MATH. SCAND. 49 (1981), 61–85

APPROXIMATING CONTINUOUS MAPS OF METRIC SPACES INTO MANIFOLDS BY EMBEDDINGS

JOUNI LUUKKAINEN

Introduction.

1. Let $n \geqq 0$ and $m \geqq 2n+1$ be integers. Consider a separable metric space X with $\dim X \leqq n$ and a topological m-manifold M, and equip the set $C(X,M)$ of all continuous maps of X into M with the target majorant topology (to be defined in 1.2). The purpose of this paper is to investigate the problem whether embeddings are dense in $C(X,M)$.

Let N_m^n denote the n-dimensional subset of $\mathbf{R}^m$ of points at most n of whose coordinates are rational. Then in the case $M=I^m$ one has the following classical result:

THEOREM A. *The set of all embeddings $f\colon X \to I^m$ with $\overline{fX} \subset N_m^n$ contains a dense G_δ-set in $C(X,I^m)$.*

For the history of this theorem, see either [13, pp. 56, 60, and 64], [18, p. 107] or [7, pp. 128–130]. For the future we remark that it is known ([20]) that the set of all embeddings of X into I^m is generally not a G_δ-set in $C(X,I^m)$. To begin with, we will generalize Theorem A for arbitrary m-submanifolds M of $\mathbf{R}^m$ in Theorem 2.1. Previously it was known ([7, Problem 1.11.C(c)]) that embeddings are dense in $C(X,\mathbf{R}^m)$ with respect to the weaker topology of uniform convergence. Our proof is a modification of the proof of Theorem A in [13] or [18]. In Section 2 we will also give analogues of Theorem 2.1 for infinite-dimensional target spaces.

Our main result is the following, which is a simplified version of Theorem 5.6:

THEOREM B. *Embeddings are dense in $C(X,M)$.*

Note that Theorem B is not valid for the source majorant topology (defined in 1.2) even if $n=0$ and $M=\mathbf{R}^m$. For example, let $f\colon \mathbf{N} \to \mathbf{R}^m$, $m \geqq 1$, be any map

Received August 15, 1980.

whose image is dense; then every map $g\colon \mathbf{N} \to \mathbf{R}^m$ with $|f(j)-g(j)|<1/j$ for all $j \in \mathbf{N}$ has a dense image and is thus not an embedding. We will reduce Theorem B by a simple compactification argument to the following result, which is a corollary of it and forms a part of Theorem 5.1:

THEOREM C. *Suppose that X is locally compact. Then closed embeddings are dense in the subspace* Prop (X, M) *of* $C(X, M)$ *of proper maps.*

Morlet [17, IV, Corollaire 5, p. 7-01] proved Theorem C in the case that X is a manifold. It is a corollary of results of Dancis [5, Topological General Position Lemma 1, General Position Lemma 2, and Remark on p. 255] that Theorem C holds if X is either a manifold or a polyhedron, provided that $m \geqq n+3$ if $n \leqq 1$. Moreover, here all the cases $(n, m)=(0,1)$, $(0,2)$ or $(1,3)$, excluded by the additional dimensional restriction, follow from well-known PL general position results ([12, Lemma 4.8, p. 102]) because every q-manifold, $q \leqq 3$, is homeomorphic to a PL manifold (the case $n=0$ is trivial). The thus known special case of Theorem C where X is a polyhedron easily implies the general case as it will appear in Remark 5.4. However, we will give a proof for Theorem C which is independent of [5].

In Section 4 we will prove Theorem C in the case that M is a Lipschitz manifold in the sense of [15]. In the proof of the main lemma (according to which maps in Prop (X, M) with small point-inverses are dense) we first remetrize X and approximate maps by locally Lipschitz maps and then use general position techniques for locally Lipschitz maps developed by Väisälä and the author in [15]. Now, every q-manifold, $q \neq 4$, without boundary is homeomorphic to a Lipschitz manifold by a recent deep result of Sullivan [23]. This will imply Theorem C for $m \neq 4$. In Section 3 we will consider the case $n \leqq 1$. We will utilize a relative version of Theorem A for $n \leqq 1$ due to Bothe [4] to prove, along with other results, a relative version of Theorem C for $n \leqq 1$, Theorem 3.6. This will imply Theorem C for $m=4$.

2. While completing this paper, the author was informed by H. Toruńczyk that Theorem C is known as it is a consequence of the following recent result. Before stating it we define the notion of a Z^n-set. A set A in a metric space Y is called a *Z^n-set* in Y if every continuous map of I^n into Y can be uniformly approximated by continuous maps whose images are disjoint from A. An embedding of a space into Y is called a *Z^n-embedding* if its image is a Z^n-set in Y.

THEOREM D (Heisey and Toruńczyk). *Suppose that above X is locally compact, X_0 is a closed subset of X and $f_0\colon X_0 \to M$ is a closed Z^n-embedding.*

Then the set of all closed Z^n-embeddings $f\colon X \to M$ with $f\,|\,X_0=f_0$ is dense (in fact, contains a dense G_δ-set) in the space of all proper maps $f\colon X \to M$ with $f\,|\,X_0=f_0$.

It is clear that Theorem D for $X_0=\emptyset$ is stronger than Theorem C. Theorem D also implies the above-mentioned relative form of Theorem C for $n\leqq 1$. Theorem D is obtained as follows. Say that a metric space Y has the *disjoint n-cube property* if every continuous map of $I^n\times\{1,2\}$ into Y is uniformly approximable by continuous maps sending $I^n\times\{1\}$ and $I^n\times\{2\}$ to disjoint sets. By [25, Lemma], an ANR Y has this property if it has it locally. (The proof in [25] fails. We could give a proof only under the additional assumption, not too strong for the sequel, that Y is completely metrizable and separable. However, Toruńczyk later gave a new, correct proof (to appear probably in Topology Proc.).) By PL general position, R^m and R^m_+ have the disjoint n-cube property. Hence so does every m-manifold M. Therefore Theorem D is a special case of [10, Corollary 5], according to which Theorem D is valid if M is replaced by any locally compact separable ANR having the disjoint n-cube property (the remark on G_δ-sets in Theorem D follows from the proof of [10, Proposition 4]). The proof of Heisey and Toruńczyk is quite elementary.

