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THE DUALS OF GENERIC HYPERSURFACES

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Introduction.

The concept of transversality had its origin in general position arguments in algebraic geometry. Sard's theorem on regular values, Thom's transversality theorem and Mather's refinement have made the concept well understood in the smooth category. The general result is that any transversality condition should be satisfied by an often open always dense subset of the given smooth mappings. The density is proved (see [9]) by showing that we can embed any map in a family almost all of whose members are transverse to the manifold in question.

A natural problem is that of returning to the algebraic case (real or complex) and trying to apply the above results in this setting. One of the major difficulties is that one does not have such a large class of maps as before from which to construct the family mentioned above. Indeed the collection of relevant mappings is usually finite dimensional, and consequently transversality conditions involving higher jets may not be generically satisfied. Thus the question arises: in any given situation what sort of transversality conditions can we ensure? In the complex case the problem is further complicated by the fact that there is no transversality theorem in the complex analytic category.

In this paper we consider this problem in the context of tangent singularities of smooth hypersurfaces i.e. the type of singularity a height function in the normal direction to a hypersurface at a point has at that point. In section 1 we discuss the results available in the smooth category (mainly due to Looijenga) and, in part, show to what extent the height functions are relevant to the differential geometry. In particular we shall find that the dual of the hypersurface and the Gauss map arise naturally in an analysis of these functions. In section 2 we consider analogous problems for real or complex algebraic hypersurfaces in a projective space P^{n+1}. We shall, under certain hypotheses; obtain local structure theorems concerning the duals of general complex algebraic hypersurfaces. Throughout our notation will generally follow that used in [9].

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1. The smooth case.

The basic fact about transversality is the following crucial observation due to R. Thom.

LEMMA 1.1. *Let M, N, P be smooth manifolds, $W \subset P$ a submanifold and F: $N \times M \to P$ a smooth map. If F is transverse to W then for almost all $y \in M$ the map F_y: $N \to P$, $x \mapsto F(x,y)$ is transverse to W.*

PROOF. Consider the natural projection π: $F^{-1}(W) \to M$. By Sards theorem the set of critical values of π has measure zero. But it is easy to see that y is a regular value of π if and only if F_y: $N \to P$ is transverse to W.

Thus for transversality theorems one can prove density of transversal maps by embedding any given map f: $N \to P$ in a family F: $N \times M \to P$ which is transverse to $W \subset P$ (e.g. F a submersion).

Keeping this observation in mind we turn to the situation that interests us here. Let $S^n = S$ denote the unit sphere in R^{n+1} and let N be a smooth manifold. Given an embedding f: $N \to \mathsf{R}^{n+1}$ consider the family of functions

$$N \times S \xrightarrow{H_f} \mathsf{R} \times S \xrightarrow{\pi} S$$

where $H_f(x,a) = (f(x) \cdot a, a)$, $\pi(c,a) = a$ for $(x,a,c) \in N \times S \times \mathsf{R}$ and where the "dot" denotes the usual inner product on R^{n+1}. If $\mathfrak{X} = \bigcup X_j$ is a stratified subset of the multijet space ${}_r\mathsf{J}^k(N,\mathsf{R})$ we say that f is tangent transverse to $\mathfrak{X}$ if the obvious jet map ${}_rj_1^k H_f$: $N^{(r)} \times S \to {}_rJ^k(N,\mathsf{R})$ is transverse to each $X_j \subset \mathfrak{X}$.

THEOREM 1.2 (Looijenga). *Given $\mathfrak{X} = \bigcup X_j$ as above the set of embeddings $f \in C^\infty(N, \mathsf{R}^{n+1})$ with H_f tangent transverse to $\mathfrak{X}$ is residual.*

PROOF. This result is proved in [9 p. 743] (essentually using the idea of Lemma 1). Note that (with the notation of [9]) since in our case $s=0$, $e \neq si(m)$, so we do not need the restriction W invariant under addition of constants.

Note that the critical set ΣH of H_f is $\{(x,a): x$ is a critical point of $H_f(a)$: $N \to \mathsf{R}\}$ which is clearly the unit normal bundle to N. Suppose we now restrict to the case N compact of dimension n. What relevance has the family of functions H (dropping the subscript f) to the study of the differential geometry of $N^n \subset \mathsf{R}^{n+1}$? Well, N separates R^{n+1} and the projection π_1: $\Sigma H \to N$ is a trivial double cover, since one can always select the normal vector pointing out of N. Let the collection of such outward normal vectors be denoted by Σ^+. We define the *dual* of N to be the set of critical values $H(\Sigma^+)$. Here we are using the word dual in the sense of the locus of (oriented) tangent planes of N; given $H(\Sigma^+)$ one can easily recover the set of all tangent planes to N. On the other hand our duals do not have the usual duality property of those in algebraic geometry where for example the dual of a dual of a curve is the original curve. Also the Gauss map is closely associated with H being the composite

$$G\colon N \xrightarrow{\pi_1^{-1}} \Sigma^+ \to \mathbf{R} \times S \xrightarrow{\pi} S\,.$$

As defined the family of functions H depends on the position of N with respect to the origin. However it is easy to check that any translate of N gives an equivalent family of functions in the following sense. Two smooth families of functions $f_i\colon N \times M \to \mathbf{R}$, $i=1,2$ are equivalent if there are diffeomorphisms h_1, h_2, h_3 making the following diagram commute

$$\begin{array}{ccccc} N \times M & \xrightarrow{F_1} & \mathbf{R} \times M & \xrightarrow{\pi} & M \\ \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 \\ N \times M & \xrightarrow{F_2} & \mathbf{R} \times M & \xrightarrow{\pi} & M \end{array}$$

where $F_i(x,y) = (f_i(x,y), y)$ and π is the obvious projection. It follows that there is a diffeomorphism of $\mathbf{R} \times S$ taking the dual of N to the dual of the translate of N. The Gauss map of course is unaltered.

We may now ask various questions. For example what type of height functions H_a should one expect, what does the dual of N look like generically? We divide our discussion into two parts. Initially we shall consider the case $r=1$ in Theorem 1 above; we shall then consider applications of the multitransversality results, needed to obtain information on the dual of N. We shall only consider stratified subsets $\mathfrak{X} = \bigcup X_j \subset J^k(N, \mathbf{R})$ obtained in the obvious way from $\mathscr{R}^{(k)} \times \mathscr{L}^{(k)}$ invariant stratifications of $J^k(n, 1)$. Note that one advantage of considering the hypersurface case is that we can think of the pull back of singular strata X_j as lying on N itself via the projection $\pi_1\colon \Sigma \to N$.

Now to business: first the case $r=1$. It follows from theorem 1 that we will not be able to avoid strata X_j of codimension $\leqq 2n$. For $n \leqq 5$ we have the ideal situation: provided $k \geqq 8$ the jet space $J^k(n,1)$ has a stratification by simple orbits (of sufficient jets) and strata of codim $\geqq 6+n$. Thus for $n \leqq 5$ we expect the height functions to have simple singularities only. Moreover if f is simple with Milnor number μ the corresponding subset of $J^k(N, \mathbf{R})$ will have codimension $n+\mu-1$ so we expect such singularities when $\mu - 1 \leqq n$. If (as in the case just discussed) the local jet map germ

$$j_l^k H \;:\; (N \times S, (x,a)) \to (J^k(n,1), j_l^k H(x,a))$$

$j_l^k H(y,b) = k$ jet of $(z \to H(z,b) - H(y,b))$ at $z=y$ is transverse to the $\mathscr{R}^{(k)}$ orbit of $j_l^k H(x,a)$ we say that the singular jet $j_l^k H(x,a)$ appears transversally on N (provided $j_l^k H(x,a)$ is a k sufficient jet). In this case the dual of a small neighbourhood of $x \in N$ is a familiar object. For

$$H\ :\ (N\times S,(x,a))\to(\mathsf{R}\times S,(H(x,a),a))$$

is a versal unfolding of the germ $H\colon (N,x)\to(\mathsf{R},H(x,a))$ and the dual is consequently locally the discriminant set of the versal unfolding H i.e.

$$(\{(t,b)\in\mathsf{R}\times S : \exists\,(y,b)\in\Sigma \text{ with } H(y,b)=t\};\ (H(x,a),a))\ .$$

The following discussion shows that given any k sufficient jet $j^k g$ (which by the splitting lemma we may suppose is 2 flat) with Milnor number μ this jet does appear transversally as a height function on a manifold of $\mu-1$ in R^μ. If g is a function of r variables $x_1,\ldots,x_r$ since $j^2g=0$ it is clear that $r<\mu$. Let $\{1,x_1,\ldots,x_r,\varphi_{r+1},\ldots,\varphi_{\mu+1}\}$ be a polynomial basis for the complex vector space $\hat{\mathscr{E}}/J_g$ where $\hat{\mathscr{E}}$ is the ring of formal power series in $x_1,\ldots,x_r$ and $J_g\lhd\hat{\mathscr{E}}$ is the Jacobian ideal of g; clearly we may suppose that $\deg\varphi_i\geqq 2$. Set

$$h(x_1,\ldots,x_r,y_{r+1},\ldots,y_{\mu-1})\ =\ g+\sum\pm(y_i+\varphi_i)^2\ .$$

We claim that $z=h(x,y)$ (a hypersurface in R^μ) exhibits $j^k g$ transversally. Locally the map H is given by

$$H\ :\ (\mathsf{R}^{\mu-1}\times\mathsf{R}^{\mu-1},(0,0))\to(\mathsf{R}\times\mathsf{R}^{\mu-1},H(0,0))\ ,$$

$$(x,y,a)\mapsto\left(\sum a_ix_i+\sum a_jy_j-h(x,y),a\right)\ .$$

The local jet map

$$j_l^kH\ :\ (\mathsf{R}^{\mu-1}\times\mathsf{R}^{\mu-1},(0,0))\to(J^k(\mu-1,1),j_l^kH(0,0))$$

