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HECKE OPERATORS AND LAMBERT SERIES

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0. Introduction

By considering the action of Hecke operators on the logarithm of the Dedekind eta function in conjunction with the transformation formula for this function, Knopp [11] proved an extension of an identity for classical Dedekind sums originally proved by Dedekind. In this article, it will be shown that certain Lambert series studied by Apostol [1] are eigenfunctions for certain Hecke operators. Using this information together with the transformation formulae for these Lambert series, an identity for a type of generalized Dedekind sums will be established. This new identity has as a special case an identity previously proved by Carlitz [7].

1. Dedekind's identity and Knopp's extension.

The *classical Dedekind eta function* is defined for $\operatorname{Im}\tau > 0$ by

$$\eta(\tau) = e^{\pi i t/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau}) .$$

The function $\eta(\tau)$ has no zeros in the upper half-plane so it has a single-valued logarithm given by

$$\log \eta(\tau) = \frac{\pi i \tau}{12} - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} e^{2\pi i m n\tau}, \qquad \operatorname{Im}\tau > 0 .$$

Now let

$$\begin{pmatrix} u & w \\ k & h \end{pmatrix} \in SL_2(\mathbf{Z})$$

with $k > 0$. The well-known transformation formula for $\log \eta$ (proved, for instance, in [10]) is

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$$\log \eta\left(\frac{u\tau + w}{k\tau + h}\right) = \log \eta(\tau) + \tfrac{1}{2}\log[-i(k\tau + h)] + \frac{\pi i}{12k}(h+u) - \pi i s(h,k)$$

where $s(h,k)$ is the *classical Dedekind sum* defined as follows. Let

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbf{Z} \\ 0 & \text{if } x \in \mathbf{Z} \end{cases}.$$

Then the classical Dedekind sum is

$$s(h,k) = \sum_{\mu (\mathrm{mod}\, k)} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right).$$

Dedekind's work [8] contains the identity

$$s(ph,k) + \sum_{m=0}^{p-1} s(h+mk, pk) = (p+1)s(h,k) \tag{1.1}$$

for p prime and $k>0$. An elementary proof of (1.1) may be found in Rademacher and Whiteman [17]. Recently, Knopp proved

$$\sum_{\substack{ad=n \\ d>0}} \sum_{b (\mathrm{mod}\, d)} s(ah+bk, dk) = \sigma(n)s(h,k) \tag{1.2}$$

where $k>0$ and $\sigma(n)$ is the sum of the positive divisors of the integer n. It is easy to see that when n is prime (1.2) reduces to (1.1).

To briefly explain the idea behind Knopp's proof of (1.2), it is first necessary to define the Hecke operators. For a function f the *Hecke operators of weight k* are defined by

$$T_k(n)f(\tau) = n^{k-1} \sum_{\substack{ad=n \\ d>0}} \sum_{b (\mathrm{mod}\, d)} d^{-k} f\left(\frac{a\tau + b}{d}\right).$$

In his proof of (1.2), Knopp first showed that for $n \in \mathbf{Z}^+$,

$$T_0(n)\log \eta(\tau) = \frac{1}{n}\sigma(n)\log \eta(\tau) + C(n) \tag{1.3}$$

where $C(n)$ is a constant (depending only on n). Using the transformation formula for $\log \eta(\tau)$ to study the transformation properties of both sides of (1.3), Knopp showed that (1.3) has a consequence the identity (1.2).

2. The Lambert series and the effect of Hecke operators.

Apostol [1] has studied the Lambert series

$$G_q(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m^{-q} x^{mn}$$

for $|x|<1$ and q an odd integer, $q>1$. Let

$$H_q(\tau) = G_q(e^{2\pi i\tau}) .$$

The function $H_q(\tau)$ is an eigenfunction of the Hecke operators of weight $1-q$ as the following theorem demonstrates.

THEOREM 2.1. *Let $T(n)=T_{1-q}(n)$ be a Hecke operator of weight $1-q$ with q and odd integer, $q>1$, and $n \in \mathbf{Z}^+$. Then*

$$T(n)H_q(\tau) = n^{-q}\sigma_q(n)H_q(\tau) \tag{2.2}$$

where $\sigma_q(n)$ is the sum of the q-th powers of the positive divisors of n.