Theorem D can easily be generalized, see Remark 3.10, to the following relative version of Theorem B:

THEOREM E. *Let X be a separable metric space with* $\dim X\leqq n\geqq 0$, *let M be a topological m-manifold, $m\geqq 2n+1$, let X_0 be a locally compact closed subset of X and let $f_0\colon X_0 \to M$ be a closed Z^n-embedding. Then the set*

$$\{f \mid f\colon X \to M \text{ is an embedding, } f\,|\,X_0 = f_0,\ \overline{fX} \text{ is a } Z^n\text{-set, } \dim \overline{fX}=\dim X\}$$

is dense in the subspace $\{f\in C(X,M) \mid f\,|\,X_0=f_0\}$ of $C(X,M)$.

1. Preliminaries.

In this section we will fix notation and state some known facts which will be needed later.

1.1. NOTATION AND TERMINOLOGY. The letters n and m denote non-negative integers. Let $I^m=[0,1]^m$ and $\mathsf{N}=\{1,2,\dots\}$. If X is a separable metric space, $\dim X$ denotes the dimension of X in the sense of [13]. For subspaces of the euclidean m-space R^m we use the euclidean metric. Every other metric is denoted by d if not otherwise stated. In a metric space, $B(x,r)$ is the open ball and $\bar{B}(x,r)$ the closed ball with center x and radius r. The distance between two sets A, B in a metric space (X,d) is denoted by $d(A,B)$, the diameter of A by $d(A)$ and the closure of A sometimes also by $\mathrm{cl}_X A$.

A map $f\colon X \to Y$ between metric spaces is called an embedding if it defines a homeomorphism $X \to fX$. An m-manifold is a separable metric space such that each point has a neighborhood homeomorphic to I^m. A submanifold of $\mathbf{R}^m$ is a subspace which is a manifold. If M is an m-submanifold of $\mathbf{R}^m$, there is no ambiguity in our use of $\operatorname{int} M$ both for the interior of M as a manifold and for the interior of M as a subspace because these coincide. An ANR means a metric absolute neighborhood retract for the class of metric spaces; similarly for an AR. Every polyhedron is supposed to be a locally compact separable metric space.

1.2. FUNCTION SPACES. Let X and Y be metric spaces. Let $C(X, Y)$ denote the set of all continuous maps of X into Y. Let $C_+(X) = C(X, (0,\infty))$. A map $f\colon X \to Y$ is called *proper* if it is continuous and the inverse image of every compact set is compact; then f is closed. If X is compact, then every continuous map $f\colon X \to Y$ is proper. We denote by $\operatorname{Prop}(X, Y)$, $\operatorname{Emb}(X, Y)$ and $\operatorname{CEmb}(X, Y)$ the subsets of $C(X, Y)$ consisting, respectively, either of all proper maps, of all embeddings or of all closed embeddings. Obviously, a map $f\colon X \to Y$ is a closed embedding if and only if it is a proper injection. Suppose that the letter F denotes any of C, Prop, Emb or CEmb. If M is a manifold or if X_0 is a closed subset of X and $f_0 \in F(X_0, Y)$, we will use the notations

$$F^{\mathrm{int}}(X,M) = \{f \in F(X,M) \mid \overline{fX} \subset \operatorname{int} M\},$$

$$F(X,Y;\, f_0) = \{f \in F(X,Y) \mid f|X_0 = f_0\}.$$

Two continuous maps $f, g\colon X \to Y$ are said to be *$\mathscr{U}$-near*, where $\mathscr{U}$ is an open cover of Y, if for every $x \in X$ there is $U \in \mathscr{U}$ such that $f(x), g(x) \in U$. In the *target majorant topology* $\mathscr{T}$ of $C(X, Y)$ a neighborhood basis of $f \in C(X, Y)$ is given by the sets

$$N(f,\mathscr{U}) = \{g \in C(X, Y) \mid f \text{ and } g \text{ are } \mathscr{U}\text{-near}\}$$

where $\mathscr{U}$ is an open cover of Y. We will use this topology for $C(X, Y)$ and its subsets if not otherwise stated. By [14, Fact 4, p. 47], another neighborhood basis of $f \in C(X, Y)$ in $\mathscr{T}$ is given by the sets

$$V(f,\delta) = \{g \in C(X, Y) \mid d(f(x), g(x)) < \delta(f(x)) \quad \text{for all } x \in X\}$$

where $\delta \in C_+(Y)$. Occasionally we will use the *source majorant topology* $\mathscr{T}_s$ for $C(X, Y)$. In the space $C_s(X, Y) = (C(X, Y), \mathscr{T}_s)$ an open neighborhood basis of $f \in C(X, Y)$ is given by the sets

$$U(f,\varepsilon) = \{g \in C(X,Y) \mid d(f(x),g(x)) < \varepsilon(x) \quad \text{for all } x \in X\}$$

where $\varepsilon \in C_+(X)$. By [28, (5.2)], $\mathcal{T}_s$ is equal to the *graph topology*, a basis of which consists of the sets $W_U = \{f \mid f \subset U\}$ where U is an open set in $X \times Y$. Thus $\mathcal{T}_s$ is independent of the metric of the space Y. Since $V(f,\delta) = U(f,\delta f)$, we have $\mathcal{T} \subset \mathcal{T}_s$. If X is compact, then obviously $\mathcal{T}$ and $\mathcal{T}_s$ are equal and given by the supremum metric

$$d(f,g) = \sup\{d(f(x),g(x)) \mid x \in X\} \qquad \text{for } f,g \in C(X,Y)\,.$$

Also, if Y is compact, then $\mathcal{T}$ is given by this metric. If a metric will be used for $C(X,Y)$, it is this metric.

If $f \in \operatorname{Prop}(X,Y)$ and $\varepsilon \in C_+(X)$, there is $\delta \in C_+(Y)$ such that $\delta(f(x)) \leqq \varepsilon(x)$ for all $x \in X$ by [14, Lemma, p. 47]. Hence $\mathcal{T}$ and $\mathcal{T}_s$ induce the same topology for $\operatorname{Prop}(X,Y)$. If X and Y are locally compact, then $\operatorname{Prop}(X,Y)$ is open (and closed) in $C(X,Y)$, and thus in $C_s(X,Y)$, too, because, as it is easy to see, if $\mathcal{U}$ is a locally finite cover of Y by open sets with compact closure and if $f \in \operatorname{Prop}(X,Y)$, then $N(f,\mathcal{U}) \subset \operatorname{Prop}(X,Y)$.