has a tangent map at $(0,0)$ and its image is spanned by the vectors $\{x_1,\ldots,x_r,y_{r+1},\ldots,y_{\mu-1},\partial h/\partial x_j,\varphi_i\}$. To show that this is transverse to the $\mathscr{R}^{(k)}$ orbit of $j_l^kH(0,0)$ we have to show that they span $\mathscr{m}/\mathscr{m}J_h$ where $\mathscr{m}<\hat{\mathscr{E}}$ is the maximal ideal. Now the φ_i and x_j span $\mathscr{m}/J_{h'}$ where $h'=g+\sum y_i^2$, so changing co-ordinates by $y_i\mapsto y_i+\varphi_i$ we find that they also span $\mathscr{m}/J_h$. But

$$J_h\ =\ \mathscr{m}J_h+\mathrm{Sp}\left\{\frac{\partial h}{\partial x_i}\right\}+\mathrm{Sp}\left\{\frac{\partial h}{\partial y_i}\right\}$$

so

$$\mathrm{Sp}\{x_i\}+\mathrm{Sp}\{y_i\}+\mathrm{Sp}\left\{\frac{\partial h}{\partial x_i}\right\}+\mathrm{Sp}\{\varphi_i\}+\mathscr{m}J_h$$

$$=\ \mathrm{Sp}\{x_i\}+\mathrm{Sp}\left\{\frac{\partial h}{\partial x_i}\right\}+\mathrm{Sp}\left\{\frac{\partial h}{\partial y_i}\right\}+\mathrm{Sp}\{\varphi_i\}+J_h\ =\ \mathscr{m}$$

as required. Note that in general the hypersurface $\{x_{n+1}=h(x_1,\ldots,x_n)\}\subset\mathsf{R}^{n+1}$ need not display the height function h transversally. For example $x_3=x_1^2\pm x_2^4$ (respectively $x_4=x_1^2+x_2^3+x_3^3$) does *not* display A_3 (respectively D_4)

transversally while $x_3=(x_1+x_2)^2 \pm x_2^4$ (respectively $x_4=x_1^2x_2^3+x_3^3+x_1x_2x_3$) does.

Of course one cannot expect, for $n \geqq 6$, that generically H will versally unfold its height functions i.e. that the height functions will appear transversally on N. This is because for $n \geqq 6$ one finds codimension $2n$ submanifolds of $J^k(n,1)$ consisting of uncountably many orbits. Thus although one can always ensure that $j_1^k H$ is transverse to the submanifold one cannot ensure that $j^k H$ is transverse to the orbits (which have codim $\geqq 2n+1$). In this case instead of stratifying by simple orbits (as we do for $n \leqq 5$) one would give the k sufficient jets in $J^k(n,1)$ the canonical stratification described by Looijenga in [6]. It is not too difficult to see that the simple orbits are strata in the Looijenga stratification (see [1] and [2] for a discussion in a slightly different context) so this stratification does indeed generalize that by simple orbits. Unfortunately the actual nature of this stratification is something of a mystery. The simple orbits are strata for fairly trivial reasons. See the paper [10] of Wall (and following Walls methods [3]) for discussion of the partition of the simple elliptic $\tilde{\mathrm{E}}_6$ (respectively $\tilde{\mathrm{E}}_7$) stratum, where some very strange exceptional values of the modulus are found.

Let us now consider some applications to the differential geometry of $N \subset \mathbf{R}^{n+1}$. One of the crudest invariants of a singular jet is its corank and the $\mathscr{R}^{(2)} \times \mathscr{L}^{(2)}$ orbits

$$X_i = \{j^2 g \in J^2(n,1) : j^1 g = 0, \quad \operatorname{corank} j^2 g = i\}$$

give a manifold partition (indeed a Whitney stratification) of $\Sigma \subset J^2(n,1)$. Theorem 1 shows that for generic $N \subset \mathbf{R}^{n+1}$ there is an open set of points whose normal height function is of type A_1 (i.e. has a Morse singularity) and a "nice" set (a Whitney stratified and hence triangulable set [5]) of codimension 1 whose normal height functions have corank $\geqq 1$. (Codim X_i in $J^2(n,1)$ is $\frac{1}{2}i(i+1)+n$). The following lemma (due I believe to Milnor [7]) shows that this set is the parabolic set of points of N where the Gaussian curvature vanishes. In what follows we orient $\mathbf{R}^{n+1}$ and give N and S the induced orientation using outward normal unit vectors.

Lemma 1.3. *The point $y \in N$ is a regular point of $G\colon N \to S$ with $G(y)=a$ if and only if the height function $H_a\colon N \to \mathbf{R}$ has an A_1 singularity at y. Moreover with respect to correctly oriented co-ordinates at y and a,*

$$\operatorname{sign}\left(\det\left(\frac{\partial^2 H_a}{\partial x_i \partial x_j}\right)\right) = (-1)^n \operatorname{sign}\left(\det\left(\frac{\partial G_i}{\partial x_j}\right)\right).$$

Proof. It is clearly enough to consider the case $a=(0,\ldots,0,1)$ i.e. H_a is the height function x_{n+1}. Clearly we can then write N locally as $x_{n+1}=h(x_1,\ldots,x_n)$ for some smooth $h=H_a$ in terms of local co-ordinates on N. So

$$G(x_1,\ldots,x_n,h(x_1,\ldots,x_n)) = p\left(-\frac{\partial h}{\partial x_1},\ldots,-\frac{\partial h}{\partial x_n},1\right)$$

where p: $\mathsf{R}^{n+1}-0 \to S$ is the map $x \to x\cdot\|x\|^{-1}$, $\|\cdot\|$ the usual Euclidean norm. When restricted to any hyperplane not through the origin p is a local diffeomorphism, so a is a regular value of G if and only if $(-\partial^2 h/\partial x_i\partial x_j)$ is non singular. The assertion about signs follows easily. Note that det $(\partial G_i(y)/\partial x_j)$ is the Gauss curvature of N at y.

Of course this lemma is the first indication of a deeper connection between the singularities of the Gauss map and the height functions (see [9] for further details). Despite its simplicity it can be used to prove a large number of well known results (see for example [4]). We list some below:

(i) For n even the degree of G is $\frac{1}{2}$(Euler characteristic of $N)=\frac{1}{2}\chi(N)$ (Hopf).

(ii) G: $N \to S$ is surjective.

(iii) The curvature $\varkappa$ is >0 somewhere on N. If $\varkappa \neq 0$, N is diffeomorphic to the sphere S.

(iv) For n even if dV is the element of volume on N, $\int_N \varkappa\, dV=\frac{1}{2}\chi(N)$ vol S^n (Gauss Bonnet).

(v) For n even if $H=\{x \in N\colon \varkappa(x)>0\}$ then

$$\left|\int_H \varkappa\, dV\right| \geqq \tfrac{1}{2}(2-\chi(N))\,\mathrm{vol}\, S^n\,.$$

Thus $\chi(N)<2$ implies $\varkappa<0$ somewhere on N.

(vi) For n even if $H_q(N)\neq 0$ for q odd or $H_q(N)$ not free for q even or N not simply connected (H_q singular homology with Z coefficients) then $\varkappa<0$ somewhere on N.

The proofs are fairly straightforward. For example to prove (v) setting $E=\{x \in N : \varkappa(x)>0\}$ we obtain

$$\int_E \varkappa - \left|\int_H \varkappa\right| = \tfrac{1}{2}\chi(N)\,\mathrm{vol}\, S^n$$

by (iv) (an easy corollary of (i)). But every height function has an absolute maximum, which if it is non degenerate has $\varkappa>0$ by the lemma. So $G(E)$ is the complement of a set of measure zero in S^n so $\int_E \varkappa\, dV \geqq \mathrm{vol}\, S^n$ from which the result follows. To prove (vi) suppose $\varkappa \geqq 0$; choosing a generic height function it is easy to show that M can be built up from even dimensional cells only.

The results (v) and (vi) above provide sufficient criteria for the curvature to change sign on an embedded hypersurface of even dimension. The following argument shows that for an open dense subset of the non convex embeddings of $N \to \mathsf{R}^{n+1}$ the curvature changes sign (note $N \neq S$ means any embedding of N is non convex!) Well suppose we choose N tangent transverse to the X_j above (since X_j are Whitney (A) regular over each other and $\bigcup X_j$ is closed this is an open condition see [9]). Then either $\varkappa > 0$ on N whence the embedding $N = S \to \mathsf{R}^{n+1}$ is convex or $\varkappa = 0$ at say $y \in N$. If $b = (0,\ldots,0,1)$ is the normal vector at y the local jet map

$$j_l^2 H\colon (\mathsf{R}^{n+1} \times \mathsf{R}^{n-1}, (0,0)) \to J^2(n-1,1)$$

takes $(y,b) \equiv (0,0)$ to an orbit with some representative

$$\varepsilon_1 x_1^2 + \ldots + \varepsilon_r x_r^2, \qquad \varepsilon_i = \pm 1,\ r \leqq n-1 .$$

It follows that the orbits with representatives

$$(\varepsilon_1 x_1^2 + \ldots + \varepsilon_r x_r^2 + x_{r+1}^2 + \ldots + x_{n-1}^2 + \varepsilon x_n^2), \qquad \varepsilon = \pm 1 ,$$

contains $j_l^2 H(0,0)$ in their closure. Since $j_l^2 H$ is transverse to the $j_l^2 H(0,0)$ orbits this means that points arbitrarily near y have normal height functions with singularities of type

$$\varepsilon_1 x_1^2 + \ldots + \varepsilon_r x_r^2 + x_{r+1}^2 + \ldots + x_{n-1}^2 + \varepsilon x_2^2 ,$$

which are 2 determined. In particular it follows from the lemma above that $\varkappa$ changes sign near y.