PROOF. To prove (2.2) one first deals with the case when n is prime. Then using an induction argument the identity is proved for prime powers. Finally, using the multiplicativity of the Hecke operators, (2.2) is established for arbitrary n.

When $n=p$, p prime, one has

$$\begin{aligned} T(p)H_q(\tau) &= p^{-q}H_q(p\tau)+p^{-1}\sum_{b=0}^{p-1} H_q\left(\frac{\tau+b}{p}\right) \\ &= p^{-q}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m^{-q}e^{2\pi i mnp\tau} + \\ &\quad +p^{-1}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m^{-q}e^{2\pi i mn\tau/p}\sum_{b=0}^{p-1} e^{2\pi i mnb/p} . \end{aligned}$$

Since

$$\sum_{b=0}^{p-1} e^{2\pi i mnb/p} = \begin{cases} 0 & \text{if } p \nmid mn \\ p & \text{if } p \mid mn \end{cases},$$

the second sum is

$$\begin{aligned} p^{-1}\sum_{\substack{m,n \\ p\mid mn}} pm^{-q}e^{2\pi i mn\tau/p} &= \sum_{\substack{m,n \\ p\mid m}} m^{-q}e^{2\pi i mn\tau/p} + \sum_{\substack{m,n \\ p\mid n}} m^{-q}e^{2\pi i mn\tau/p} - \\ &\quad - \sum_{\substack{m,n \\ p\mid m \text{ and } p\mid n}} m^{-q}e^{2\pi i mn\tau/p} . \end{aligned}$$

However,

$$\sum_{\substack{m,n \\ p\mid m}} m^{-q}e^{2\pi i mn\tau/p} = p^{-q}\sum_{m',n} m'^{-q}e^{2\pi i m'n\tau}, \qquad \text{where } pm'=m ,$$

$$\sum_{\substack{m,n \\ p|n}} m^{-q} e^{2\pi i m n \tau / p} = \sum_{m,n'} m^{-q} e^{2\pi i m n' \tau}$$

where $pn' = n$, and

$$\sum_{\substack{m,n \\ p|m \text{ and } p|n}} m^{-q} e^{2\pi i m n \tau / p} = p^{-q} \sum_{m',n'} m'^{-q} e^{2\pi i m' n' p \tau} .$$

Hence one concludes that

$$T(p)H_q(\tau) = (1+p^{-q})H_q(\tau) = p^{-q}\sigma_q(p)H_q(\tau) ,$$

since $\sigma_q(p) = 1 + p^q$.

To prove (2.2) for prime powers one needs to use the following identity for Hecke operators of weight $1-q$ (see [2] or [9] for a discussion of this and other properties of Hecke operators)

$$T(p^s) = T(p)T(p^{s-1}) - p^{-q}T(p^{s-2}) \tag{2.3}$$

for all positive integers $s \geqq 2$. Now note that $T(p^0)H_q(\tau) = (p^0)^{-q}\sigma_q(p^0)H_q(\tau)$ is trivially true, so that the identity

$$T(p^k)H_q(\tau) = p^{-kq}\sigma_q(p^k)H_q(\tau) \tag{2.4}$$

holds for $k=0$ and $k=1$. To complete the induction argument assume (2.4) holds for $k-1$ and $k-2$. Then from (2.3),

$$\begin{aligned} T(p^k)H_q(\tau) &= T(p)T(p^{k-1})H_q(\tau) - p^{-q}T(p^{k-2})H_q(\tau) \\ &= p^{-q}\sigma_q(p)(p^{k-1})^{-q}\sigma_q(p^{k-1})H_q(\tau) - p^{-q}(p^{k-2})^{-q}\sigma_q(p^{k-2})H_q(\tau) \\ &= p^{-kq}[\sigma_q(p)\sigma_q(p^{k-1}) - p^q\sigma_q(p^{k-2})]H_q(\tau) . \end{aligned}$$

Employing the easily verified identity $\sigma_q(p)(p^m) = \sigma_q(p^{m+1}) + p^q\sigma_q(p^{m-1})$ with $m = k-1$, one obtains (2.4).