The following lemma gives a basic property of function spaces. A *Baire space* is a topological space in which the intersection of every countable family of open dense sets is dense.

1.3. Lemma. *Let X and Y be metric spaces. If Y is complete, then $C(X,Y)$ and $C_s(X,Y)$ are Baire spaces. If X and Y are locally compact, then* $\operatorname{Prop}(X,Y)$ *is a Baire space.*

Proof. If Y is complete, then $C_s(X,Y)$ is a Baire space by [11, Theorem 2.4.2]. A modification of the proof of the quoted result shows that $C(X,Y)$ is also a Baire space. Since every locally compact metric space is completely metrizable and since every open subspace of a Baire space is a Baire space, the last assertion follows.

1.4. LIP manifolds. Let X and Y be metric spaces and let $f\colon X \to Y$. If there is $L \geqq 0$ with $d(f(x),f(y)) \leqq L\, d(x,y)$ for all $x,y \in X$, then f is said to be *Lipschitz*. If every point of X has a neighborhood on which f is Lipschitz, then f is said to be *locally Lipschitz*, abbreviated LIP. If f is a bijection and both f and f^{-1} are LIP, then f is called a LIP *homeomorphism*. A *Lipschitz m-manifold*, also called a LIP *m-manifold*, is a separable metric space M such that every point of M has a neighborhood LIP homeomorphic to I^m. We refer to [15] for LIP maps and LIP manifolds.

We now consider briefly the problem whether a given manifold is always homeomorphic to a LIP manifold. Let M be an m-manifold and let $\mathscr{U} = ((U_j, h_j))_{j \in J}$ be an atlas on M, i.e., the sets U_j form an open cover of M and h_j is a homeomorphism of U_j onto an open set U'_j of $\mathbf{R}^m_+ = \{x \in \mathbf{R}^m \mid x_m \geqq 0\}$. Suppose that

$$h_i h_j^{-1} : h_j(U_i \cap U_j) \to h_i(U_i \cap U_j)$$

is a LIP homeomorphism for all $i, j \in J$. Then $\mathscr{U}$ is called a LIP *atlas* on M, and by [15, 3.3] there is a topologically compatible metric d' on M such that $h_j\colon (U_j, d') \to U'_j$ is a LIP homeomorphism for every j. Hence (M, d') is a LIP manifold. It follows from this or from the embeddability into a euclidean space that every PL manifold and every differentiable manifold is homeomorphic to a LIP manifold. It is well-known that every m-manifold, $m \leqq 3$, is homeomorphic to a PL manifold. For a proof for 2- and 3-manifolds without boundary, see [16, Theorems 4.8, 8.3, 23.1, and 35.3]. On the other hand, Siebenmann [22, p. 137] (=[14, p. 311]) has constructed for every $m \geqq 7$ a closed LIP m-manifold which is not homeomorphic to any PL manifold. Sullivan [23, Corollary 3, p. 549] proved recently that for $m \geqq 5$ every m-manifold without boundary has a LIP atlas. These remarks imply the following lemma to be used in the proof of Theorem 5.1. As it will then appear, it is only Sullivan's case $m \geqq 5$ that will really be needed.

1.5. LEMMA. *For $m \neq 4$, every m-manifold without boundary is homeomorphic to a* LIP *manifold.*

2. Approximating continuous maps of metric spaces into codimension zero submanifolds of euclidean spaces.

We let N^n_m, $0 \leqq n \leqq m \geqq 1$, denote the subset of $\mathbf{R}^m$ consisting of all points at most n of whose coordinates are rational; then $\dim N^n_m = n$ by [13, Example IV.1].

2.1. THEOREM. *Let X be a separable metric space with* $\dim X \leqq n$ *and let M be an m-submanifold of* $\mathbf{R}^m$, $m \geqq 2n+1$. *Then the set*

$$\{f \in \operatorname{Emb}(X, M) \mid \operatorname{cl}_M fX \subset N^n_m \cap \operatorname{int} M\}$$

contains a dense G_δ-set in the Baire space $C(X, M)$.

For $M = I^m$, Theorem 2.1 is classical; see [13, Theorems V.3 and V.5] or [18, Theorem IV.8]. Our proof is a modification of the one in [18, pp. 101–108]. The theorem follows after a sequence of lemmas.

2.2. LEMMA. *Let X and Y be metric spaces and let $F \subset Y$ be closed. Then the set of all continuous maps $f\colon X \to Y$ with $\overline{fX} \cap F = \emptyset$ is open in $C(X, Y)$.*

PROOF. Let $f \in C(X, Y)$, $\overline{fX} \cap F = \emptyset$. Choose an open neighborhood U of F with $\overline{fX} \cap \bar{U} = \emptyset$. Then $\mathscr{U} = (Y \setminus \bar{U},\ Y \setminus \overline{fX})$ is an open cover of Y. If $g \in N(f, \mathscr{U})$, then $gX \subset Y \setminus \bar{U}$, whence $\overline{gX} \subset Y \setminus U$, and so $\overline{gX} \cap F = \emptyset$.

2.3. LEMMA. *Let M be a manifold and let $\delta\colon M \to [0, \infty)$ be continuous with $\partial M \subset \delta^{-1}(0, \infty)$. Then there exists a closed embedding $h\colon M \to M$ such that $hM \subset \operatorname{int} M$ and $d(h(x), x) \leqq \delta(x)$ for all $x \in M$.*

PROOF. By [21, Theorem 1.7.4, p. 40, or Theorem 1.7.7, p. 42] there is a closed collar $c\colon \partial M \times [0, 1] \to M$ of ∂M in M, that is, c is a closed embedding with $c(x, 0) = x$ for all $x \in \partial M$ and with $c(\partial M \times [0, 1))$ open in M. It is easy to modify c in such a way that one has $d(cI_x) \leqq \min \delta c I_x$, where $I_x = \{x\} \times [0, 1]$, for all $x \in \partial M$. Define $h\colon M \to M$ by $h(c(x, t)) = c(x, (t+1)/2)$ if $(x, t) \in \partial M \times [0, 1]$ and by $h(x) = x$ if $x \in M \setminus \operatorname{im} c$. Then h satisfies the requirements.

2.4. LEMMA. *If X is a metric space and M a manifold, then $C^{\mathrm{int}}(X, M)$ is open and dense in $C(X, M)$.*

PROOF. The openness follows from Lemma 2.2 and the denseness from Lemma 2.3.