To discuss the dual of N we need to describe the results one can obtain from Theorem 1 by applying it to multijet spaces. We shall be concerned with stratifications of the jet spaces ${}_rJ^k(N,\mathsf{R})$ constructed as follows. If $\mathfrak{X} = \bigcup X_i$ is some natural stratification of $J^k(N,\mathsf{R})$ (as described above) give ${}_rJ^k(N,\mathsf{R}) \subset (J^k(N,\mathsf{R}))^r$ the product stratification. There is a natural projection

$$\Pi^r : {}_rJ^k(N,\mathsf{R}) \to \mathsf{R}^r,\ \Pi^r(j^k f_1,\ldots,j^k f_r) = (\text{target}\, j^k f_1,\ldots,\text{target}\, j^k f_r)$$

which is a submersion restricted to strata. Give R^r the natural "diagonal" stratification $\mathscr{D}(\mathsf{R}^n) = \bigcup D_i$ i.e. so that with the natural action of the symmetric group $S(r)$ on R^r by permutation of co-ordinates two points are in the same stratum if and only if they have the same isotropy group. New refine the product stratification of ${}_rJ^k(N,\mathsf{R})$ by taking intersections with the $(\Pi^r)^{-1}D_i$. We denote the resulting stratification by $\mathfrak{X}^{(r)} = \bigcup X_j^{(r)}$. The use of this stratification is best explained in terms of *regular intersections*. The following discussion is a watered down version of one given in [6].

Let $\mathfrak{X} = \bigcup X_i$ be some natural stratification of $\Sigma \subset J^k(N,\mathsf{R})$. By Theorem 1 for generic embeddings $f\colon N \to \mathsf{R}^{n+1}$ we can pull back this stratification to one of $\Sigma H \subset N \times S$, which we denote by $\mathfrak{X}_f$. Now $H\colon N \times S \to \mathsf{R} \times S$ is said to have regular intersections with respect to the stratification $\mathfrak{X}_f \subset N \times S$ if the r fold product $H^r\colon (N \times S)^{(r)} \to (\mathrm{R} \times S)^r$ is transverse to the diagonal stratification of $(\mathrm{R} \times S)^r$ when restricted to the product stratification $\mathfrak{X}_f \times \ldots \times \mathfrak{X}_f \subset (N \times S)^{(r)}$ for some $r \geqq n+2$. If for any $(t,a) \in \mathrm{R} \times S$,

$$\Sigma(t,a) = \{(x,a) \in \Sigma H : H(x,a) = (t,a)\},$$

then H has regular intersections if and only if $\Sigma(t,a)$ is finite and the planes $\{(T_{(x,a)}H(T_{(x,a)}X_{(x,a)}))\}_{(x,a)\in\Sigma(t,a)}$ (where $X_{(x,a)}$ is the stratum containing (x,a)) are in general position in $T_{(t,a)}(\mathrm{R} \times S)$. We say that $H\colon N \times S \to \mathrm{R} \times S$ is multi transverse to $\mathfrak{X}$ if

(i) the jet map $j^kH\colon N \times S \to J^k(N,\mathrm{R})$ is transverse to the X_j *of* $\mathfrak{X}$,

(ii) the mapping $H\colon N \times S \to \mathrm{R} \times S$ has regular intersections with respect to $\mathfrak{X}_f$.

Proposition 1.4 (Looijenga [6]). *The following are equivalent*:

(i) *H is multi transverse to $\mathfrak{X}$.*

(ii) ${}_rj^kH\colon N^{(r)} \times S \to {}_rJ^k(N,\mathrm{R})$ *is transverse to* $\mathfrak{X}^{(r)}$.

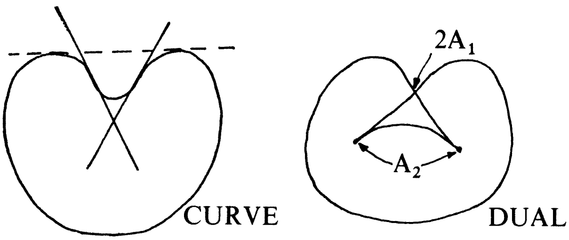
We note that for strata $X \subset \Sigma \subset J^k(N,\mathrm{R})$ of the Looijenga stratification the map $H\colon (j^kH)^{-1}(X) \to \mathrm{R} \times S$ is an immersion, provided H is transverse to the stratification. So the regular intersections condition above means that the corresponding immersed submanifolds of $\mathrm{R} \times S$ all meet in general position.

We now apply these results to the consideration of examples in low dimensions. In particular we note that if $\chi_1,\ldots,\chi_r$ are types of simple singularities in $J^k(n,1)$ (k sufficiently large) the set $\{\chi_1,\ldots,\chi_r\} = \{(j^kf_1,\ldots,j^kf_r)$: target $j^kf_1 =$ target j^kf_i, $1 \leqq i \leqq r$, j^kf_i is of type $\chi_{j(i)}$ for some reordering $j(i)$ of $1,\ldots,r\}$ is one stratum in the Looijenga stratification of ${}_rJ^k(N,\mathrm{R})$. A short computation shows that it has codimension $rn + \Sigma\mu_i - 1$. In what follows $\mathrm{A}_1(p)$ is the set of functions $\mathscr{R} \times \mathscr{L}$ equivalent to $x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+1}^2$ (note $\mathrm{A}_1(p) = \mathrm{A}_1(n-p)$).

Examples 1.5. The duals.

(a) $n=1$. For generic plane curves there is an open subset of non parabolic points where the tangent does not cross the curve, and a finite number of parabolic or flex points where the tangent does cross the curve. Considering multi jets we find that there are a finite number of double tangents (both points non parabolic). The corresponding duals are curves with cusps or ordinary double points as singularities. (See dig. 1.)

b) $n=2$. Here there are two types of A_1 singularity $\mathrm{A}_1(1)$ (saddle), $\mathrm{A}_1(2)$ (max/min). The open set of points for which the tangent plane meets the surface in an $\mathrm{A}_1(1)$ singularity are the hyperbolic points, so named because a

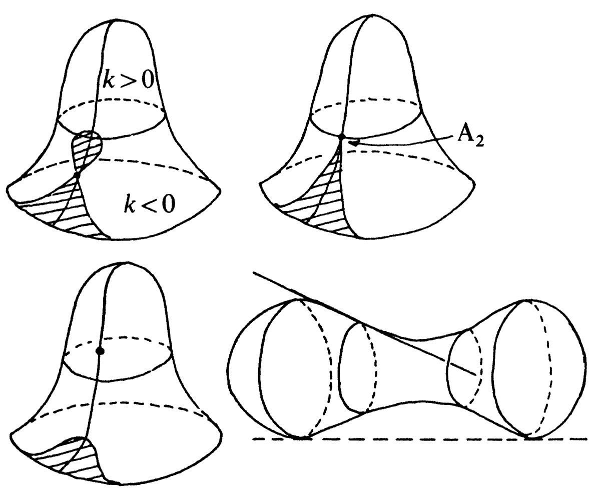


dig. 1.

small deformation of the tangent plane meets N locally in a hyperbola; they are also the points where $\varkappa<0$. Type $A_1(2)$ appear at elliptic points where $\varkappa>0$. Note that both types occur if $N \neq S^2$.

There are three types of parabolic points where $\varkappa=0$. Those of type A_2, where the tangent planes meet N in a cusp, form a curve on N. On the parabolic curve (which in this case $(n=2)$ is smooth) there will be a discrete set of points where the tangent planes meet N in a "real" tacnode $A_3(1)$ $(x^2-y^4=0)$ or an "imaginary" tacnode $(x^2+y^2=0)$ $A_3(2)$. Considering multijets we find that there are curves of double tangents whose tangency points are of type (i) $A_1(1)$, $A_1(1)$; (ii) $A_1(1)$, $A_1(2)$; (iii) $A_1(2)$, $A_1(2)$. There are a discrete set of double tangents whose tangency points are of type (iv) $A_1(1), A_2$; (v) $A_1(2), A_2$. Finally there are a finite number of triple tangents of types (vi) $A_1(1), A_1(1), A_1(1)$; (vii) $A_1(1), A_1(1), A_1(2)$; (viii) $A_1(1), A_1(2), A_1(2)$; (ix) $A_1(2), A_1(2), A_1(2)$. (See dig. 2.)

The corresponding dual singularities are cusp $\times$ line, double point $\times$ line, triple points and dovetails (corresponding to $A_3(1)$ and $A_3(2)$). (See dig. 3.)



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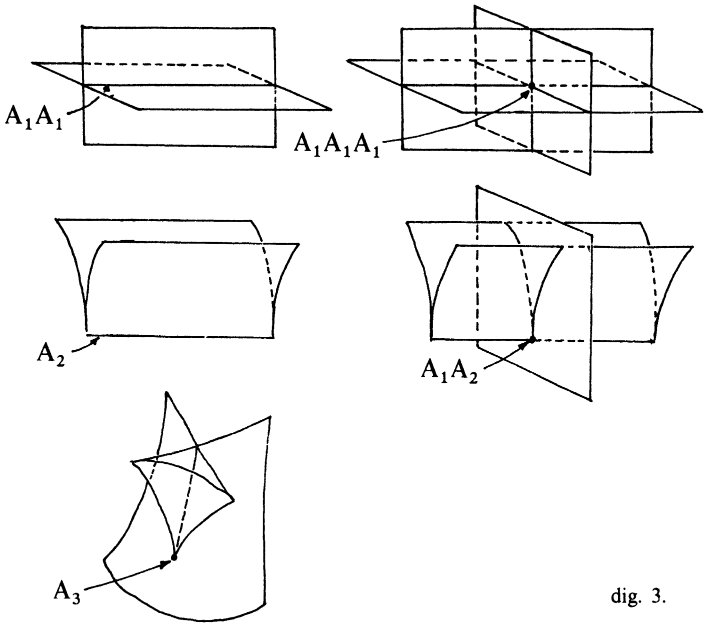
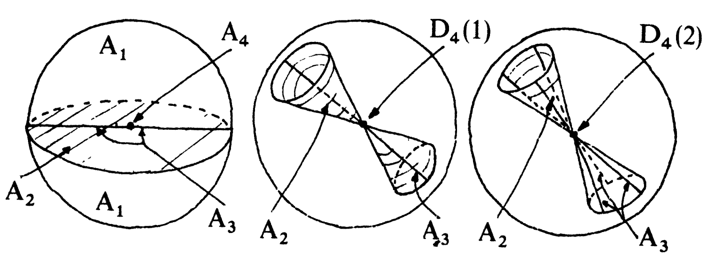


dig. 3.