To prove (2.2) for arbitrary n, one uses the identity

$$T(mn) = T(m)T(n) \quad \text{for } (m,n) = 1 . \tag{2.5}$$

Employing (2.5) as well as the multiplicativity of σ_q, (2.2) follows immediately for all n from (2.4).

3. Generalized Dedekind sums and some of their properties.

The *Bernoulli polynomials* $B_r(x)$ are defined by

$$ze^{xz}/e^z - 1 = \sum_{r=0}^{\infty} B_r(x)z^r/r! .$$

The *Bernoulli numbers* are given by $B_r = B_r(0)$. One defines the *periodic Bernoulli function of order r* by $\bar{B}_r(x) = B_r(x - [x])$. Now let $q, r \in \mathbf{Z}^+$ with $0 \leqq r \leqq q+1$. The *generalized Dedekind sum* with $h, k \in \mathbf{Z}$, $k > 0$ is

$$c_r(h,k) = \sum_{\mu \pmod{k}} \bar{B}_{q+1-r}\left(\frac{\mu}{k}\right)\bar{B}_r\left(\frac{h\mu}{k}\right).$$

(Note that the dependence of c_r on q is not reflected by the notation.) These generalized Dedekind sums arise in the transformation formulae for the functions $H_q(\tau)$ (given in section 4). They have been studied by many authors including Apostol [3] who proved a reciprocity law for them, Carlitz [5], [6], [7] who has shown that they satisfy many properties analogous to those satisfied by classical Dedekind sums, and K. Barner [4] and H. Lang [13], [14] who have shown that these generalized Dedekind sums arise in the evaluation of certain zeta-functions and L-functions for real quadratic fields at integral points.

Some properties of the sums $c_r(h,k)$ will now be mentioned. First note that $c_r(h+k,k) = c_r(h,k)$. Since $\bar{B}_r(-x) = (-1)^r \bar{B}_r(x)$ one sees that $c_r(h,k) = 0$ if q is even and $0 \leqq r \leqq q+1$. Also $c_r(-h,k) = (-1)^r c_r(h,k)$. One also easily sees that when $(h,k) = 1$,

$$c_0(h,k) = c_{q+1}(h,k) = k^{-q} B_{q+1}.$$

The following simple property of $c_r(h,k)$ has been proved by Carlitz [7; Theorem 3].

LEMMA 3.1. *Let n, q, and r be positive integers with* $0 \leqq r \leqq q+1$. *Then*

$$c_r(nh, nk) = n^{r-q} c_r(h,k).$$

Carlitz [7; Theorem 6] has also proved the following identity in a manner analogous to the proof given by Rademacher and Whiteman for (1.1).

THEOREM 3.2. *Let* $q \in \mathbf{Z}^+$, $r \in \mathbf{Z}$, $0 \leqq r \leqq q+1$, *and let p be prime. Then*

$$\sum_{m=0}^{p-1} c_r(h+mk, pk) = (p + p^{1-q}) c_r(h,k) - p^{1-r} c_r(ph,k). \tag{3.3}$$

In the next section, the following new identity for these generalized Dedekind sums, containing (3.3) as a special case, will be established.

THEOREM 3.4. *Let* $q, r \in \mathbf{Z}$ *with* $q > 1$, $0 \leqq r \leqq q+1$, *and* $n \in \mathbf{Z}^+$. *Then*

$$(3.5)\qquad \sum_{\substack{ad=n\\ d>0}} \sum_{b(\mathrm{mod}\, d)} d^{r-1}c_r(ah+bk, dk) = n^{r-q}\sigma_q(n)c_r(h,k)\,.$$

When n is prime it is easy to see that (3.5) reduces to (3.3).

4. Transformation formulae for $H_q(\tau)$ and a new identity for generalized Dedekind sums.

Apostol [1], Iseki [10], and Mikolás [15] have proved the following transformation formula $H_q(\tau)$, given below in a form convenient for its subsequent use.

