

3) For any $\alpha > 0$ the function $f(x) = 1/\cosh(\alpha x)$ is positive definite. In fact

$$\frac{1}{\cosh(\alpha x)} = \frac{1}{2\alpha}\int_{-\infty}^{\infty} e^{ixt}\cosh\left(\frac{\pi t}{2\alpha}\right)^{-1} dt .$$

By spectral theory we can replace x by the selfadjoint operator $\log \Delta_\xi$. Putting $\alpha = \frac14$ we get

$$(\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})^{-1} = \int_{-\infty}^{\infty} \Delta_\xi^{it}\cosh(2\pi t)^{-1}\,dt .$$

As $\Delta_\xi^{it}P^\natural = P^\natural$, $t \in \mathbf{R}$ it follows that $(\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})^{-1}P^\natural \subseteq P^\natural$. Hence by 2) $Q_\xi^\circ \subseteq P^\natural$. Taking the dual cones we get $P^\natural \subseteq Q_\xi$.

For each projection $p \in M$, $F_p = p(JpJ)P^\natural$ is a closed face in $P^\natural$. Moreover each closed face in $P^\natural$ is of the form F_p (cf. [2, Theorem 4.2]). Put $\varphi = \omega_\xi$ (on M). Then for any projection p in M and $t \in \mathbf{R}$:

$$\begin{aligned} F_{\sigma_t^\varphi(p)} &= \Delta_\xi^{it}p\Delta_\xi^{-it}J\Delta_\xi^{it}p\Delta_\xi^{-it}JP^\natural = \Delta_\xi^{it}p(JpJ)P^\natural \\ &= \Delta_\xi^{it}F_p . \end{aligned}$$

In particular the face F_p is Δ_ξ^{it}-invariant if and only if p belongs to the centralizer M_φ of φ.

Proposition 1.2. *Let F be a closed face in $P^\natural$. The following conditions are equivalent*:

1) *F is Δ_ξ^{it}-invariant.*
2) *There exists a vector $\eta \in Q_\xi^\circ$, such that $F = \{\xi \in P^\natural \mid (\xi \mid \eta) = 0\}$.*

Proof. 1) $\Rightarrow$ 2): Let $p \in M$ be the σ_t^φ-invariant projection in M, for which $F = F_p$. Put $\eta = (1-p)J(1-p)J\xi$. Clearly $\Delta_\xi^{it}\eta = \eta$, $t \in \mathbf{R}$. Thus $\eta = 2(\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})^{-1}\eta$, which proves that $\eta \in (\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})^{-1}P^\natural \subseteq Q_\xi^\circ$. Since ξ is cyclic and separating for M, the face in $P^\natural$ generated by ξ is dense in $P^\natural$. Therefore the face in $P^\natural$ generated by $\eta = (1-p)J(1-p)J\xi$ is dense in $F_{1-p} = (1-p)J(1-p)JP^\natural$. Hence

$$\{\zeta \in P^\natural \mid (\zeta \mid \eta) = 0\} = \{\zeta \in P^\natural \mid (\zeta \mid \eta') = 0,\ \forall \eta' \in F_{1-p}\} = F_p = F$$

(cf. [2, § 4]).

2) $\Rightarrow$ 1): Let $\eta \in Q_\xi^\circ$, and put $F = \{\zeta \in P^\natural \mid (\zeta \mid \eta) = 0\}$. We shall prove that F is Δ_ξ^{it}-invariant. Consider the operator

$$T = 1 + 2(\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})^{-1}$$

Clearly $T(P^\natural) \subseteq P^\natural$. Since $1 \leqq T \leqq 2$, $T(P^\natural)$ is a closed subset of $P^\natural$. We will show that $Q_\xi^\circ \subseteq T(P^\natural)$. By Proposition 1.1, (2) it is enough to show that every $\zeta \in Q_\xi^\circ$ of the form $\zeta = (\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})^{-1}\zeta'$, $\zeta' \in P^\natural$, is in $T(P^\natural)$.