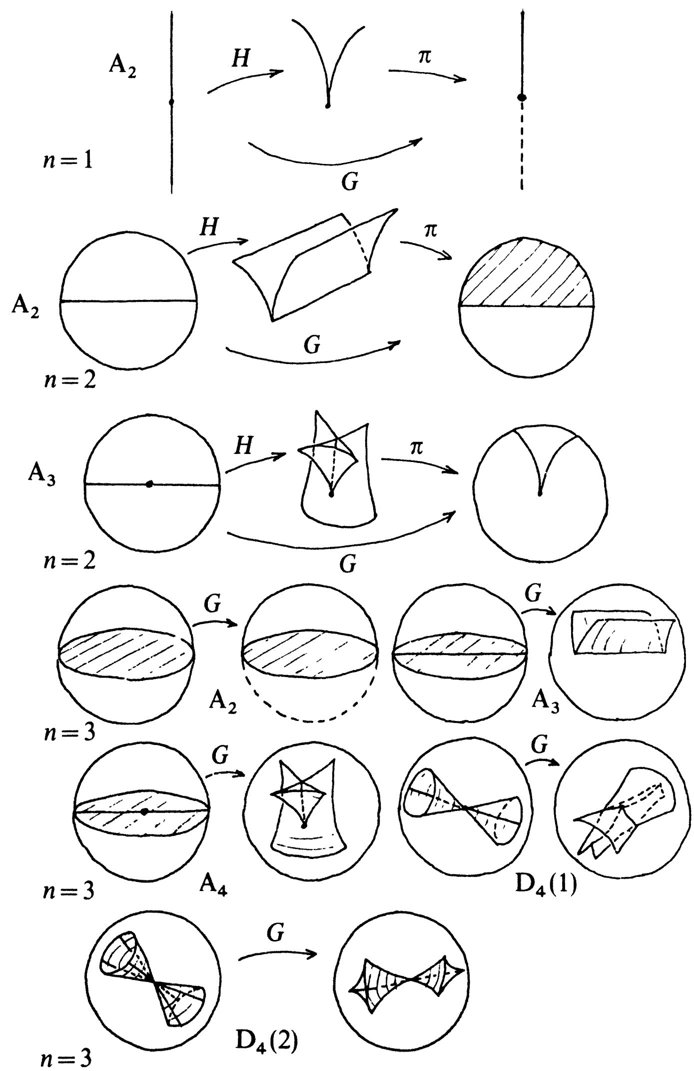
(c) $n=3$. Again there are two types of A_1, $A_1(1)$ and $A_1(3)$. There are also two types of A_2, $A_2(1)$: $x^2+y^2+z^3$, $A_2(2)$: $-x^2+y^2+z^3$, and three types of A_3, $A_3(1)$: $x^2+y^2+z^4$, $A_3(2)$: $-x^2+y^2+z^4$, $A_3(3)$: $-x^2-y^2+z^4$. Of course the quadratic terms are irrelevant as far as the local picture in N and the dual are concerned. There are two new singularities A_4 and D_4 which appear in this dimension, both have two types $A_4(1)$: $x^2+y^2+z^5$, $A_4(2)$: $-x^2+y^2+z^5$, $D_4(1)$: $x^2+y^3-z^3$, $D_4(2)$: $x^2+yz(y-z)$. The induced stratification of N has the same sort of picture at an A_4 as at A_2 and A_3; in particular the parabolic set is locally smooth (see dig. 4.) The pictures for $D_4(1)$ and $D_4(2)$ are more interesting: unfortunately we can offer no pictures for the dual!

EXAMPLES 1.5. The Gauss maps. The discussion of regular intersections given above is not as general as that in Looijengas thesis [6] where account is made



dig. 4.

for the projection map Π: $\mathbb{R}^{(r)} \times S \to S$. One can (and Looijenga does) redo the above with regular intersections relative to Π, and Theorem 1.2 then allows a description (at least for $n \leqq 5$) of the local singularities of Gauss maps. Wall proves in [9] that they are generic singularities of Lagrange maps, and they correspond in low dimensions with the elementary catastrophe maps of Thom. (Of course they can be considered as the composite of the dual map with projection to a suitable codimension 1 subspace.) Below we list the types and draw pictures of the singular points ΣG and singular values $G(\Sigma G)$ of generic Gauss maps G: $M^n \to S^n$ for $n \leqq 3$. Note that for $n \leqq 2$ the types are exactly the



dig. 5.

stable maps, but for $n=3$ we obtain two generic Lagrange germs which are not stable as maps (see dig. 5). The set of singular values $G(\Sigma G)$ is locally that subset of the bifurcation variety of the height function concerned with degenerate (non A_1) singular points.

As Terry Gaffney pointed out, when $n=2$ the diagram for the A_3 case shows that in any neighbourhood of a point which gives rise to a cusp of the Gauss map there are two points which share the same tangent plane, and conversely. (For $n=3$, ΣG and $G(\Sigma G)$ are shaded in dig. 5.) Before turning to the algebraic case we note the following consequence of Looijengas work in [6], firstly because it is such a beautiful result, and secondly because it is not stated explicitly in [6] or [9]. It is the following stability theorem:

THEOREM 1.6 (Looijenga [6]), *For an open dense subset of embeddings* $D \subset \mathrm{Emb}^{\infty}(N^n, \mathbf{R}^{n+1})$, $f \in D$ *means that we can find a neighbourhood* W *of* f *such that for* $g \in W$ *there are homeomorphisms* h_1, h_2, h_3 *making the following diagram commute*

$$\begin{array}{ccccc} N \times S & \xrightarrow{H_f} & \mathbf{R} \times S & \xrightarrow{\Pi} & S \\ {\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle h_3} \\ N \times S & \xrightarrow[H_g]{} & \mathbf{R} \times S & \xrightarrow{\Pi} & S \end{array}$$

where H_f, H_g *and* Π *are as above. Moreover the construction of* h_1, h_2, h_3 *is such that* h_1 *maps* ΣH_f *homeomorphically to* ΣH_g *and* h_2 *maps* $H_f(\Sigma H_f)$ *homeomorphically to* $H_g(\Sigma H_g)$. *Consequently the duals and Gauss maps of* $f(N)$ *and* $g(N)$ *are topologically equivalent.*

2. The algebraic case.

In this section we shall prove some results similar to those obtained in section 1, but for real and complex projective algebraic hypersurfaces. Below the symbol K will denote the field of real or complex numbers, $\mathbf{P}^{n+1}$ real or complex projective $n+1$ space, and K^M the vector space of homogeneous polynomials of degree d in $n+2$ variables $\left(M=\binom{n+d+1}{d}\right)$. Let $D \subset K^M$ be the algebraic subset of forms defining singular hypersurfaces, and let $F \in K^M - D$. If

$$\Gamma_F = \{(x, L) \in \mathbf{P}^{n+1} \times \mathbf{P}^{n+1} : x \in L \cap \{F=0\}\},$$

and $\Pi_F\colon \Gamma_F \to \mathbf{P}^{n+1}$ is the obvious projection the singular set of Π_F, $\Sigma\Pi_F = \{(x, L)\colon L$ is the tangent plane to $\{F=0\}$ at $x\}$, and the set of critical values of Π_F, $\Pi_F(\Sigma\Pi_F)$ is the dual of $\{F=0\}$. (Since the obvious projections $\mathbf{P}^a \times \mathbf{P}^b \to \mathbf{P}^a$ are Zariski closed when $K=\mathbf{C}$ in this case the duals are algebraic). We want eventually to discuss the structure of the duals of "generic" hypersurfaces $\{F=0\}$, but the maps Π_F are not really well suited to our purpose. So we imitate the methods of section 1 where we used our better understanding of the singularities of functions (as contrasted with mappings).

We first discuss ordinary jet transversality (without multijets). Of course since we are working in projective space we have to be careful: height functions are Euclidean objects. However transversality is a local phenomenon so we can, and do, give a local definition of tangent transversality. To do this we have to select an affine chart in $\boldsymbol{P}^{n+1}$ and of course it is essential that our definition of tangent transversality is independent of the chart and any other choices required. As in section 1 we shall only consider natural singular submanifolds of the jet space arising from $\mathscr{R}^{(k)} \times \mathscr{L}^{(k)}$ invariant submanifolds $X_j \subset \Sigma \subset J^k(n, 1)$, and definition of tangent transversality will ensure that for a non singular hypersurface $V=\{F=0\}$ tangent transverse to X_j the set $\{x \in V$: normal height function at x is of type $X_j\}$ is a smooth submanifold of V of the correct codimension. Now to the definition.