2.5. DEFINITION. Let X and Y be metric spaces and let $\mathscr{U}$ be an open cover of X. A *$\mathscr{U}$-map* $f\colon X \to Y$ is a continuous map such that every point of Y has a neighborhood whose inverse image is contained in some member of $\mathscr{U}$.

2.6. LEMMA. *Let X and Y be metric spaces and let $\mathscr{U}$ be an open cover of X. Then the set of all $\mathscr{U}$-maps of X into Y is open in $C(X, Y)$.*

PROOF. Let $f\colon X \to Y$ be a $\mathscr{U}$-map. Then there is an open cover $\mathscr{V}$ of Y such that the cover $f^{-1}\mathscr{V}$ of X refines $\mathscr{U}$. By [6, VIII, Theorem 3.5] choose an open (strong) star-refinement $\mathscr{W}$ of $\mathscr{V}$. Let $g \in N(f, \mathscr{W})$. Let $W \in \mathscr{W}$ and choose $V \in \mathscr{V}$ with $\operatorname{st}(W, \mathscr{W}) \subset V$. Choose $U \in \mathscr{U}$ with $f^{-1}V \subset U$. Let $x \in g^{-1}W$. Choose $W' \in \mathscr{W}$ with $f(x), g(x) \in W'$. Then $f(x) \in \operatorname{st}(W, \mathscr{W})$, whence $x \in U$. Thus $g^{-1}W \subset U$, and so g is a $\mathscr{U}$-map.

2.7. LEMMA. *Let X be a separable metric space. Then there exist open covers $\mathscr{B}_j$, $j \in \mathbb{N}$, of X, each consisting of two sets, such that every continuous map of X into a metric space Y which is a $\mathscr{B}_j$-map for all $j \in \mathbb{N}$ is an embedding.*

PROOF. Take a countable basis $(U_j)_{j\in\mathsf{N}}$ for X and construct, for every pair i, j with $\bar{U}_i\subset U_j$, an open cover $(U_j, X\setminus\bar{U}_i)$ of X. It is easy to see that these covers satisfy the lemma; cf. [18, p. 108].

2.8. DEFINITION. Let X and M be as in Theorem 2.1. If $\mathscr{B}=(B_1, B_2)$ is an open cover of X and if L is an $(m-n-1)$-dimensional affine subspace of R^m, we denote by $D(\mathscr{B}, L)$ the set of all $\mathscr{B}$-maps $f\colon X\to M$ for which $\mathrm{cl}_M fX\cap L=\emptyset$.

2.9. LEMMA. *$D(\mathscr{B}, L)$ is open and dense in $C(X, M)$.*

PROOF. The openness follows from Lemmas 2.6 and 2.2. For the denseness it suffices by Lemma 2.4 to prove that $C^{\mathrm{int}}(X, M)\subset\overline{D(\mathscr{B}, L)}$. Thus let $f\colon X\to M$ be continuous with $\mathrm{cl}_M fX\subset\mathrm{int}\, M$ and let $\mathscr{U}$ be an open cover of M. We must find a map g in $N(f,\mathscr{U})\cap D(\mathscr{B}, L)$. We may assume that $\mathscr{U}$ is locally finite, $\mathrm{cl}_M U$ is compact for each $U\in\mathscr{U}$ and every $U\in\mathscr{U}$ which meets fX is an open ball of R^m. Choose a locally finite open pointwise star-refinement $\mathscr{V}$ of $\mathscr{U}$ by non-empty sets. Let the points $p_1,\ldots, p_{m-n}\in L$ be affinely independent. Then there are disjoint countable sets P_1 and P_2 in $\mathsf{R}^m\setminus L$ which are dense in R^m such that the set

$$P_1\cup P_2\cup\{p_1,\ldots,p_{m-n}\}$$

is in general position in R^m. We choose for every $V\in\mathscr{V}$ points $p_i(V)\in P_i\cap V$, $i=1,2$. Since $\dim X\leqq n$, by [7, Proposition 3.2.2 or Problem 1.7E] there is a locally finite open cover $\mathscr{W}=(W_j)_{j\in\mathsf{N}}$ of X which refines both $\mathscr{B}$ and $f^{-1}\mathscr{V}$ and which is of order $\leqq n$, i.e., each point of X belongs to W_j for at most $n+1$ indexes $j\in\mathsf{N}$. We choose for every index $j\in\mathsf{N}$ a number $l(j)\in\{1,2\}$ and a set $V_j\in\mathscr{V}$ such that $W_j\subset B_{l(j)}$ and $fW_j\subset V_j$, and then we define $q_j=p_{l(j)}(V_j)$. Obviously the set $\{q_j \mid j\in\mathsf{N}\}$ is discrete and closed in M.

Let $x\in X$. The set

$$\{q_j \mid j\in\mathsf{N} \quad\text{and}\quad x\in W_j\}$$

is the set of the vertices of an at most n-dimensional (closed) simplex $S_x\subset\mathsf{R}^m$ and it is contained in $\mathrm{st}\,(f(x),\mathscr{V})$. Choose $U_x\in\mathscr{U}$ containing $\mathrm{st}\,(f(x),\mathscr{V})$; then U_x is a ball. It follows that $S_x\subset U_x\subset M$. Let $(\varphi_j)_{j\in\mathsf{N}}$ be a partition of unity on X subordinated to $\mathscr{W}$. We set

$$g(x) = \sum_{j\in\mathsf{N}}\varphi_j(x)q_j \quad\text{for } x\in X\,.$$

Let $x\in X$; then $g(x)\in S_x$, which implies that $g(x)\in M$ and that $f(x), g(x)\in U_x$. Clearly $g\colon X\to M$ is continuous. We have yet to show that $g\in D(\mathscr{B}, L)$.

We first claim that the family

$$\mathscr{S} = \{S \mid \exists x \in X: S = S_x\}$$

is locally finite in M. Let $y \in M$ and let F be a compact neighborhood of y in M. Then

$$F_1 = \bigcup\{\mathrm{cl}_M U \mid U \in \mathscr{U},\ U \cap F \neq \emptyset\}$$

is compact. Consider the simplices S_x with $S_x \cap F \neq \emptyset$. Let q be a vertex of such a simplex S_x. Then $q \in U_x$ and $U_x \cap F \neq \emptyset$, whence $q \in F_1$. Therefore there are only finitely many such vertices q and, consequently, only finitely many simplices $S \in \mathscr{S}$ meeting F. This proves our claim.