However,

$$\begin{aligned} T^{-1}\zeta &= (1+2(\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})^{-1})^{-1}(\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})^{-1}\zeta' , \\ &= (2+\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})^{-1}\zeta' = (\Delta_\xi^{\frac{1}{8}}+\Delta_\xi^{-\frac{1}{8}})^{-2}\zeta' . \end{aligned}$$

Since $1/\cosh(\frac{1}{8}x)$ is a positive definite function on $\mathbf{R}$, we conclude as in the proof of Proposition 1.1 (3), that $(\Delta_\xi^{\frac{1}{8}}+\Delta_\xi^{-\frac{1}{8}})^{-1}P^\natural \subseteq P^\natural$. Therefore $T^{-1}\zeta \in P^\natural$, or $\zeta \in T(P^\natural)$. Hence we have proved that $Q_\zeta^\circ \subseteq T(P^\natural)$. In particular $\eta \in T(P^\natural)$. Put now $\eta' = T^{-1}\eta \in P^\natural$. Since

$$T = 1+2(\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})^{-1} = 1+2\int_{-\infty}^{\infty} \Delta_\xi^{it} \frac{dt}{\cosh(2\pi t)},$$

we get that for any $\zeta \in F$:

$$(\zeta \mid \eta')+2\int_{-\infty}^{\infty} (\zeta \mid \Delta_\xi^{it}\eta')\frac{dt}{\cosh(2\pi t)} = (\zeta \mid \eta) = 0 .$$

However $(\zeta \mid \Delta_\xi^{it}\eta') \geqq 0$, because $\Delta_\xi^{it}\eta' \in P^\natural$. Therefore $(\zeta \mid \Delta_\xi^{it}\eta')=0$, $t \in \mathbf{R}$. Let $s \in \mathbf{R}$. Then

$$\begin{aligned}(\Delta_\xi^{is}\zeta \mid \eta) &= (\Delta_\xi^{is}\zeta \mid \eta')+2\int_{-\infty}^{\infty} (\Delta_\xi^{is}\zeta \mid \Delta_\xi^{it}\eta')\frac{dt}{\cosh 2\pi t)}\\ &= (\zeta \mid \Delta_\xi^{-is}\eta')+2\int_{-\infty}^{\infty} (\zeta \mid \Delta_\xi^{i(t-s)}\eta')\frac{dt}{\cosh(2\pi t)}\\ &= 0 .\end{aligned}$$

Hence $\zeta \in F \Rightarrow \Delta_\xi^{is}\zeta \in F$ for all $s \in \mathbf{R}$, i.e. F is Δ_ξ^{is}-invariant.

Corollary 1.3. *If $Q_\xi = P^\natural$, then $\varphi = \omega_\xi$ is a trace on M.*

Proof. Any closed face F in $P^\natural$ is of the form

$$F = \{\zeta \in P^\natural \mid (\zeta \mid \eta)=0\}$$

for some $\eta \in P^\natural$. Indeed if $F=F_p$ one can use $\eta=(1-p)J(1-p)J\xi$ (cf. proof of 1) $\Rightarrow$ 2) in Proposition 1.2). Hence if $Q_\xi = P^\natural$, we get by Proposition 1.2, that every face in $P^\natural$ is Δ_ξ^{it}-invariant, or equivalently, every projection in M is σ_t^φ-invariant. Hence σ_t^φ is the identity on M.

Corollary 1.4. *Let ξ and η be two cyclic and separating vectors in $P^\natural$. If $Q_\xi = Q_\eta$ the centralizers of $\varphi = \omega_\xi$ and $\psi = \omega_\eta$ coincide.*

Proof. By Proposition 1.2, $Q_\xi = Q_\eta$ implies that a closed face F in $P^\natural$ is Δ_ξ^{it}-invariant iff it is Δ_η^{it}-invariant. Hence the centralizers M_φ and M_ψ for ω_ξ and ω_η must have the same projections, i.e. $M_\varphi = M_\psi$.