Let $x \in V$ and choose an affine chart $K^{n+1} \subset \boldsymbol{P}^{n+1}$ containing x, set $V_1 = V \cap K^{n+1}$; without loss of generality we suppose that $x=0 \in K^{n+1}$; one easily checks that the family of functions to be constructed is independent of the position of x in K^{n+1}. The irreducible affine variety V_1 will be the zero set of some irreducible polynomial f and we can choose a local parameterization of V_1 at 0, $h: K^n \rightarrowtail V_1$ (so $f \circ h(\alpha) \equiv 0$). If $(K^{n+1})^*$ is the dual of K^{n+1} let $df(0) \in (K^{n+1})^*$ denote the differential of f and 0 and choose a linear map $L: K^n \rightarrow (K^{n+1})^*$ so that

$$\operatorname{Im} L \oplus \operatorname{Sp}\{df(0)\} = (K^{n+1})^* .$$

We define a germ of an n parameter family of functions $H_a: K^n \rightarrow K$ by

$$H(f, h, L): (K^n \times K^n, 0) \rightarrow (K, 0)$$

$$(\alpha, a) \mapsto (df(0) + L(a))(h(\alpha)) .$$

Lemma 2.1. (1) *H is independent of the choices of equation f, parameterization h and linear map L* i.e. *any two choices give equivalent families.*

(2) *If $\Sigma H = \{(\alpha, a) : \alpha$ is a critical point of $H_a\}$ then ΣH is smooth and the projection $\Sigma H \rightarrow K^n$, $(\alpha, a) \rightarrow \alpha$ is a local diffeomorphism.*

Proof. (1). (a) Since V_1 is an irreducible affine variety any other irreducible equation defining V_1 is of the form $\lambda f = 0$ for some $\lambda \in K - \{0\}$. If we change coordinates by $\Psi(\alpha, a) = (\alpha, \lambda a)$, $\Phi(c, a) = \lambda c$ we find that

$$\Phi \circ H(f, h, L)(\alpha, a) = \lambda(df(0) + L(a))(h(\alpha)) = H(\lambda f, h, L) \circ \Phi(\alpha, a) .$$

(b) Any two parameterizations h_1, h_2 of V_1 at 0 differ by a diffeomorphism $h: (K^n, 0) \circlearrowleft$ that is $h_2 = h_1 \circ h$. A change of co-ordinates $\Psi(\alpha, a) = (h(\alpha), a)$ gives $H(f, h_1, L) \circ \Psi = H(f, h_2, L)$.

(c) Suppose we have two linear maps L_1, L_2 as above. We can write $L_1(a) = \theta(a)df(0) + \varphi(a)$ for some linear maps $\theta: K^n \to K$, $\varphi: K^n \to \operatorname{Im} L_2$. Note that φ is an isomorphism. Changing co-ordinates by

$$\Psi(\alpha, a) = \big(\alpha, L_2^{-1}(\varphi(a)/(1+\theta(a)))\big), \qquad \Phi(c, a) = \big(c/(1+\theta(a))\big)$$

we have

$$\begin{aligned}\Phi(H(f, h, L_1)(\alpha, a), a) &= \Phi\big((df(0) + L_1(a))(h(\alpha)), a\big) \\ &= (1+\theta(a))^{-1}\big((df(0)(1+\theta(a)) + \varphi(a))(h(\alpha))\big) \\ & (df(0) + \varphi(a)\cdot(1+\theta(a))^{-1})h(\alpha) ,\end{aligned}$$

while

$$\begin{aligned}H(f, h, L_2) \circ \Psi(\alpha, a) &= H(f, h, L_2)\big(\alpha, L_2^{-1}(\varphi(a)/(1+\theta(a)))\big) \\ &= \big(df(0) + \varphi(a)\cdot(1+\theta(a))^{-1}\big)h(\alpha)\end{aligned}$$

as required.

(2) The critical set

$$\Sigma = \{(\alpha, a) : (T_\alpha h)^*(df(0) + L(a))\} = 0$$

where $(T_\alpha h)^*: (K^{n+1})^* \to (K^n)^*$ is the dual of $T_\alpha h$. Writing

$$\xi(\alpha, a) = (T_\alpha n)^*(df(0) + L(a))$$

we find that $\partial\xi/\partial a(0,0) = (T_0 h)^* L: K^n \to (K^n)^*$ is invertible so by the implicit function theorem we can parameterize Σ as $(\alpha, \beta(\alpha))$ for some β whence the result.

Given $(\alpha, a) \in \Sigma$ we say that the function $\alpha \to H(\alpha, a)$ is the *normal height* function at α. Note that if we choose a co-ordinate system $x_1, \ldots, x_{n+1}$ for K^{n+1} there is an isomorphism $x^*: K^{n+1} \to (K^{n+1})^*$ given by $y \to \Sigma x_i(y) x_i$. Given a co-ordinate system x we can always choose the linear map $L: K^n \to (K^{n+1})^*$ to be $L = x^* \circ T_0 h$. We now drop the brackets (f, h, L) and consider the map

$$\begin{aligned} j_1^k H_x : (K^n \times K^n, (0,0)) &\to (J^k(K^n, K), j_1^k H(0,0)) \\ (\alpha, a) &\mapsto j^k H_a(\alpha) .\end{aligned}$$

DEFINITION 2.2. *A projective hypersurface $V \subset \boldsymbol{P}^{n+1}$ is said to be tangent transverse to the $\mathscr{R}^{(k)} \times \mathscr{L}^{(k)}$ imvariant stratification $\mathfrak{X} = \bigcup X_j \subset \Sigma \subset J^k(n, 1)$ if for* *each* $x \in V$ *we can choose an affine chart and a neighbourhood* $A_1 \times A_2$ *of* $(0,0) \in K^n \times K^n$ *so that the map* $j_1^k H_x \colon A_1 \times A_2 \to J^k(K^n, K)$ *is transverse to the corresponding submanifold* $\tilde{X}_j \subset J^k(K^n, K)$.

It follows from Lemma 2.1 above that if $j_1^k H_x \colon A_1 \times A_2 \to J^k(K^n, K)$ is transverse to X_j then for y sufficiently close to x, $j_1^k H_y$ will also be transverse to X_j. The following proposition also follows from Lemma 2.1.

PROPOSITION 2.3. *If* X *is an* $\mathscr{R}^{(k)} \times \mathscr{L}^{(k)}$ *invariant submanifold of* $\Sigma \subset J^k(n,1)$ *and* $V \subset \boldsymbol{P}^{n+1}$ *is tangent transverse with respect to* X *the set*

$$V(X) = \{x \in V\colon \text{the normal height function at } x \text{ is of type } X\}$$

is smooth and codim $V(X)$ *in* $V =$ codim *of* X *in* Σ.

We want to show that those hypersurfaces $\{F=0\}$ which are *not* tangent transverse to some invariant $X \subset \Sigma$ are scarce. To obtain a strong result we shall assume that in the real case X is semialgebraic, in the complex case X is constructible (no real restriction in practice). We can then prove.

THEOREM 2.4. *Each* $F \in K^M - D$ *has a neighbourhood* $U \subset K^M - D$ *and in case* $K = \mathsf{R}$ *(respectively* $K = \mathsf{C}$*) a subanalytic subset* (respectively *real subanalytic subset*) $B \subset U$ *of real codimension* $\geqq 1$, *such that for all* $G \in U - B$ *the hypersurface* $\{G=0\}$ *is tangent transverse to* $X \subset J^k(n,1)$ *provided* $k \leqq d$.

We shall prove a corresponding multi transversality result later on. Using the fact that $\{F=0\} \subset \boldsymbol{P}^{n+1}$ is compact the proof of Theorem 2 follows from the following assertion.

ASSERTION. *For each* $F \in K^M - D$, *and* $x(0) \in \{F=0\}$ *we can find neighbourhoods* A *(respectively* U*) of* $x(0) \in \boldsymbol{P}^{n+1}$ *(respectively* $F \in K^M - D$*) and a bad set* $B \subset U$ *as above, so that for* $G \in U - B$ *and* $y \in A \cap \{G=0\}$ *the map germ* $j_1^k H_y$ *(for* $\{G=0\}$*) is transverse to* $\tilde{X}$.

Of course to prove this assertion we shall use Lemma 1.1 of Thom. In this case however the family of deformations are obtained by varying the hypersurfaces $\{G=0\}$.

PROOF OF ASSERTION. Without loss of generality let us suppose that $x(0) = (1:0:0:\ldots:0)$ and its tangent plane is $x_1 = 0$; we use the obvious affine chart at $(1:0:\ldots:0) \in \boldsymbol{P}^{n+1}$.

Consider the map $g\colon K^m \times K^n \to K$, $g(u,x) = F(1,x) + \sum_1^M u_i \varphi_i(1,x)$, where

$\varphi_1, \ldots, \varphi_M$ are basis monomials for the vector space of homogeneous forms in $n+2$ variables of degree d. For a fixed $u \in K^M$ the set $g(u,x)=0$ is the affine part of a hypersurface in $\boldsymbol{P}^{n+1}$. We wish to parameterize these hypersurfaces (at least near $0 \in K^{n+1}$) so we seek $h\colon K^M \times K^n \rightarrowtail K$ so that $g(u, h(u,\alpha), \alpha)) \equiv 0$. But $\partial g/\partial x_1(0,0) \neq 0$, so by the implicit function theorem such an analytic h exists. Thus for small u the hypersurface $g(u,x)=0$ is, near 0, given by $(h(u,\alpha), \alpha)$. We now have to choose the linear map L; but we have a co-ordinate system so set

$$L(a_1, \ldots, a_n) = L(a) = \sum_1^n a_j x_{j+1} .$$

Clearly this is an admissible choice for $g_u = 0$ near $x(0)$ provided u is sufficiently small. Thus we have a family of function $H\colon K^M \times K^n \times K^n \to K$,

$$H(u,\alpha,a) = \left(dg_u(0) + \sum_1^n a_j x_{j+1}\right)(h(u,\alpha),\alpha) = \frac{\partial g_u}{\partial x_1}(0) \cdot h(u,\alpha) + \sum_1^n a_j \alpha_j$$

which gives rise to a map

$$\eta = j^k H_{(u,a)}(\alpha)\colon K^M \times K^n \times K^n \to J^k(K^n, K) = K^n \times K \times J^k(n,1) .$$

The fundamental result we need is that η *is a submersion at* $(0,0,0)$ *provided* $k \leqq d$.