Next we show that if $A \subset X$ and if the simplices S_x, $x \in A$, have a common point, then they have a common vertex. Choose $y \in \bigcap \{S_x \mid x \in A\}$. We denote the vertices of a simplex S_x, $x \in X$, by $q_{x,0}, q_{x,1}, \ldots, q_{x,r(x)}$ $(r(x) \leqq n)$. We have for each $x \in A$ that

$$y = \sum_{i=0}^{r(x)} t_{x,i} q_{x,i}$$

where $t_{x,i} \in I^1$ with $\sum_{i=0}^{r(x)} t_{x,i} = 1$. Assuming $A \neq \emptyset$ fix $a \in A$ and choose k with $t_{a,k} \neq 0$. If $x \in A$, then in the sum

$$\sum_{i=0}^{r(x)} t_{x,i} q_{x,i} + \sum_{i=0}^{r(a)} (-t_{a,i}) q_{a,i} = 0$$

there are at most $2n+2 \leqq m+1$ terms and the sum of the coefficients is equal to zero. Therefore $q_{x,i} = q_{a,k}$ for some i, so $q_{a,k}$ is the desired vertex.

Now we can prove that g is a $\mathscr{B}$-map. Let $y \in M$. Let $A = \{x \in X \mid y \in S_x\}$. Then

$$N = M \setminus \bigcup \{S_x \mid x \in X \setminus A\}$$

is an open neighborhood of y in M. The simplices S_x, $x \in A$, have a common vertex, q. There is $i \in \{1,2\}$ with $q \in P_i$. We show that $g^{-1}N \subset B_i$. Let $x \in g^{-1}N$. Then $g(x) \in S_x \cap N$, whence $x \in A$. Thus there is $j \in \mathbb{N}$ with $q = q_j$ and $x \in W_j$. Then $l(j) = i$, so $x \in W_j \subset B_i$. Hence g is a $\mathscr{B}$-map.

We finally assert that $\mathrm{cl}_M gX \cap L = \emptyset$. Let $x \in X$. Since the number of the points $q_{x,0}, \ldots, q_{x,r(x)}, p_1, \ldots, p_{m-n}$ is $\leqq m+1$, they are affinely independent. Hence $S_x \cap L = \emptyset$. Let $T = \bigcup_{x \in X} S_x$. Then $T \cap L = \emptyset$, T is closed in M and $gX \subset T$. The assertion follows. Thus g is the required map.

2.10. Proof of Theorem 2.1. First, $C(X, M)$ is a Baire space by Lemma 1.3 because M, being locally compact, is completely metrizable. Let $\mathscr{B}_1, \mathscr{B}_2, \ldots$ be the open covers of X given by Lemma 2.7. Obviously, $\mathbb{R}^m \setminus N_m^n = \bigcup_{j \in \mathbb{N}} L_j$, where the L_j's are the $(m-n-1)$-dimensional affine subspaces of $\mathbf{R}^m$ of the form $x_{i_1} = r_1, \ldots, x_{i_{n+1}} = r_{n+1}$ where $1 \leqq i_1 < \ldots < i_{n+1} \leqq m$ and every r_k is rational. By Lemmas 2.9 and 2.4, the sets $D(\mathscr{B}_i, L_j)$ for $i, j \in \mathbf{N}$ and $C^{\text{int}}(X, M)$ are open and dense in $C(X, M)$. Hence the set

$$D = \left(\bigcap \{D(\mathscr{B}_i, L_j) \mid i, j \in \mathbf{N}\}\right) \cap C^{\text{int}}(X, M)$$

is a dense G_δ-set in $C(X, M)$. Consider $f \in D$. By Lemma 2.7, f is an embedding. Since clearly $\text{cl}_M fX \subset N_m^n \cap \text{int}\, M$, the proof is complete.

2.11. Corollary. *For each $n \in \mathbf{N} \cup \{0\}$ there exists an n-dimensional closed set $P^n \subset \mathbf{R}^{2n+1}$ such that every locally compact separable metric space X with $\dim X \leqq n$ is homeomorphic to a closed subset of P^n.*

Proof. Write $N = N_{2n+1}^n$. By Theorem 2.1 there is an embedding $h: N \to \mathbf{R}^{2n+1}$ such that $\overline{hN} \subset N$ and $|h(x) - x| < 1$ for all $x \in N$. Then $P^n = \overline{hN}$ is n-dimensional. One can find a proper map $\varphi: X \to \mathbf{R}^1$; for example, if $(\varphi_j)_{j \in \mathbf{N}}$ is a partition of unity on X with compact supports, one sets $\varphi = \sum_{j \in \mathbf{N}} j\varphi_j$. Combining φ with the inclusion map $\mathbf{R}^1 \to \mathbf{R}^{2n+1}$ one gets a proper map $f: X \to \mathbf{R}^{2n+1}$. Theorem 2.1 gives an embedding $g_0: X \to N$ with $|g_0(x) - f(x)| < 1$ for all $x \in X$. Consider the embedding $g = hg_0: X \to P^n$. Since

$$|g(x) - f(x)| < 2 \quad \text{for all } x \in X,$$

it follows that g is proper as a map into $\mathbf{R}^{2n+1}$. Therefore gX is closed in $\mathbf{R}^{2n+1}$ and thus in P^n, too.

2.12. Remarks. 1. As observed in [2, Lemma 3.1], every locally compact separable metric space is homeomorphic to a closed subset of the space P^∞ which one obtains of the Hilbert cube Q by deleting a point; this follows at once from Urysohn's embedding theorem and the homogeneity of Q by a one-point compactification argument. If $n = 0$, Corollary 2.11 also follows analogously from properties of the Cantor set.

2. Engelking [8] gave a new proof for Corollary 2.11. With his permission, we present it here, in a slightly simplified form. For every point $x = (x_1, \ldots, x_{2n+1})$ in $N = N_{2n+1}^n$ one can find an embedding $h: N \to N$ with $h(x) = p = (\pi, \ldots, \pi)$; in fact, applying [7, Problem 1.3.G(a)] one finds for each $i \leqq 2n+1$ a homeomorphism $h_i: \mathbf{R}^1 \to \mathbf{R}^1$ such that

$$h_i(\mathbf{Q} \setminus \{x_i\}) = \mathbf{Q} \quad \text{and} \quad h_i(x_i) = \pi,$$

and then one lets $h = (h_1 \times \ldots \times h_{2n+1}) | N$. By Theorem A there is an embedding $g: N \to \mathbf{R}^{2n+1} \cup \{\infty\}$ such that $\dim \overline{gN} = n$ and $g(p) = \infty$. Hence a one-point compactification argument (cf. [13, Corollary 2, p. 32]) and Theorem A give Corollary 2.11 with $P^n = \overline{gN} \cap \mathbf{R}^{2n+1}$.