THEOREM 4.1. *Let*

$$V = \begin{pmatrix} u & w \\ k & h \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$$

and let q be an odd integer. Then

$$(k\tau+h)^{q-1}H_q\left(\frac{u\tau+w}{k\tau+h}\right) = H_q(\tau)+f_V(\tau)$$

where

$$f_V(\tau) = \tfrac{1}{2}\zeta(q)\left[1-(-1)^{\frac{1}{2}(q-1)}\left(\frac{k\tau+h}{i}\right)^{q-1}\right]+$$
$$+\frac{(2\pi)^q}{2(q+1)!}\left(\frac{k\tau+h}{i}\right)^q \sum_{r=0}^{q+1}\binom{q+1}{r}(k\tau+h)^{-r}(-1)^r c_r(h,k)\,.$$

Note that Theorem 4.1 shows that $H_q(\tau)$ is a modular integral of weight $q-1$ with rational period function $f_V(\tau)$, as discussed in [12] and [16] for instance.

To establish a new identity for generalized Dedekind sums, one evaluates

$$T(n)H_q(V\tau) \qquad \text{for } V = \begin{pmatrix} u & w \\ k & h \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$$

in two different ways. Comparing the results the identity is found.

First, note that from Theorems 2.1 and 4.1 one has

$$(4.2)\qquad \begin{aligned} T(n)H_q(V\tau) &= n^{-q}\sigma_q(n)H_q(V\tau) \\ &= n^{-q}\sigma_q(n)(k\tau+h)^{1-q}(H_q(\tau)+f_V(\tau))\,. \end{aligned}$$

On the other hand, when one begins by using the definition of $T(n)$ and then employs the transformation formula one obtains

$$T(n)H_q(V\tau) = n^{-q} \sum_{\substack{ad=n \\ d>0}} d^{q-1} \sum_{b \pmod d} H_q(M_{b,d}V\tau)$$

where

$$M_{b,d} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

For each $M_{b,d}$ there exists

$$M' = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}, \quad a'd' = n,\ d' > 0\,,$$

such that $M_{b,d}V = V'M'$ where

$$V' = \begin{pmatrix} u' & w' \\ k' & h' \end{pmatrix} = \begin{bmatrix} \dfrac{d'(au+bk)}{n} & \dfrac{-aub'-bb'k+aa'w+ba'h}{n} \\ \dfrac{dd'k}{n} & \dfrac{a'dh-b'dk}{n} \end{bmatrix}$$

with $V' \in \mathrm{SL}_2(\mathbf{Z})$. Furthermore, as the $M_{b,d}$ run through all matrices subject to the conditions $ad=n$, $d>0$, $b \pmod d$, the M' do as well. Hence

$$\begin{aligned} T(n)H_q(V\tau) &= n^{-q} \sum d^{q-1} H_q(V'M'\tau) \\ &= n^{-q} \sum d^{q-1}(k'M'\tau+h')^{1-q}(H_q(M'\tau)+f_V(M'\tau)) \end{aligned}$$

where the unindexed summations here and below are over the set of (a,b,d) with $ad=n$, $d>0$, and $b \pmod d$. Next note that since

$$k'M'\tau+h' = k'\frac{a'\tau+b'}{d'}+h' = \frac{d}{d'}(k\tau+h) \tag{4.3}$$

one has

$$\begin{aligned} T(n)H_q(V\tau) &= n^{-q}(k\tau+h)^{1-q} \sum d'^{q-1}(H_q(M'\tau)+f_{V'}(M'\tau)) \\ &= (k\tau+h)^{1-q}T(n)H_q(\tau)+n^{-q}(k\tau+h)^{1-q} \sum d'^{q-1} f_{V'}(M'\tau) \\ &= n^{-q}\sigma_q(n)(k\tau+h)^{1-q}H_q(\tau)+n^{-q}(k\tau+h)^{1-q} \sum d'^{q-1} f_{V'}(M'\tau) \end{aligned}$$

where Theorem 2.1 has been used. By comparing this expression for $T(n)H_q(V\tau)$ with that of (4.2) it follows that

$$\sigma_q(n) f_V(\tau) = \sum d'^{q-1} f_{V'}(M'\tau)\,. \tag{4.4}$$

To discover an identity for generalized Dedekind sums one next substitutes the definition of $f_V(\tau)$ into (4.4). The left hand side is

$$\sigma_q(n)\left[\tfrac{1}{2}\zeta(q)\left(1-(-1)^{\frac{1}{2}(q-1)}\left(\frac{k\tau+h}{i}\right)^{q-1}\right)+\right.$$
$$\left.+\frac{(2\pi)^q}{2(q+1)!}\left(\frac{k\tau+h}{i}\right)^q\sum_{r=0}^{q+1}\binom{q+1}{r}(k\tau+h)^{-r}(-1)^r c_r(h,k)\right].$$

