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RADON–NIKODYM THEOREM IN SPACES OF MEASURES

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This paper is concerned with the Radon–Nikodým derivative in spaces of measures. The existence of a derivative of a bounded non-negative measure valued measure that is compact in the sense of Marczewski [1] and is absolutely continuous with respect to a complete σ-finite measure is proved. The proof is carried out constructively except for the use of the lifting theorem. Pachl's result ([2, 3.4]) enables us to obtain a derivative with some additional properties. The disintegration theorem of J. K. Pachl ([2, 3.5.]) is essentially a special case of the main theorem.

Throughout we adhere to the terminology and notation of Pachl's paper [2].

The following lemma can be proved in a standard way (cf. [3, Lemma 2.1, Th. 4.1 (ii)]).

LEMMA 1. *Suppose that $\mathscr{K}$ is a semicompact lattice on X, and β is a monotone supermodular function on $\mathscr{K}$. Denote*

$$\mathscr{M} = \{M \subseteqq X \ ; \ (\forall T \subseteqq X) \ \beta_* T = \beta_*(T \cap M) + \beta_*(T - M)\} .$$

Then the following holds:

(1) *β_* is a σ-additive measure on the algebra $\mathscr{M}$.*

(2) *$\mathscr{M} = \{M \subseteqq X \ ; \ \beta X = \beta_* M + \beta_*(X - M)\}$.*

(3) *$\mathscr{M}$ is a σ-algebra on X provided that, in addition, $\mathscr{K} \subseteqq \mathscr{M}$ and $\mathscr{K}$ is closed under countable intersections.*

The following result is proved in Section 3 of [2].

LEMMA 2. *Suppose that $\mathscr{K}$ is a lattice on X, while β is a monotone modular function on $\mathscr{K}$.*

Then there is a monotone modular function γ on $\mathscr{K}$ such that $\gamma \geqq \beta$, $\gamma X = \beta X$ and

$$\gamma K + \gamma_*(X - K) = \gamma X$$

for each $K \in \mathscr{K}$.

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Lemma 3. *Suppose that $\mathscr{K}$ is a semicompact lattice on X that is closed under countable intersections, μ is monotone and modular on $\mathscr{K}$, μ_* is modular on an algebra $\mathscr{M}$ on X. Then $\mu_* \upharpoonright \mathscr{M}$ can be extended to a complete measure λ on a σ-algebra containing $\mathscr{K} \cup \mathscr{M}$ such that $\mathscr{K}$ approximates λ.*

Proof. By the lemma 2, there is a monotone modular function γ on $\mathscr{K}$ such that $\gamma \geqq \mu$ on $\mathscr{K}$, $\gamma X = \mu X$ and $\mathscr{K} \subseteqq \mathscr{B}$, where

$$\mathscr{B} = \{M \subseteqq X \ ; \ \gamma X = \gamma_* M + \gamma_*(X - M)\} .$$

Put $\lambda = \gamma_* \upharpoonright \mathscr{B}$. Then

$$\mu_* M \leqq \gamma_* M \leqq \gamma X - \gamma_*(X - M) \leqq \mu X - \mu_*(X - M) = \mu_* M$$

for each $M \in \mathscr{M}$, $\lambda = \gamma_* = \mu_*$ on $\mathscr{M}$, $\mathscr{M} \subseteqq \mathscr{B}$. By the lemma 1, λ is a complete measure on the σ-algebra $\mathscr{B} \supseteqq \mathscr{K} \cup \mathscr{M}$ such that $\mathscr{K}$ approximates λ.

Definition. Let Σ be a σ-algebra on X. Then $M^+(X, \Sigma)$ denotes the set of all bounded non-negative measures on Σ, $M^+(X)$ denotes the set of all bounded non-negative measures defined on a σ-algebra on X.

Let $\mathscr{A}$ be a σ-algebra on Ω. We say that m: $\mathscr{A} \to M^+(X, \Sigma)$ is a bounded non-negative measure if

$$m\left(\bigcup_{i=1}^{\infty} A_i\right)(B) = \sum_{i=1}^{\infty} m(A_i)(B)$$

for each sequence of pairwise disjoint sets $A_i \in \mathscr{A}$ and for each $B \in \Sigma$.