By slightly modifying the proof of Theorem 2.1 one obtains results on approximation of continuous maps of separable metric spaces into open subsets of infinite-dimensional convex sets in topological vector spaces by embeddings. We give two theorems. The first improves [13, Theorem V.4] for finite-dimensional spaces X and the second is a generalization of the latter result. We define the Hilbert cube Q as the set of points $(x_j)_{j\in\mathbf{N}}$ in the Hilbert space l^2 for which $0 \leqq x_j \leqq 1/j$ for all j. For $n \in \mathbf{N} \cup \{0\}$ we denote by N^n_∞ the subset of Q of points at most n of whose coordinates are rational. By [13, Example IV.2], $\dim N^n_\infty = n$.

2.13. THEOREM. *Let X be a separable metric space with* $\dim X \leqq n$ *and let* $U \subset Q$ *be open. Then the set*

$$\{f \in \operatorname{Emb}(X, U) \mid \operatorname{cl}_U fX \subset N^n_\infty\}$$

contains a dense G_δ-set in $C(X, U)$.

PROOF. The proof is almost exactly the same as that of Theorem 2.1. Clearly, $N^n_\infty = Q \setminus \bigcup_{j\in\mathbf{N}} L_j$, where the L_j's are the closed affine subspaces of l^2 of codimension $n+1$ of the form $x_{i_1} = r_1, \ldots, x_{i_{n+1}} = r_{n+1}$ where $1 \leqq i_1 < \ldots < i_{n+1}$ and every r_k is rational. We now only choose for every j the disjoint sets $P_1, P_2 \subset Q \setminus L_j$, dense in Q, such that the set $P_1 \cup P_2$ is in general position in Q, i.e., every finite subset of it is affinely independent, and such that every simplex spanned by a subset of $P_1 \cup P_2$ of at most $n+1$ points is disjoint from L_j.

2.14. THEOREM. *Let E be a locally convex metric vector space, let C be an infinite-dimensional locally compact convex set in E (C is then separable), let $U \subset C$ be open in C and let X be a separable metric space. Then* $\operatorname{Emb}(X, U)$ *contains a dense G_δ-set in $C(X, U)$.*

PROOF. The proof is the same as that of Theorem 2.13, except for the simplification that we are concerned neither with the planes L_j nor with the order of the cover $\mathcal{W}$ of X.

3. Relatively approximating continuous maps of zero- or one-dimensional metric spaces into manifolds.

In this section we will apply the following theorem due to Bothe. We prove analogous or more general results on extending (closed) embeddings of locally compact closed subspaces of zero- or one-dimensional separable metric spaces into manifolds. We also approximate relatively continuous extensions of such embeddings by embeddings. We will need only Theorem 3.6 (for $X_0=\varnothing$) and Lemma 3.8 later in this paper.

3.1. THEOREM (Bothe [4, Theorem 1 and Remark 1]). *Let* $n=0$ *or* 1. *Let* X *be an n-dimensional separable metric space,* $X_0\subset X$ *compact and* $f_0\colon X_0 \to I^m$, $m \geqq 2n+1$, *an embedding. Then* f_0 *can be extended to an embedding of* X *into* I^m.

Moreover, $\operatorname{Emb}(X,I^m;f_0)$ *is dense in* $C(X,I^m;f_0)$.

3.2. REMARKS 1. In [4, pp. 130–131] the first, extension part of Theorem 3.1 is reduced to the special case of the second, approximation part where X is compact, which then is proved in [4], by the aid of the fact ([13, Theorem V 6]) that every separable metric space has a dimension-preserving metric compactification and of Tietze's theorem. However, the second part obviously does not follow from the special case by the aid of this compactification theorem, contrary to [4]. Instead, one has to use a stronger compactification result ([7, Lemma 1.13.3] or Lemma 3.8); cf. the proof of Theorem 3.9. We will need Theorem 3.1 only for a compact X.

2. The first part of Theorem 3.1 does not hold if X_0 is supposed to be merely a closed subset of X; see [4, Example 1]. For this reason, in this section we shall always suppose that the set corresponding X_0 in Theorem 3.1 is locally compact.

3. For $n\geqq 2$, Theorem 3.1 does not hold ([4, Example 2]); however, by [4, Theorem 2 and Remark 1] it holds for embeddings f_0 with the property that $\mathsf{R}^m\setminus f_0X_0$ is 1-ULC.

The following extension result is a simple consequence of the first part of Theorem 3.1.

3.3. THEOREM. *Let* $n=0$ *or* 1. *Let* X *be an n-dimensional separable metric space,* X_0 *a locally compact closed subset of* X *and* $f_0\colon X_0 \to \mathsf{R}^m$, $m\geqq 2n+1$, *a closed embedding. Then there exists an embedding* $f\colon X \to \mathsf{R}^m$ *extending* f_0.

Moreover, if X *is locally compact,* f *can be chosen to be closed.*

PROOF. One can embed X as a subspace of an n-dimensional locally compact separable metric space S such that X_0 is closed in S. To see this, embed X as a subspace of an n-dimensional compact metric space T. Since X_0 is locally compact, X_0 is open in $\operatorname{cl}_T X_0$. Hence $S=T\setminus(\operatorname{cl}_T X_0\setminus X_0)$ is the required space. Therefore we may assume that X is locally compact.

Choose a compact metric space Y containing X topologically as a subspace with a point $p\in Y$ such that $Y\setminus X=\{p\}$. By [13, Corollary 2, p. 32], $\dim Y=n$.