Clearly we may take $k=d$. Now $g(u, h(u,\alpha), \alpha) \equiv 0$ so differentiating with respect to the u_i we get

$$\text{(i)} \qquad \frac{\partial g}{\partial x_1}(u, h(u,\alpha), \alpha) \frac{\partial h}{\partial u_i}(u,\alpha) + \frac{\partial g}{\partial u_i}(u, h(u,\alpha), \alpha) \equiv 0 .$$

We have now to show that the tangent map $T_0\eta\colon K^M \times K^n \times K^n \to K^n \times K \times J^d(n,1)$ is onto (here we are identifying a vector space with its tangent space at a point). The tangent space to $0 \times K^n \times 0 \subset K^M \times K^n \times K^n$ takes care of $K^n \times 0 \times 0$ in the target of $T_0\eta$. Also if u_1 is the co-ordinate corresponding to the monomial x_0^d,

$$\frac{\partial H}{\partial u_1}(0,0) = \frac{\partial f}{\partial x_1}(0) \cdot \frac{\partial h}{\partial u_1}(0,0) = \frac{-\partial g}{\partial u_1}(0,0) = -1 \neq 0 .$$

Thus it is enough to show that $L = \{u \in K^M\colon \sum u_i \varphi_i$ has no x_1 terms, nor an x_0^d term$\}$ maps under $T_0\eta$ onto $J^d(n,1)$. (Note that both spaces $L, J^d(n,1)$ have the same dimension.)

Well,

$$\text{(ii)} \quad \frac{\partial}{\partial u_i}(H(u,\beta,0) - H(u,0,0)) = \frac{\partial}{\partial u_i}\left(\frac{\partial g}{\partial x_1}(u,0) \cdot h(u,\beta) - \frac{\partial g}{\partial x_1}(0,0) \cdot h(0,\beta)\right)$$

$$= \frac{\partial^2}{\partial u_i \partial x_1}(u,0)\cdot h(u,\beta) + \frac{\partial g}{\partial x_1}(u,0)\cdot\frac{\partial h}{\partial u_i}(u,\beta)$$

$$= \frac{\partial \varphi_i}{\partial x_1}(u,0)\cdot h(u,\beta) + \frac{\partial g}{\partial x_1}(u,0)\cdot\frac{\partial h}{\partial u_i}(u,\beta)\ .$$

So (ii) as a polynomial in β is, at $u=0$

$$\frac{\partial f}{\partial x_1}(0)\cdot\frac{\partial h}{\partial u_i}(0,\beta) = \frac{-\partial g}{\partial u_i}(0,h(0,\beta),\beta) = -\varphi_i(\beta) \quad \text{by (i)}\ .$$

Whence the result, $T_0\eta$ is onto.

It follows that η is a submersion on some neighbourhood $U\times A_1\times A_2$ of $0\in K^M\times K^n\times K^n$. Choose U, A_1, A_2 to be closed, real subanalytic neighbourhoods (e.g. closed balls), and U' an open neighbourhood of $U\times A_1\times A_2$ on which η is a submersion. Now η: $U'\to J^d(K^n,K)$ is transverse to $\tilde{X}$, so $\eta^{-1}(\tilde{X})\cap U'$ is a smooth submanifold of U'. If π: $K^M\times K^n\times K^n\to K^M$ is the obvious projection let Σ denote the set of singular points of $\pi|\eta^{-1}(x)\cap U'$. Since η and π are analytic and for $K=\mathbf{R}$, $\tilde{X}$ is semialgebraic (for $K=\mathbf{C}$ $\tilde{X}$ is constructible) Σ is semi analytic. Now π: $U\times A_1\times A_2\to U$ is proper so $B=\pi(\Sigma\cap(U\times A_1\times A_2))$ is real subanalytic. This set B has, by Sards theorem, measure zero so consequently has codimension $\geqq 1$. It follows from Lemma 1 that for $u\in U-B$ the map η_U: $A_1\times A_2\to J^d(K^n,K)$ is transverse to $\tilde{X}$, so for all $\alpha\in A_1$ the point $y=(h(u,\alpha),\alpha)\in\{g_u=0\}$ satisfies the tangent transversality condition given in the definition above. For the neighbourhood A of $x(0)=(1:0:0:\ldots:0)$ choose A so that $(h,1)^{-1}(A)\subset U\times A_1$.

COROLLARY 2.5. *Let f be k determined, with Milnor number μ (so $k\leqq\mu+1$). If $d\geqq k$ then for almost every hypersurface $\{G=0\}=V\subset \boldsymbol{P}^{n+1}$ of degree d the set $V(f)=\{x\in V$: normal height function at x has a singularity at x right left equivalent to $f\}$ is smooth and has dimension $n+1-\mu$.*

EXAMPLES 2.6.

(i) $n=1$. Here, provided $d\geqq 3$, generically the only tangent singularities are ordinary (A_1) and cuspidal (A_2). (The cases $d=1,2$ are trivial).

(ii) $n=2$. Here, provided $d\geqq 4$, generically the only tangent singularities are ordinary (A_1) (two types in the real case), cuspidal (A_2) and tacnodal (A_3) (again two types in the real case).

(Again the cases $d=1,2$ are easy. One can make the arguments work for $d=3$ as well: A_1, A_2 are 3 determined and the orbit of the 2 jet $x^2\in J^3(2,1)$ has codimension 2. The other orbits have codimension >2. So for G tangent transverse to the singular orbits of $J^3(2,1)$ we get ordinary tangencies (A_1) on an open set, cuspidal tangencies (A_2) on a curve and isolated points where the height function has 2 jet x^2. But as $\{G=0\}$ is a non singular cubic any plane section can only have singularities of types A_1, A_2, A_3, D_4 so the isolated points are of A_3 type.)

Although Theorem 2.4 above describes the type of tangent singularities occuring on generic hypersurfaces of sufficiently large degree it is insufficient for a discussion of their duals. For this we shall need the corresponding multitransversality result. In proving this multi transversality theorem the first step is that of showing that tangency points generically occur in general position on tangent hyperplanes. In what follows we shall work over $\mathbf{C}=K$, but see the remarks below.

Recall that k points $p_1,\dots,p_k \in \boldsymbol{P}^n$ are in general position if the corresponding lines in $\mathbf{C}^{n+1}$ form linearly independent subspaces, so in particular $k \leqq n+1$. Let $L(p_1,\dots,p_k)$ denote the $(k-1)$ plane in $\boldsymbol{P}^n$ spanned by $p_1,\dots,p_k$. The point $p_0 \in L(p_1,\dots,p_k)$ is in general position with respect to $p_1,\dots,p_k$ if it does not lie in any subspace determined by a proper subset of $\{p_1,\dots,p_k\}$.

DEFINITION 2.7. *A hypersurface $\{F=0\} \subset \boldsymbol{P}^{n+1}$ is said to have its tangencies in general position if given a hyperplane L tangent at $\{p_1,\dots,p_k\} \subset \{F=0\}$ the points $p_1,\dots,p_k$ are in general position on L.*

LEMMA 2.8. *The set of F whose tangencies are not in general position form a constructible set of codimension $\geqq 1$.*

PROOF. We need to prove a number of subsidiary results. In what follows we suppose $d \geqq 3$.

(1) $X_1=\{F\colon F=0$ has k singularities in general position$\}$ is a constructible set of codimension k whose closure is irreducible.

If e_i denotes the ith vertex of reference in $\boldsymbol{P}^{n+1}$ the subset of $\mathbf{C}^M$ consisting of hypersurfaces with singularities at $e_1,\dots,e_k$ is a linear-space L. If

$$\varphi\colon \mathrm{Gl}_{n+2} \times \mathbf{C}^M \to \mathbf{C}^M$$

is the natural group action of $\mathrm{Gl}_{n+2}=\mathrm{Gl}\,(n+2,\mathbf{C})$, then

$$X_1 = \varphi(\mathrm{Gl}_{n+2} \times L)$$

which is constructible by Chevalleys theorem (see [8 p. 37]). If we think of $\mathrm{Gl}_{n+2} \subset \mathbf{C}^{(n+2)^2}$ then there is a natural extension of the action of φ to $\mathbf{C}^{(n+2)^2}$, and of course

$$\overline{\varphi(\mathrm{Gl}_{n+2} \times L)} \subset \overline{\varphi(\overline{\mathrm{Gl}}_{n+2} \times L)}$$

(where $^-$ denotes closure with respect to the Zariski topology). Now $\overline{\mathrm{Gl}}_{n+2}$ and L are irreducible so

$$\overline{\mathrm{Gl}}_{n+2} \times L \quad \text{and} \quad \overline{\varphi(\overline{\mathrm{Gl}}_{n+2} \times L)}$$

are irreducible. Clearly $\mathrm{Gl}_{n+2} \times L \subset \overline{\mathrm{Gl}}_{n+2} \times L$ is Zariski open so by [9 p. 37], $\varphi(\mathrm{Gl}_{n+2} \times L)$ contains a Zariski open subset of $\overline{\varphi(\overline{\mathrm{Gl}}_{n+2} \times L)}$. Hence

$$\dim \overline{\varphi(\mathrm{Gl}_{n+2} \times L)} = \dim \overline{\varphi(\overline{\mathrm{Gl}}_{n+2} \times L)}$$

and hence

$$\overline{\varphi(\overline{\mathrm{Gl}}_{n+2} \times L)} = \overline{\varphi(\mathrm{Gl}_{n+2} \times L)}$$

is irreducible. The codimension is obtained by computing the rank of $T\varphi$ at $(1, F)$ for each $F \in L$, using the fact that the tangent space to the orbit of F is $\mathrm{Sp}\{x_i \, \partial F/\partial x_j\}$.