2. The equation $Q_\xi = Q_\eta$.

Consider a σ-finite factor M on standard form, and let $\xi, \eta \in P^\natural$ be cyclic and separating vectors for M. In [6, Lemma 3.3 and Lemma 3.4] it is proved that

1) $P^\#_\eta = P^\#_\xi \Leftrightarrow \eta = \lambda\xi$ for a $\lambda \in \mathbf{R}_+$,
2) $P^\#_\eta = P^\flat_\xi \Leftrightarrow M$ is finite, and $\eta = \lambda\xi^{-1}$ for a $\lambda \in \mathbf{R}_+$.

(The interpretation of the equation $\eta = \lambda\xi^{-1}$ was clarified in the introduction.)

Since the projected images of $P^\#_\xi$ and $P^\flat_\xi$ on $H^\natural$ both are equal to Q_ξ, the "if-part" of the following Theorem is immediate:

THEOREM 2.1. *Let M be a factor not of type* III_1, *and let $\xi, \eta \in P^\natural$ be cyclic and separating vectors for M. Then $Q_\xi = Q_\eta$ if and only if, either*

1) $\eta = \lambda\xi$ *for a* $\lambda \in \mathbf{R}_+$,

or

2) *M is finite, and $\eta = \lambda\xi^{-1}$ for a* $\lambda \in \mathbf{R}_+$.

To prove the "only if-part" we need a series of lemmas. In the following M denotes (unless specified) an arbitrary σ-finite von Neumann algebra.

LEMMA 2.2. *Let $\xi, \eta \in P^\natural$ be cyclic and separating vectors for M. If $\varphi = \omega_\xi$ and $\psi = \omega_\eta$ commute then*

1) $\Delta_\xi^{it}\eta = \eta$ *and* $\Delta_\eta^{it}\xi = \xi$, $\quad t \in \mathbf{R}$

2) $\Delta_\xi^{is}\Delta_\eta^{it} = \Delta_\eta^{is}\Delta_\xi^{it}$, $\quad s, t \in \mathbf{R}$.

PROOF. 1) By definition φ and ψ commute iff ψ is σ_t^φ-invariant, or equivalently, φ is σ_t^ψ-invariant (cf. [5]). Hence, when φ and ψ commute, η and $\Delta_\xi^{it}\eta$ induce the same vector-functional on M. As both vectors are in $P^\natural$ it follows that $\Delta_\xi^{it}\eta = \eta$, $t \in \mathbf{R}$ [2, Theorem 2.7 (f)]. Similarly $\Delta_\eta^{it}\xi = \xi$, $t \in \mathbf{R}$. 2) When φ and ψ commute, σ^φ and σ^ψ are commuting automorphism groups. Since by 1) ξ is both Δ_ξ^{is}- and Δ_η^{it}-invariant, we have for $x \in M$, that

$$\Delta_\xi^{is}\Delta_\eta^{it}x\xi \;=\; \sigma_s^\varphi \circ \sigma_t^\psi(x)\xi \;=\; \sigma_t^\psi \circ \sigma_s^\varphi(x)\xi \;=\; \Delta_\eta^{it}\Delta_\xi^{is}x\xi\,.$$

This proves 2), because ξ is cyclic for M.

LEMMA 2.3. *Let $\xi, \eta \in P^\natural$ be* cyclic and separating for M, such that $\varphi = \omega_\xi$ and $\psi = \omega_\eta$ commute. If $Q_\xi = Q_\eta$ then

$$(\Delta_\eta^{\frac14} + \Delta_\eta^{-\frac14})^{-1}(\Delta_\xi^{\frac14} + \Delta_\xi^{-\frac14})[0, \xi] \;=\; [0, \xi]\,.$$

PROOF. By proposition 1.1 (1), Q_ξ is the closure of

$$\bigcup_{\lambda > 0} \lambda(\Delta_\xi^{\frac14} + \Delta_\xi^{-\frac14})[0, \xi]\,.$$

Therefore

$$(*) \qquad (\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})^{-1}Q_\xi \subseteq \left(\bigcup_{\lambda>0} \lambda[0,\xi]\right)^- \subseteq P^\natural .$$