Further, $Y_0 = X_0 \cup \{p\}$ is closed in Y. Let $\bar{\mathsf{R}}^m = \mathsf{R}^m \cup \{\infty\}$. Then we can define an embedding g_0: $Y_0 \to \bar{\mathsf{R}}^m$ by $g_0(x) = f_0(x)$ for $x \in X_0$ and $g_0(p) = \infty$. Now, $f_0 X_0$ is a proper closed subset of R^m, so, we may assume that $f_0 X_0 \cap I^m = \emptyset$. Since $\bar{\mathsf{R}}^m \setminus \operatorname{int} I^m$ is homeomorphic to I^m, Theorem 3.1 gives an embedding g: $Y \to \bar{\mathsf{R}}^m$ that extends g_0. Then g defines a closed embedding f: $X \to \mathsf{R}^m$ with $f \mid X_0 = f_0$.

3.4. EXAMPLE. For $n=1$, in the first part of Theorem 3.3 the supposition that f_0 is closed cannot be omitted (for $n=0$ it can, as we will prove in Theorem 3.12). To see this, let $X = \mathsf{N} \cup [1,2]$, $X_0 = \mathsf{N}$ and $m \geqq 1$. Then there is an embedding f_0: $X_0 \to \mathsf{R}^m$ such that R^{m-1} separates $f_0(1)$ and $f_0(2)$ and $\mathsf{R}^{m-1} \subset \overline{f_0 X_0}$. Clearly no continuous extension of f_0 to X can be an embedding. This example also shows that in Theorem 3.9 it is essential that f_0 is closed.

3.5. LEMMA. *Let X be a metric space, Y an* ANR, *$f \in C(X, Y)$ and $\varepsilon \in C_+(X)$. Then there exists $\delta \in C_+(X)$ with the property that for every closed $A \subset X$, every $g \in U(f \mid A, \delta \mid A)$ has an extension $\bar{g} \in U(f, \varepsilon)$.*

PROOF. The lemma is probably known. Anyway, it can be proved like [15, Theorem 5.17]. Only observe that one can embed Y isometrically as a closed subset of a normed linear space E by [24, p. 192] and that E is an AR by Dugundji's theorem [6, IX, Theorem 6.1].

3.6. THEOREM. *Let $n=0$ or 1. Let X be a locally compact separable metric space with* $\dim X \leqq n$, *M an m-manifold, $m \geqq 2n+1$, $X_0 \subset X$ closed and f_0: $X_0 \to M$ a closed embedding. Then* CEmb $(X, M; f_0)$ *is dense in* Prop $(X, M; f_0)$.

PROOF. *Special case*: X compact. Let f: $X \to M$ be continuous with $f \mid X_0 = f_0$ and let $\varepsilon > 0$. We must construct an embedding g: $X \to M$ with $g \mid X_0 = f_0$ and $d(f, g) < \varepsilon$. Choose sets $A_1, \ldots, A_k \subset M$ homeomorphic to I^m such that

$$fX \subset \bigcup_{j=1}^{k} \operatorname{int}_M A_j \,.$$

Choose compact sets $B_1, \ldots, B_k \subset X$ which cover X such that $fB_j \subset \operatorname{int}_M A_j$ for each j. We may assume that $\varepsilon \leqq d(fB_j, M \setminus A_j)$ for each j. Since M is an ANR by [9, Theorem 3.3], by Lemma 3.5 there are numbers $0 < \delta_1 \leqq \ldots \leqq \delta_k = \varepsilon$ such that if $1 \leqq j < k$, $A \subset X$ is closed and h: $A \to M$ is continuous with $d(h, f \mid A) < \delta_j$, then h has a continuous extension $\bar{h}$: $X \to M$ with $d(\bar{h}, f) < \delta_{j+1}$.

Let $X_j = X_0 \cup B_1 \cup \ldots \cup B_j$. We will recursively define embeddings f_j: $X_j \to M$, $1 \leqq j \leqq k$, such that $f_j \mid X_{j-1} = f_{j-1}$ and $d(f_j, f \mid X_j) < \delta_j$. Then, since $X_k = X$, $g = f_k$ is the required embedding. So suppose that $1 \leqq j \leqq k$ and that

$f_0, \ldots, f_{j-1}$ are defined satisfying the requirements. There is a continuous extension $g_j\colon X \to M$ of f_{j-1} with $d(g_j, f) < \delta_j$ (take $g_1 = f$). Then $g_j B_j \subset A_j$. Consider the compact set $C_j = B_j \cup f_{j-1}^{-1} A_j$. We have

$$X_j = X_{j-1} \cup C_j,\ g_j C_j \subset A_j \quad \text{and} \quad X_{j-1} \cap C_j = f_{j-1}^{-1} A_j .$$

Therefore, by Theorem 3.1, there is an embedding $h_j\colon C_j \to M$ such that $h_j C_j \subset A_j$, $h_j = f_{j-1}$ on $X_{j-1} \cap C_j$ and

$$d(h_j, g_j \mid C_j) < \delta_j - d(g_j, f) ,$$

whence $d(h_j, f \mid C_j) < \delta_j$. Define a continuous map $f_j\colon X_j \to M$ by $f_j \mid X_{j-1} = f_{j-1}$ and $f_j \mid C_j = h_j$. Then $d(f_j, f \mid X_j) < \delta_j$. Finally, f_j is clearly injective and thus an embedding.

General case. Let $f \in \operatorname{Prop}(X, M)$ with $f \mid X_0 = f_0$ and let $\varepsilon \in C_+(X)$. We must construct a closed embedding $g\colon X \to M$ with $g \mid X_0 = f_0$ and $g \in U(f, \varepsilon)$. There are compact sets $K_j \subset M$, $j \in \mathbf{N}$, covering M such that $K_j \subset \operatorname{int} K_{j+1}$ for each j. Set $K_0 = \emptyset$. Then the sets

$$A_j = K_j \setminus \operatorname{int} K_{j-1}, \quad j \in \mathbf{N} ,$$

are compact and cover M. There is a compact neighborhood U_j of A_j in M for every j such that $U_i \cap U_j = \emptyset$ if $|i-j| \geqq 2$. Then $(U_j)_{j \in \mathbf{N}}$ is locally finite in M. The sets $B_j = f^{-1} A_j$ are compact and form a locally finite cover of X. By [28, Lemma 5.1] we may assume that $\varepsilon(x) \leqq d(f(x), M \setminus U_j)$ for all j and $x \in B_j$. Let

$$\mathbf{N}_1 = \{ j \in \mathbf{N} \mid j \text{ odd} \} \quad \text{and} \quad \mathbf{N}_2 = \{ j \in \mathbf{N} \mid j \text{ even} \} .$$

Consider the closed set $A = X_0 \cup (\bigcup_{j \in \mathbf{N}_1} B_j)$ in X. Lemma 3.5 gives $\delta \in C_+(A)$, $\delta \leqq \varepsilon \mid A$, such that every map $h \in U(f \mid A, \delta)$ has an extension $\bar{h} \in U(f, \varepsilon)$.