(2) Let X_2 be that subset of X_1 consisting of hypersurfaces which have an additional singularity which lies in the $k-1$ plane spanned by the previous k, and is in general position with respect to them. We claim that X_2 is constructible of codimension $\geqq k+1$. Constructibility follows as in (1). To show that X_2 has larger codimension than X_1 we claim that it is enough to show that $X_1 - X_2$ has codimension k. To see this we first note that for any constructible set X we have $\dim(\bar{X} - X) < \dim X = \dim \bar{X}$. Now suppose that $\dim X_1 = \dim X_2$; then $\dim \bar{X}_1 = \dim \bar{X}_2$ so as $\bar{X}_1$ is irreducible $\bar{X}_1 = \bar{X}_2$, and so $\bar{X}_1 - X_2 = \bar{X}_2 - X_2$. But then

$$N-k = \dim(X_1 - X_2) \leqq \dim(\bar{X}_1 - X_2) = \dim(\bar{X}_2 - X_2) < \dim \bar{X}_2 = N-k,$$

a contradiction. So given that $\operatorname{codim}(X_1 - X_2) = k$ it follows that $\operatorname{codim} X_2 > k$. To prove that $\operatorname{codim}(X_1 - X_2) = k$ it is in fact enough to prove (using (1)).

(3) Given any $1 \leqq k \leqq n+2$ there are hypersurfaces $F \in \mathbf{C}^N$ with $\{F=0\}$ having k singularities in general position, and no other singularites. (Actually for $n=0$ we need $d \geqq 4$.) The proof is by induction on n. For $n=0$ the forms

$$F = x_1^2 \cdot \prod_{j=1}^{d-2} (x_1 - jx_2) \qquad \left(\text{respectively } F = x_1^2 \cdot x_2^2 \cdot \prod_{j=1}^{d-4} (x_1 - jx_2)\right)$$

have repeated roots at 1 (respectively 2) points. For the special case $d=3$ we need $n=1$, and consider

$$F = x_1 x_2 x_3 + x_1^3 + x_2^3 \qquad (\text{respectively } F = x_3(x_3^2 - x_1 x_2),\ F = x_1 x_2 x_3).$$

Now we wish to produce hypersurfaces $\{F=0\} \subset \boldsymbol{P}^{n+1}$ having only k singularities in general position, $1 \leqq k \leqq n+2$. By induction we can find $G_k(x_1, \ldots, x_{n+1})$ with $\{G_k = 0\} \subset \boldsymbol{P}^n$ having only k singularities in general position, $1 \leqq k \leqq n+1$. The form $G_k + x_{n+2}^d$ takes care of the cases $1 \leqq k \leqq n+1$. For $k = n+2$ we claim that

$$G_{n+2} = G_{n+1} + x_{n+2}^{d-2}\left(\sum^{n+1} a_i x_i^2\right)$$

for generic $(a_1, \ldots, a_{n+1})$ has precisely $n+2$ singularities. Clearly we have only to show that

$$\left\{G_{n+1} + \sum^{n+1} a_i x_i^2 = 0\right\} \subset \mathbf{C}^{n+1}$$

has only one singularity (at the origin) for generic $(a_1, \ldots, a_{n+1})$. Well, consider the map

$$G\colon (\mathbf{C}^{n+1} - 0) \times \mathbf{C}^{n+1} \to \mathbf{C}, \qquad (x, a) \mapsto G_{n+1}(x) + \sum a_i x_i^2 = G_a(x) .$$

Clearly $0 \in \mathbf{C}$ is a regular value of $G_a\colon \mathbf{C}^{n+1} - \{0\} \to \mathbf{C}$ for almost all a by Lemma 1, since 0 is a regular value of G. So $\{G_a = 0\} \subset \mathbf{C}^{n+1} - 0$ has no singular points, as required.

(4) Suppose $\{F=0\}$ has a tangent plane T with $k+1$ tangent points $p_0, \ldots, p_k$, with $p_1, \ldots, p_k$ in general position and $p_0 \in L(p_1, \ldots, p_k)$ in general position with respect to $p_1, \ldots, p_k$. We start by considering the special case $p_i = e_i$, $1 \leqq i \leqq k$, $p_0 = (1: \ldots :1:0: \ldots :0)$ (k ones) and $T = \{x_{n+2} = 0\}$. Then we can write

$$F = f(x_1, \ldots, x_k) + \sum_{k+1}^{n+2} x_j f_j(x_1, \ldots, x_{n+2})$$

and we have

$$F = 0, \ \frac{\partial f}{\partial x_i} + \sum \left(\delta_{ij} f_j + x_j \frac{\partial f_j}{\partial x_i}\right) = 0 ,$$

$1 \leqq i \leqq n+2$ at $p_0, e_1, \ldots, e_k$ (here δ_{ij} is the Kronecker symbol). In particular $f = \partial f / \partial x_i = 0$ for $1 \leqq i \leqq k$ at $p_0, e_1, \ldots, e_k$, so if $\mathbf{C}^N \subset \mathbf{C}^M$ is the subset of forms of degree d in $x_1, \ldots, x_k$ and X_2 is the subset of $\mathbf{C}^N$ discussed in (2) (put $n+2=k$) $f \in X_2$. Also $f_j = 0$ at $e_1, \ldots, e_k$ for $k+1 \leqq j \leqq n+1$ so

$$f_j = \sum_{k+1}^{n+2} x_i f_{ij} \quad \text{and} \quad F = f + \sum_{j=k+1}^{n+1} \sum_{i=k+1}^{n+2} x_j x_i f_{ij} + x_{n+2} f_{n+2} .$$

The collection of all such F is clearly a product $X_2 \times L$ for some linear space L. space to $M_p \times L$ at $F = (f, l)$. Clearly T is contained in

$$T_f M_p + \mathrm{Sp}\{x_i \psi \ : \ \deg \psi = d-1, \ k+1 \leqq i \leqq n+2\} ,$$

so the image $T_{(F,1)}\varphi(T) = T'$ is spanned by

$$T_f(\varphi(\mathrm{Gl}_k\times M_p))+\mathrm{Sp}\,\{x_i\psi, k+1\leqq i\leqq n+2\}+\mathrm{Sp}\left\{x_i\frac{\partial F}{\partial x_j}\ :\ 1\leqq i,\ j\leqq n+2\right\},$$

where $\mathrm{Gl}_k\subset \mathrm{Gl}_{n+2}$ is the obvious subgroup. Intersecting with $\mathbf{C}^N$ we find that

$$C^N\cap T'\ \subset\ T_f(\varphi(\mathrm{Gl}_k\times M_p))+\mathrm{Sp}\,\{x_if_{n+2}(x_1,\ldots,x_k,0,\ldots,0),\ 1\leqq i\leqq k\}\ .$$

But by (2) codim $\varphi(\mathrm{Gl}_k\times M_p)$ in $\mathbf{C}^N$ is $>k$ so codim of $\mathbf{C}^N\cap T'$ in $\mathbf{C}^N$ is $\geqq 1$. Consequently codim $(\varphi(\mathrm{Gl}_{n+2}\times M_p\times L))$ is $\geqq 1$. The lemma now easily follows.

REMARK 2.9 We note that the proof of (2) in the lemma above is the only part which does not work in the real case. Hopefully the lemma is still true over the real numbers and when proved the multi transversality result which follows will also hold for real algebraic hypersurfaces.

Now to the statement and proof of the multi transversality theorem. The statement is of course rather more local than that given in section 1.

Suppose $V=\{F=0\}\subset \boldsymbol{P}^{n+1}$ is a nonsingular hypersurface and $(x(1),\ldots,x(r))\in V^{(r)}$ with the $x(i)$ sharing a common tangent plane T. We can choose an affine chart $K^{n+1}\subset \boldsymbol{P}^{n+1}$ containing all of the $x(i)$ and if $V_1=\{f=0\}=V\cap K^{n+1}$ the vectors $df(x(i))$ are non zero multiples of each other. Now choose local parameterizations h_i: $(K^n,0)\rightarrowtail(V_1,x(i))$ and a linear map L: $K^n\to(K^{n+1})^*$ with

$$\mathrm{Sp}\,\{df(x(i))\}\oplus \mathrm{Im}\,L\ =\ (K^{n+1})^*\ .$$

If H_i: $(K^n\times K^n,(0,0))\to K$ is the map

$$(\alpha(i),a)\ \to\ (df(x(1))+L(a))(h_i(\alpha(i)))$$

consider the map

$$j_1^kH\ =\ j_1^kH(x(1),\ldots,x(r))\ :\ ((K^n)^r\times K^n,0)\to\left(\prod_1^r J^k(K^n,K),j_1^kH(0)\right)$$

$$(\alpha(1),\ldots,\alpha(r),a)\ \to\ (j_1^kH_1(\alpha(1),a),\ldots,j_1^kH_r(\alpha(r),a))\ .$$

Now suppose we have a stratification $\mathfrak{X}=\bigcup X_j\subset\Sigma\subset J^k(n,1)$. We define the diagonal stratification $\Delta^r(\mathfrak{X})$ as follows. Give $\prod_1^r\mathfrak{X}\subset\prod_1^r J^k(K^n,K)$ the product stratification. If

$$\Pi^r\colon {}_rJ^k(K^n,K)\ \to\ K^r$$

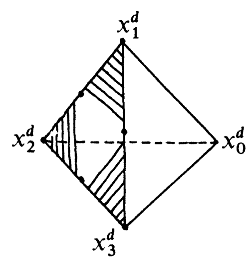
is the projection

$$(j^kf_1,\ldots,j^kf_r)\ \to\ (\mathrm{target}\,j^kf_1,\ldots,\mathrm{target}\,j^kf_r)\ ,$$

then Π^r is a submersion on strata, so if $\Delta = \{(c, c, \ldots, c) \in K^r\}$, then $(\Pi^r)^{-1}(\Delta)$ has an induced stratification which we denote by $\Delta^r(\mathfrak{X})$. We say that $\{F=0\} = V$ is tangent multi transverse to $\mathfrak{X}$ if for each r and all $(x(1), \ldots, x(r))$ as above the map $j_1^k H$ is transverse to $\Delta^r(\mathfrak{X})$. Note that tangent multi transversality ensures regular intersections (compare section 1). We also note that Lemma 2 shows that for most hypersurfaces $(x(1), \ldots, x(r))$ as above occur only if $r \leqq n+1$.