Theorem. *Suppose that Σ is a σ-algebra on X, $\mathscr{A}$ is a σ-algebra on Ω, μ is a complete σ-finite non-negative measure on $\mathscr{A}$, $\mathscr{K} \subseteqq \Sigma$ is a semicompact lattice on X that is closed under countable intersections. Let m: $\mathscr{A} \to M^+(X, \Sigma)$ be a bounded non-negative measure such that $\mathscr{K}$ approximates $m(\Omega)$.*

Then the following holds:

(1) *There are unique bounded non-negative measures m_a, m_s: $\mathscr{A} \to M^+(X, \Sigma)$ such that $m = m_a + m_s$,*

$$\mu A = 0 \Rightarrow m_a(A)(B) = 0$$

for each $A \in \mathscr{A}$, $B \in \Sigma$, there is a set $E \in \mathscr{A}$ such that

$$\mu(\Omega - E) = 0, \ m_s(E)(B) = 0$$

for each $B \in \Sigma$, and $\mathscr{K}$ approximates $m_a(A)$ for each $A \in \mathscr{A}$.

(2) *There is a mapping T: $\Omega \to M^+(X)$ such that*

$$\int_A T(\cdot)(B)\,d\mu = m_a(A)(B)$$

for each $A \in \mathscr{A}$, $B \in \Sigma$.

(3) *Moreover* T *can be taken in such a way that* $T(\omega)$ *is a complete measure on a* σ*-algebra containing* $\mathscr{K}$, *and* $\mathscr{K}$ *approximates* $T(\omega)$ *for each* $\omega \in \Omega$.

PROOF. (1) By Lebesgue's decomposition theorem, there are bounded non-negative measures m_1, m_2 on $\mathscr{A}$ such that

$$m(\cdot)(X) = m_1 + m_2\,,$$

and m_1 is absolutely continuous and m_2 is singular with respect to μ. Thus there is a set $E \in \mathscr{A}$ such that $\mu(\Omega - E) = 0$, $m_2(E) = 0$. Put

$$m_a(A)(B) = m(A \cap E)(B),\ m_s(A)(B) = m(A - E)(B) \quad \text{for } A \in \mathscr{A},\ B \in \Sigma\,.$$

Then obviously $m = m_a + m_s$,

$$m_a(A)(B) = m(A \cap E)(B) \leqq m(A \cap E)(X) = m_1(A \cap E) \leqq m_1(A)\,,$$

hence $m_a(\cdot)(B)$ is absolutely continuous with respect to μ for each $B \in \Sigma$. Further we have

$$m_s(E)(B) = m(E - E)(B) = m(\varnothing)(B) = 0$$

for each $B \in \Sigma$. Hence m_a is absolutely continuous and m_s is singular with respect to μ. $\mathscr{K}$ approximates $m_a(A)$ for each $A \in \mathscr{A}$, for it holds

$$0 \leqq m_a(A)(B - K) \leqq m(A)(B - K)$$

for each $A \in \mathscr{A}$, $B \in \Sigma$, $K \in \mathscr{K}$.

The uniqueness follows immediately from Lebesgue's decomposition theorem.

(2) By the Radon–Nikodým theorem, there is a finite non-negative function $h \in \mathscr{L}(\Omega, \mathscr{A}, \mu)$ such that

$$\int_A h\,d\mu = m_a(A)(X)$$

for each $A \in \mathscr{A}$. It is easily seen that there are pairwise disjoint sets $\Omega_i \in \mathscr{A}$ such that

$$\mu\Omega_i < \infty,\ \bigcup_{i=1}^{\infty} \Omega_i = \Omega$$

and h is bounded on each Ω_i. Put

$$\mathscr{A}_i = \{A \in \mathscr{A}\ ;\ A \subseteqq \Omega_i\}$$

and let μ_i be the measure μ restricted to the σ-algebra $\mathscr{A}_i$. Then μ_i is a complete measure with $\mu_i \Omega_i < \infty$.

Choose liftings ϱ_i on each $(\Omega_i, \mathscr{A}_i, \mu_i)$ (see [4, IV-Th. 3]). Let Q be the set of all $\mathscr{A}$-measurable functions g such that $|g| \leqq c \cdot h$ for some positive number c. Define a "modified lifting" ϱ on Q by the formula

$$\varrho g(\omega) = \varrho_i(g \upharpoonright \Omega_i)(\omega)$$

for each $\omega \in \Omega_i$ $(i=1,2,\ldots)$, ϱ is well defined, for g is bounded on each Ω_i.

By the Radon–Nikodým theorem, for each $B \in \Sigma$, there is an $\mathscr{A}$-measurable function h_B such that

$$\int_A h_B \, d\mu = m_a(A)(B)$$

for each $A \in \mathscr{A}$. Obviously $0 \leqq h_B \leqq h$ a.e. on Ω (for $0 \leqq \int_A h_B \, d\mu = m_a(A)(B) \leqq m_a(A)(X) = \int_A h \cdot d\mu$), we may assume that $0 \leqq h_B \leqq h$ everywhere on Ω.