The right hand side is

$$\sum d'^{q-1}\left[\tfrac{1}{2}\zeta(q)\left(1-(-1)^{\frac{1}{2}(q-1)}\left(\frac{k'M'\tau+h'}{i}\right)^{q-1}\right)+\right.$$
$$\left.+\frac{(2\pi)^q}{2(q+1)!}\left(\frac{k'M'\tau+h'}{i}\right)^q\sum_{r=0}^{q+1}\binom{q+1}{r}(k'M'\tau+h')^{-r}(-1)^r c_r(h',k')\right]$$
$$=\tfrac{1}{2}\zeta(q)\sum d'^{q-1}-\tfrac{1}{2}\zeta(q)(-1)^{\frac{1}{2}(q-1)}\left(\frac{k\tau+h}{i}\right)^{q-1}\sum d^{q-1}+$$
$$+\frac{(2\pi/\mathrm{i})^q}{2(\mathrm{q}+1)!}\sum_{r=0}^{q+1}\binom{q+1}{r}(k\tau+h)^{q-r}(-1)^r\sum d^{q-r}d'^{r-1}c_r(h',k')$$

where (4.3) has been used. To simplify this expression note that

$$\sum_{\substack{ad=n\\d>0}}\sum_{b(\mathrm{mod}\, d)} d'^{q-1} = \sum_{\substack{ad=n\\d>0}}\sum_{b(\mathrm{mod}\, d)} d^{q-1} = \sum_{\substack{ad=n\\d>0}} d^q = \sigma_q(n)\,.$$

Also using Lemma 3.1, one has

$$c_r(h',k') = c_r\left(\frac{a'dh-b'dk}{n},\frac{dd'k}{n}\right)$$
$$= \left(\frac{d}{n}\right)^{r-q} c_r(a'h-b'k,d'k)\,.$$

From this one finds that

$$\begin{aligned}\sum d^{q-r}d'^{r-1}c_r(h',k') &= n^{q-r}\sum d'^{r-1}c_r(a'h-b'k,d'k)\\ &= n^{q-r}\sum d'^{r-1}c_r(a'h+(d'-b')k,d'k)\\ &= n^{q-r}\sum d'^{r-1}c_r(a'h+b'k,d'k)\\ &= n^{q-r}\sum d^{r-1}c_r(ah+bk,dk)\,.\end{aligned}$$

Consequently, the right hand side of (4.4) is

$$\tfrac{1}{2}\sigma_q(n)\zeta(q)-\tfrac{1}{2}\sigma_q(n)\zeta(q)(-1)^{\frac{1}{2}(q-1)}\left(\frac{k\tau+h}{i}\right)^{q-1}+$$
$$+\frac{(2\pi/i)^q}{2(q+1)!}\sum_{r=0}^{q+1}\binom{q+1}{r}(k\tau+h)^{q-r}(-1)^r n^{q-r}\sum_{\substack{ad=n\\d>0}}\sum_{b(\mathrm{mod}\, d)} d^{r-1}c_r(ah+bk,dk)\,.$$

By comparing coefficients of powers of $k\tau+h$ in the two sides of (4.4) one finds for $0 \leqq r \leqq q+1$ that

$$\sum_{\substack{ad=n \\ d>0}} \sum_{b(\bmod d)} d^{r-1} c_r(ah+bk, dk) = n^{r-q} \sigma_q(n) c_r(h,k) .$$

This is the identity of Theorem 3.4. To prove the theorem note that for q odd if $(h,k)=1$ one can find u and w so that

$$\begin{pmatrix} u & w \\ h & k \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$$

and from the above work the theorem is true, and if $(h,k)>1$, the theorem is also valid as is seen by the use of Lemma 3.1. When q is even the theorem is vacuously true, since in this case all the generalized Dedekind sums vanish.

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