Similarly

$$(\Delta_\eta^{\frac{1}{4}}+\Delta_\eta^{-\frac{1}{4}})^{-1}Q_\eta \subseteq P^\natural .$$

Thus, when $Q_\xi = Q_\eta$

$$(**) \qquad (\Delta_\eta^{\frac{1}{4}}+\Delta_\eta^{-\frac{1}{4}})^{-1}(\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})[0,\xi] \subseteq (\Delta_\eta^{\frac{1}{4}}+\Delta_\eta^{-\frac{1}{4}})^{-1}Q_\xi \subseteq P^\natural .$$

Since ξ is both Δ_ξ^{it} and Δ_η^{it}-invariant by Lemma 2.2, we have

$$(***) \qquad (\Delta_\eta^{\frac{1}{4}}+\Delta_\eta^{-\frac{1}{4}})^{-1}(\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})\xi = (\Delta_\eta^{\frac{1}{4}}+\Delta_\eta^{-\frac{1}{4}})^{-1}2\xi = \xi .$$

Put now

$$A = (\Delta_\eta^{\frac{1}{4}}+\Delta_\eta^{-\frac{1}{4}})^{-1}(\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}}) .$$

From (**) and (***) it follows, that if $\zeta \in [0,\xi]$, then $A\zeta \in P^\natural$ and $\xi - A\zeta = A(\xi-\zeta) \in P^\natural$. Hence $A([0,\xi]) \subseteq [0,\xi]$. We are going to show, that A maps $[0,\xi]$ onto $[0,\xi]$. Let $\zeta \in [0,\xi]$. Put

$$f_n(x) = \exp\left(-\frac{x^2}{2n^2}\right), \qquad n \in \mathbf{N} ,$$

and put

$$\zeta_n = f_n(\log \Delta_\eta)\zeta .$$

Then clearly $\|\zeta_n - \zeta\| \to 0$ for $n \to \infty$. Moreover by spectral theory one gets

$$\zeta_n \in D(\Delta_\eta^{\frac{1}{4}}) \cap D(\Delta_\eta^{-\frac{1}{4}}), \qquad n \in \mathbf{N} .$$

Since

$$f_n(\log \Delta_\eta) = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{n^2t^2}{2}\right)\Delta_\eta^{it}\, dt ,$$

$f_n(\log \Delta_\eta)$ maps $P^\natural$ into itself. Since $\Delta_\eta \xi = \xi$, we have $f_n(\log \Delta_\eta)\xi = f_n(0)\xi = \xi$. Therefore both $\zeta_n = f_n(\log \Delta_\eta)\xi$ and $\xi - \zeta_n = f_n(\log \Delta_\eta)(\zeta - \xi)$ belong to $P^\natural$ that is $\zeta_n \in [0,\xi]$.

For $\zeta' \in P^\natural$ we have

$$((\Delta_\eta^{\frac{1}{4}}+\Delta_\eta^{-\frac{1}{4}})\zeta_n \,|\, (\Delta_\eta^{\frac{1}{4}}+\Delta_\eta^{-\frac{1}{4}})^{-1}\zeta') = (\zeta_n \,|\, \zeta') \geqq 0 .$$

Thus by lemma 1.1 (2)

$$(\Delta_\eta^{\frac{1}{4}}+\Delta_\eta^{-\frac{1}{4}})\zeta_n \in (Q_\eta^\circ)^\circ = Q_\eta = Q_\xi .$$

Using (*) we obtain

$$A^{-1}\zeta_n = (\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})^{-1}(\Delta_\eta^{\frac{1}{4}}+\Delta_\eta^{-\frac{1}{4}})\zeta_n \in P^\natural .$$

The same arguments applied to $\xi-\zeta_n$ gives

$$\xi - A^{-1}\zeta_n = A^{-1}(\xi-\zeta_n) \in P^\natural .$$

Hence $A^{-1}\zeta_n \in [0,\xi]$, or $\zeta_n \in A([0,\xi])$, $n \in \mathbf{N}$. This shows that $A([0,\xi])$ is norm dense in $[0,\xi]$. However, by the proof of Proposition 1.1 (1)

$$(\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})[0,\xi] = \{(a+JaJ)\xi \mid a \in M, \quad 0 \leqq a \leqq 1\} .$$