Put $\beta_\omega B = \varrho h_B(\omega)$ for each $\omega \in \Omega$, $B \in \Sigma$. Then β_ω is a monotone modular function on Σ. For the sake of brevity we shall simply denote $(\beta_\omega \upharpoonright \mathscr{K})_*$ by $(\beta_\omega)_*$. By the Lemma 1, $(\beta_\omega)_*$is a σ-additive measure on the algebra

$$\mathscr{M}_\omega = \{B \subseteqq X \; ; \; \beta_\omega X = (\beta_\omega)_* B + (\beta_\omega)_*(X-B)\} .$$

Thus $(\beta_\omega)_* \upharpoonright \mathscr{M}_\omega$ can be extended to a measure $T(\omega)$ on a σ-algebra containing $\mathscr{M}_\omega$.

Let $B \in \Sigma$, $A \in \mathscr{A}$. Take $K_n, L_n \in \mathscr{K}$ such that $K_n \subseteqq B \subseteqq X - L_n$ and

$$m_a(A)(K_n) \to m_a(A)(B), \qquad m_a(A)(L_n) \to m_a(A)(X-B) .$$

Then it holds

$$\beta_\omega K_n \leqq (\beta_\omega)_* B \leqq \beta_\omega X - (\beta_\omega)_*(X-B) \leqq \beta_\omega X - \beta_\omega L_n ,$$

$$\int_A \beta_\omega K_n \, d\mu(\omega) = \int_A h_{K_n} d\mu = m_a(A)(K_n)$$

$$\to m_a(A)(B) ,$$

$$\int_A (\beta_\omega X - \beta_\omega L_n) \, d\mu(\omega) = \int_A \beta_\omega(X-L_n) \, d\mu(\omega) = \int_A h_{X-L_n} d\mu$$

$$= m_a(A)(X-L_n) \to m_a(A)(B) .$$

This implies that $(\beta_\omega)_* B = \beta_\omega X - (\beta_\omega)_*(X-B)$, that is $B \in \mathscr{M}_\omega$ for μ-almost all $\omega \in \Omega$, and

$$\int_A (\beta_\omega)_* B \, d\mu(\omega) = m_a(A)(B)$$

for each $A \in \mathscr{A}$, $B \in \Sigma$.

Hence T has the desired properties.

(3) By the lemma 3, $(\beta_\omega)_* \restriction \mathscr{M}_\omega$ can be extended to a complete measure $T(\omega)$ on a σ-algebra containing $\mathscr{K} \cup \mathscr{M}_\omega$ such that $T(\omega)$ is approximated by $\mathscr{K}$.

From the main theorem one can immediately deduce the following disintegration theorem of [2, 3.5].

Corollary. *Let (X, Σ, P) and $(\Omega, \mathscr{A}, \mu)$ be two probability spaces, and let R be a probability measure on $\sigma(\Sigma \otimes \mathscr{A})$ such that*

$$R(X \times A) = \mu A, \quad R(B \times \Omega) = PB$$

for each $A \in \mathscr{A}$, $B \in \Sigma$.

Suppose that μ is complete and P is approximated by a semicompact lattice $\mathscr{K} \subseteqq \mathscr{A}$ that is closed under countable intersections.

Then there are probability spaces $(X, \Sigma_\omega, P_\omega)$, $\omega \in \Omega$, such that $\mathscr{K} \subseteqq \Sigma_\omega$ and

$$\int_A P_\omega B \, d\mu(\omega) = R(B \times A)$$

for each $A \in \mathscr{A}$, $B \in \Sigma$.

Proof. Put $m(A)(B) = R(B \times A)$ for $A \in \mathscr{A}$, $B \in \Sigma$. Then m satisfies the conditions of the preceding theorem and $m = m_a$. Hence there is a mapping $T \colon \Omega \to M^+(X)$ such that

$$\int_A T(\cdot)(B) \, d\mu = m(A)(B) = R(B \times A)$$

for each $A \in \mathscr{A}$, $B \in \Sigma$, where $T(\omega)$ is defined on a σ-algebra $\Sigma'_\omega \supseteqq \mathscr{K}$. Further we have

$$\int_A T(\cdot)(X) \, d\mu = R(X \times A) = \mu A$$

for each $A \in \mathscr{A}$. Thus there is a set $E \in \mathscr{A}$ such that $\mu E = 0$ and $T(\cdot)(X) = 1$ on $\Omega - E$. Take an arbitrary $\omega_0 \in \Omega - E$ and put

$$P_\omega = T(\omega_0), \quad \Sigma_\omega = \Sigma'_{\omega_0} \quad \text{for each } \omega \in E, \text{ and}$$
$$P_\omega = T(\omega), \quad \Sigma_\omega = \Sigma'_\omega \quad \text{for each } \omega \in \Omega - E\,.$$

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