THEOREM 2.10. *Let $K = \mathbf{C}$. Let $\{F=0\}$ be a non singular hypersurface with tangencies in general position. Then given a natural stratification $\mathfrak{X} \subset \Sigma \subset J^k(n, 1)$ there is an open neighbourhood U of F and a countable family of real subanalytic subsets $B_\alpha \subset U$, with* codim $B_\alpha \geqq 1$ *such that for $G \in U - \bigcup B_\alpha$ the hypersurface $\{G=0\}$ is tangent multi transverse to $\mathfrak{X}$ provided $k \leqq (d-1)/2$.*

Before proceeding with the proof we sketch the idea which is simple enough. As usual we have a natural family of peturbations provided by the family of hypersurfaces themselves. Now suppose that we have tangency points $x(1), \ldots, x(r)$ with common tangent plane $x_0 = 0$. Since the $x(j)$ are in general position we suppose that they are unit points e_{j+1} (1 in the $j+1$th place and 0's elsewhere). Now for the case $r=1$, $x(1) = (0:1:0:\ldots:0)$ we only used one face of the $n+1$ simplex of monomials of degree d in $x_0, \ldots, x_{n+1}$ (those not involving x_0) to show that the relevant map was a submersion. In the general case then provided $k \leqq d - 1/2$ we can, at e_j, obtain all of the monomials in $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$ of degree $\leqq k$, without using any monomial in the bottom face twice. Consequently we obtain "linearly independent variations" of the height function at $x(1), \ldots, x(r)$ and this is enough to prove the result. (See dig. 6).



dig. 6.

The major difference between this case and the case $r=1$ is that $V^{(r)}$ is not compact, only separable. So when we are working locally in $V^{(r)}$ we must always use some fixed neighbourhood U of F. This means (recalling our proof of Theorem 2) that given $x \in V$ we need to be able to find parameterizations h_u valid for $F_u \in U$. Since we are working with multijets we also want to ensure that these local parameterizations do not overlap.

PROOF OF THEOREM 2.10. We start with:

CLAIM: We can choose $Q=(n+1)^2+1$ planes $L_1,\ldots,L_Q$ so that given any $n+1$ points $p_0,\ldots,p_n \in P^{n+1}$ one of the planes L_j misses all of the p_i.

In fact choose the L_j in general position (i.e. so that any $p \leqq n+2$ of them give p planes in K^{n+2} in general position). Clearly any p_i belongs to at most $n+1$ of the planes whence the result.

Now given a hyperplane $L \subset P^{n+1}$ we have $P^{n+1}-L \equiv K^{n+1}$. Choose a sufficiently large closed ball $B_j \subset P^{n+1}-L$ so that one of the $P^{n+1}-B_j$ always misses $n+1$ points. Finally setting $V=\{F=0\}$ choose L_j and B_j as above but with the additional property that the intersection of V with the intersection of any $n+1$ complements $P^{n+1}-B_j$ is empty.

Choose a neighbourhood U_1 of F so that any $G \in U_1$ has the same property as F with respect to the covering $B_1,\ldots,B_Q$ just described. Choose a chart $x_0,\ldots,x_n$ on $P^{n+1}-L_j$ and set $B_j \cap V=V_j$ which has equation $f_j=0$; set

$$N_j = \left\{(x,v) \in V_j \times K^{n+1} \ : \ v = \lambda\left(\frac{\partial f_j}{\partial x_0}(x),\ldots,\frac{\partial f_j}{\partial x_n}(x)\right)\right\}$$

and consider the map $E\colon N_j \to K^{n+1}$, $E(x,v)=x+v$. With our previous notation E is locally the map

$$(\alpha,\lambda) \to h(\alpha)+\lambda\left(\frac{\partial f_j}{\partial x_0}(h(\alpha)),\ldots,\frac{\partial f_j}{\partial x_n}(h(\alpha))\right)$$

and clearly E is a local diffeomorphism. Using the fact that V_j is compact choose $\varepsilon>0$ so that the restriction $E\colon N_j \cap (V_j \times B_\varepsilon) \to K^{n+1}$ is an embedding, where B_ε is the ε ball at $0 \in K^{n+1}$. Now choose $U_2 \subset U_1$ so that for any $G \in U_2$ the set $W_j=B_j \cap \{G=0\}$ lies in the neighbourhood $E(N_j \cap (V_j \times B_\varepsilon))$. (Strictly speaking one should work with two large balls B_j, B'_j to avoid difficulties at the boundary.) Choose an open set $U \subset U_2$ with $\bar{U} \subset U_2$ compact so that for $G \in \bar{U}$ and $y \in \{G=0\} \cap B_j$ with $y=E(x,v)$ the vector v does not belong to T_yW_j that is G is sufficiently close to F for the tangent space not to have moved far. It follows that for any $x \in V_j$ (which we suppose has tangent space $x_0=0$) we can choose a neighbourhood A_1 of $0 \in K^n$ and a family of parameterizations

$$h \ : \ \bar{U} \times A_1 \to K^{n+1}, \qquad h(u,\alpha)=(h'(u,\alpha),\alpha)$$

with h_u parameterizing $F_u=0$. Moreover as $h(\bar{U}\times 0) \subset \{E(h(0),v)\}$ given $x(1)$, $x(2) \in V_j$ with local parameterizations $h_j\colon \bar{U}\times A_1^j \to K^{n+1}$, $j=1,2$,

$$h_1(\bar{U}\times 0) \cap h_2(\bar{U}\times 0) = \varnothing,$$

so possibly shrinking the A_1^j we may suppose $\bigcap_{j=1,2} h_j(U \times A_1^j)=\varnothing$.

Now let $(x(1),\ldots,x(r)) \in \{F=0\}^{(r)}$, $r \leqq n+1$ share the same tangent plane and let $L\colon K^n \to (K^{n+1})^*$ be as usual. Choose one B_j with $x(i) \in B_j$, $1 \leqq i \leqq r$, and local parameterizations $h_j\colon \bar{U} \times A_1^j \to K^{n+1}$ with

$$h_j(U \times A_1^j) \cap h_i(U \times A_1^i) = \emptyset, \quad i \neq j .$$

Consider the map $\eta\colon A_1^1 \times \ldots \times A_1^r \times U \times K^n \to \prod_{j=1}^r J^k(K^n, K)$

$$\begin{aligned} \eta(\alpha(1),\ldots,\alpha(r),u,a) &= \prod_1^r \big(k \text{ jet at } \alpha(i) \text{ of} \\ &\qquad \alpha \to \big(df_j(x(1)) + L(a)\big)\big(h_i(u,\alpha(i))\big)\big) \\ &= \prod_1^r \big(j_1^k H_i(u,\alpha(i),a)\big) . \end{aligned}$$

We claim that η is transverse to $\Delta^r(\mathfrak{X})$. If $j_1^k H_i(u,\alpha(i),a) \notin \Sigma$ for any i, then

$$\eta(\alpha(1),\ldots,\alpha(r),u,a) \notin \Delta^r(\mathfrak{X}) .$$

If the $j_1^k H_i(u,\alpha(i),a) \in \Sigma$, but the $h_i(u,\alpha(i))$ do not share a common tangent plane, then

$$\eta(\alpha(1),\ldots,\alpha(r),u,a) \notin \Delta^r(\mathfrak{X}) .$$

If however they do share a common tangent plane given by

$$\big(df_j(\alpha(i)) + L(a)\big)\big(h_i(u,\alpha(i))\big) = c$$

then the $h_i(u,\alpha(i))$ are in general position in this plane. It follows from the remarks above that η is a submersion at $(\alpha(1),\ldots,\alpha(r),u,a)$ and hence transverse to $\Delta^r(\mathfrak{X})$. So there is a bad real subanalytic set $B \subset U$ of codimension $\geqq 1$ such that for $u \in U - B$ the map

$$\eta_u\colon \prod A_1^j \times K^n \to \prod J^k(K^n, K)$$

is transverse to $\Delta^r(\mathfrak{X})$. We now use the fact that $\{G=0\}^{(r)}$ is separable to obtain the countable number of bad sets B_α.

REMARKS 2.11. In practice the inequality $k \leqq d - 1/2$ seems too restrictive. One can obtain better results by considering the product $\prod_{j=1}^n J^{k_j}(K^n, K)$ with the k_j possibly different.

COROLLARY 2.12. *For $n \leqq 6$ and $d \geqq 2n+5$ all of the tangent singularities of a generic hypersurface in $\mathbf{P}^{n+1}$ are simple singularities. Moreover its dual is locally the union of transversely intersecting discriminant varieties of simple singularities.*

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