Therefore this set is weakly compact in H. This implies that

$$A([0,\xi]) = (\Delta_\eta^{\frac{1}{4}}+\Delta_\eta^{-\frac{1}{4}})^{-1}(\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})[0,\xi]$$

is also weakly compact in H. In particular $A([0,\xi])$ is normclosed. Thus $A([0,\xi])=[0,\xi]$.

LEMMA 2.4. *Let $\xi,\eta \in P^\natural$ be cyclic and separating vectors for M such that $\varphi=\omega_\xi$ and $\psi=\omega_\eta$ commute. If $Q_\xi=Q_\eta$ then $\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}}=\Delta_\eta^{\frac{1}{4}}+\Delta_\eta^{-\frac{1}{4}}$.*

PROOF. Following [2, §1] a positive hermitian form s on M is called selfpolar, if the set of functionals $s(\cdot,y)$, $y \in M_+$, is a face in M_+^*. By [2, Theorem 1.3], there is only one selfpolar form s_φ on M, such that $s_\varphi(x,1)=\varphi(x)$, $x \in M$, namely

$$s_\varphi(x,y) = (\Delta_\xi^{\frac{1}{4}}x\xi \mid \Delta_\xi^{\frac{1}{4}}y\xi) .$$

Put now $A=(\Delta_\eta^{\frac{1}{4}}+\Delta_\eta^{-\frac{1}{4}})^{-1}(\Delta_\xi^{\frac{1}{4}}+\Delta_\xi^{-\frac{1}{4}})$ as in the preceeding lemma. By lemma 2.2 (2) the spectral projections of Δ_ξ and Δ_η commute. Therefore A is closable and its closure is again a positive selfadjoint operator. Put now

$$s'(x,y) = (\Delta_\xi^{\frac{1}{4}}x\xi \mid A\Delta_\xi^{\frac{1}{4}}y\xi), \qquad x,y \in M .$$

Then s' is well-defined, because $M\xi \subseteqq D(\Delta_\xi^{\frac{1}{4}})$, and s' is a positive hermitian form on M. Using

$$\Delta_\xi^{\frac{1}{4}}M_+\xi = \bigcup_{\lambda>0} \lambda[0,\xi] ,$$

we get by lemma 2.3, that also

$$A\Delta_\xi^{\frac{1}{4}}M_+\xi = \bigcup_{\lambda>0} [0,\xi] .$$

Thus the set of functionals $x \to s_\varphi(x,y)$, $y \in M_+$ is the same as the set of functionals $x \to s'(x,y)$, $y \in M_+$. However, the first of these sets is a face in

M_+^*. Therefore the set of functionals $s'(.,y)$, $y \in M_+$, is also a face in M_+^*, that is s' is selfpolar. Since $A\xi = \xi$, we have

$$s'(x,1) = (\Delta_\xi^{\frac{1}{4}} x\xi \mid \xi) = (x\xi \mid \xi) = \varphi(x), \quad x \in M .$$

Therefore $s' = s_\varphi$. Thus we have proved that

$$(\Delta_\xi^{\frac{1}{4}} x\xi \mid A\Delta_\xi^{\frac{1}{4}} y\xi) = (\Delta_\xi^{\frac{1}{4}} x\xi \mid \Delta_\xi^{\frac{1}{4}} y\xi), \quad x, y \in M .$$

Since $\Delta_\xi^{\frac{1}{4}} M\xi$ is dense in H, and since A is closable, we have $\bar{A} = 1$. Therefore

$$(\Delta_\eta^{\frac{1}{4}} + \Delta_\eta^{-\frac{1}{4}})^{-1} = A(\Delta_\xi^{\frac{1}{4}} + \Delta_\xi^{-\frac{1}{4}})^{-1} = (\Delta_\xi^{\frac{1}{4}} + \Delta_\xi^{-\frac{1}{4}})^{-1} .$$

This proves the assertion.

We are now going to apply the following recent result of Thaheem and Vanheeswijck:

Lemma 2.5 [9, Theorem 3.8]. *Let α_t and β_t be two strongly continuous one parameter groups of automorphisms on a von Neumann algebra M, such that*

$$\alpha_t + \alpha_{-t} = \beta_t + \beta_{-t}, \quad t \in \mathbf{R}$$

then there exists a central projection p in M, such that $\alpha_t = \beta_t$ on pM, and $\alpha_t = \beta_{-t}$ on $(1-p)M$.

The proof of the above result relies on Arvesons theory on spectral subspaces, and is rather technical. However, we will only need Lemma 2.5 in the case, when α_t and β_t commute, and under this extra assumption, a much simpler proof can be found in Thaheems Thesis [8].

Proposition 2.6. *Let M be a σ-finite factor. Let ξ, η be two cyclic and separating vectors in $P^\natural$ and let φ, ψ be the corresponding vector functionals on M. The following conditions are equivalent*

1) *$Q_\xi = Q_\eta$ and φ commutes with ψ,*
2) *$\Delta_\xi^{it} + \Delta_\xi^{-it} = \Delta_\eta^{it} + \Delta_\eta^{-it}$, $t \in \mathbf{R}$,*
3) *$\Delta_\xi = \Delta_\eta$ or $\Delta_\xi = \Delta_\eta^{-1}$,*
4) *$\eta = \lambda\xi$ for a $\lambda \in \mathbf{R}_+$, or M is finite and $\eta = \lambda\xi^{-1}$ for a $\lambda \in \mathbf{R}_+$.*

Proof. 1) $\Rightarrow$ 2) By Lemma 2.4, 1) implies that $\Delta_\xi^{\frac{1}{4}} + \Delta_\xi^{-\frac{1}{4}} = \Delta_\eta^{\frac{1}{4}} + \Delta_\eta^{-\frac{1}{4}}$. Hence $f(\log \Delta_\xi) = f(\log \Delta_\eta)$ where $f(x) = \cosh(\frac{1}{4}x)$. Since any even function h on $\mathbf{R}$ can be written in the form $h = g \circ f$ for some function g on $[1, \infty[$, it follows that

$$h(\log \Delta_\xi) = h(\log \Delta_\eta)$$

for any even Borel function h on $\mathbf{R}$. Putting $h(x) = \cos(xt)$ we get 2).

2) $\Rightarrow$ 1) By the proof of Proposition 1.1(3) we have

$$\begin{aligned}(\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})^{-1} &= \int_{-\infty}^{\infty} \Delta_\xi^{it}\cosh(2\pi t)^{-1}\,dt \\ &= \tfrac12\int_{-\infty}^{\infty}(\Delta_\xi^{it}+\Delta_\xi^{-it})\cosh(2\pi t)^{-1}\,dt\,.\end{aligned}$$

Hence 2) implies that $(\Delta_\xi^{\frac14}+\Delta_\xi^{-\frac14})^{-1}=(\Delta_\eta^{\frac14}+\Delta_\eta^{-\frac14})^{-1}$. Thus by Proposition 1.1(2) it follows that $Q_\xi^\circ=Q_\eta^\circ$, or equivalently $Q_\xi=Q_\eta$. Moreover

$$\begin{aligned}\|\Delta_\eta^{it}\xi-\xi\|^2 &= ((2-\Delta_\eta^{it}-\Delta_\eta^{-it})\xi\mid\xi) \\ &= ((2-\Delta_\xi^{it}-\Delta_\xi^{-it})\xi\mid\xi) = 0, \qquad t\in \mathbf{R}\,.\end{aligned}$$

Hence ξ is Δ_η^{it}-invariant, which implies that $\varphi=\omega_\xi$ is σ_t^ψ-invariant, i.e. φ and ψ commute.

2) $\Rightarrow$ 3) Assume 2) is valid. By the proof of 2) implies 1) we know that ξ is Δ_η^{it}-invariant. Hence for $x\in M$:

$$\begin{aligned}(\sigma_t^\psi(x)+\sigma_{-t}^\psi(x))\xi &= \Delta_\eta^{it}x\xi+\Delta_\eta^{-it}x\xi \\ &= \Delta_\xi^{it}x\xi+\Delta_\xi^{-it}x\xi = (\sigma_t^\varphi(x)+\sigma_{-t}^\varphi(x))\xi\,.\end{aligned}$$

Since ξ is separating for M, it follows that

$$\sigma_t^\psi(x)+\sigma_{-t}^\psi(x) = \sigma_t^\varphi(x)+\sigma_{-t}^\varphi(x), \qquad x\in M\,.$$

Using that M is a factor, we get by Lemma 2.5 that $\sigma_t^\psi=\sigma_t^\varphi$, $t\in\mathbf{R}$, or $\sigma_t^\psi=\sigma_{-t}^\varphi$, $t\in\mathbf{R}$. Since for $x\in M$:

$$\Delta_\xi^{it}x\xi = \sigma_t^\varphi(x)\xi \qquad\text{and}\qquad \Delta_\psi^{it}x\xi = \sigma_t^\psi(x)\xi$$

it follows that $\Delta_\xi^{it}=\Delta_\eta^{it}$, $t\in\mathbf{R}$ or $\Delta_\xi^{it}=\Delta_\eta^{-it}$, $t\in\mathbf{R}$. This proves 3). Since 3) $\Rightarrow$ 2) is trivial we have now proved (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Finally (3) $\Leftrightarrow$ (4) is contained in [6, Lemma 3.3 and Lemma 3.4].

To finish the proof of Theorem 2.1 we need the following lemma.

LEMMA 2.7. *Let M be a factor not of type* III_1. *If φ and ψ are two positive, normal faithful functionals on M, such that $M_\varphi\subseteq M_\psi$, then φ and ψ commute.*

PROOF. The proof relies on the fact, that for factors not of type III_1,

$M'_\varphi \cap M \subseteqq M_\varphi$ for any positive, normal faithful functional φ on M. Indeed, if M is a factor of type III_λ, $\lambda \in [0,1[$ this is stated in [1, Theorem 4.2.1(a) and Theorem 5.2.1(a)]. If M is semifinite we may write $\varphi = \tau(h\cdot)$, where τ is a normal, faithful, semifinite trace on M, and h is a positive operator affiliated with M_φ. Thus

$$M'_\varphi \cap M \subseteqq \{h\}' \cap M = M_\varphi .$$

It follows now as in the proof of [1, Theorem 4.2.1(b)] that $M_\varphi \subseteqq M_\psi$ implies $\psi = \varphi(k\cdot)$, where k is a positive selfadjoint operator affiliated with the center of M_φ. In particular φ and ψ commute.

Remark. Lemma 2.7 fails of M is of type III_1. In fact, in [4] there is an example of a III_1-factor with a normal state φ such that $M_\varphi = \mathbf{C}\cdot 1$.

End of proof of Theorem 2.1. If $Q_\xi = Q_\eta$ we get by Corollary 1.4 that $M_\varphi = M_\psi$, where $\varphi = \omega_\xi$ and $\psi = \omega_\eta$. Thus when M is a factor not of type III_1, we get by Lemma 2.7 that φ and ψ commute. Theorem 2.1 follows now from proposition 2.6 (1) $\Leftrightarrow$ (4).

Concluding remarks. We are convinced that Theorem 2.1 is also valid in the III_1-factor case. What remains to be prove is, that $Q_\xi = Q_\eta$ implies that ω_ξ and ω_η commute. At present we have been able to show this if M admits a normal state φ_0 (possibly different from both ω_ξ and ω_η), such that $M'_{\varphi_0} \cap M \subseteqq M_{\varphi_0}$. This is the case for III_1-factors coming from the group-measure space construction, and in fact for all known examples of factors of type III_1.

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