We first construct $h = g \mid A$. Let $j \in \mathbf{N}_1$. The set $C_j = B_j \cup f_0^{-1} U_j \subset A$ is compact, and $X_0 \cap C_j = f_0^{-1} U_j$. The special case gives an embedding $f_j\colon C_j \to M$ with $f_j = f_0$ on $X_0 \cap C_j$ and $d(f_j(x), f(x)) < \delta(x)$ for all $x \in C_j$. Then $f_j C_j \subset U_j$. The family $(C_j)_{j \in \mathbf{N}_1}$ is disjoint and locally finite in X because $f C_j \subset U_j$ for $j \in \mathbf{N}_1$. Thus setting $h \mid X_0 = f_0$ and $h \mid C_j = f_j$ for every $j \in \mathbf{N}_1$ one gets a continuous map $h\colon A \to M$ in $U(f \mid A, \delta)$. Clearly h is injective. It is easy to see that h is closed. Hence h is an embedding.

We finally extend h to an embedding g of X. There is an extension $\bar{h} \in U(f, \varepsilon)$ of h. Let $j \in \mathbf{N}_2$. Consider the compact set $C_j = B_j \cup h^{-1} U_j$. We have $\bar{h} B_j \subset U_j$, which implies $A \cap C_j = h^{-1} U_j$. The special case yields an embedding $f_j\colon C_j \to M$ with $f_j = h$ on $A \cap C_j$ and $d(f_j(x), f(x)) < \varepsilon(x)$ for all $x \in C_j$. Then $f_j C_j \subset U_j$. The family $(C_j)_{j \in \mathbf{N}_2}$ is disjoint and locally finite in X, because $\bar{h} C_j \subset U_j$ for $j \in \mathbf{N}_2$, and it covers $X \setminus A$. Thus setting $g \mid A = h$ and $g \mid C_j = f_j$ for every $j \in \mathbf{N}_2$ one gets a continuous map $g\colon X \to M$ in $U(f, \varepsilon)$ with $g \mid X_0 = f_0$. Clearly g is injective and closed and thus an embedding.

3.7. REMARK. It follows immediately from Lemmas 4.3 and 4.4 that CEmb $(X, M; f_0)$ in Theorem 3.6 is a G_δ-set in Prop $(X, M; f_0)$ (moreover, the latter space can be shown to be a Baire space). Further, it is easy to see that in Theorem 3.6 every closed set A in M with $\dim A \leqq n$ is a Z^n-set in M. Theorem 3.6 is thus Theorem D for $n \leqq 1$.

The following lemma generalizes slightly [7, Lemma 1.13.3], which one obtains from the lemma by supposing that Y is compact (then, $\hat{X}$ is compact, too).

3.8. LEMMA. *Let X be a separable metric space, Y a locally compact metric space and $f: X \to Y$ a continuous map. Then there exists a locally compact separable metric space $\hat{X}$ containing X topologically as a dense subspace with* $\dim \hat{X} = \dim X$ *and such that f is extendable to a proper map $\hat{f}: \hat{X} \to Y$.*

PROOF. We may replace Y by $\overline{fX}$ and thus assume that Y is separable. Then there is a compact metric space Z containing Y topologically as a subspace such that $Z \setminus Y$ is a point. By [7, Lemma 1.13.3] there exists a compact metric space $\tilde{X}$ containing X topologically as a dense subspace with $\dim \tilde{X} = \dim X$ and such that $f: X \to Z$ has an extension to a continuous map $\tilde{f}: \tilde{X} \to Z$. Then the space $\hat{X} = \tilde{f}^{-1} Y$ and the map $\hat{f}: \hat{X} \to Y, \hat{f}(x) = \tilde{f}(x)$ for $x \in \hat{X}$, satisfy the lemma.

By the aid of Lemma 3.8 it is easy to generalize Theorem 3.6 to deal with approximation of continuous maps instead of only approximation of proper maps:

3.9. THEOREM. *Let $n = 0$ or 1. Let X be a separable metric space with* $\dim X \leqq n$, *M an m-manifold, $m \geqq 2n+1$, X_0 a locally compact closed subset of X and $f_0: X_0 \to M$ a closed embedding. Then the set*

$$\{g \in \mathrm{Emb}\,(X, M; f_0) \mid \dim \overline{gX} = \dim X\}$$

is dense in $C(X, M; f_0)$.

PROOF. Let $f: X \to M$ be continuous with $f \mid X_0 = f_0$ and let $\mathcal{U}$ be an open cover of M. By Lemma 3.8 there exist a locally compact separable metric space $\hat{X}$ containing X topologically as a dense subspace with $\dim \hat{X} = \dim X$ and a proper map $\hat{f}: \hat{X} \to M$ extending f. Since $\hat{f} \mid X_0 = f_0$ is proper, X_0 is closed in $\hat{X}$, as it is easy to see. Hence Theorem 3.6 gives a closed embedding $\hat{g}: \hat{X} \to M$ in $N(\hat{f}, \mathcal{U})$ with $\hat{g} \mid X_0 = f_0$. Then $g = \hat{g} \mid X$ is an embedding in $N(f, \mathcal{U})$ with $g \mid X_0 = f_0$. Obviously $\overline{gX} = \hat{g}\hat{X}$, whence $\dim \overline{gX} = \dim X$.

3.10. REMARK. In a similar way one can deduce Theorem E from Theorem D applying Lemma 3.8. Theorem 3.9 is the same as Theorem E for $n \leqq 1$; cf. Remark 3.7.

Theorem 3.1 and the first part of Theorem 3.3 are corollaries of Theorem 3.9 by Tietze's theorem. Theorem 3.6 has the following consequence, according to which every embedding of a locally compact subset can be extended to a neighborhood of this set:

3.11. THEOREM. *Let $n=0$ or 1. Let X be a separable metric space with* $\dim X \leqq n$, *M an m-manifold, $m \geqq 2n+1$, $X_0 \subset X$ locally compact and $f_0: X_0 \to M$ an embedding. Then f_0 has an extension to an embedding $f: U \to M$ of some neighborhood U of X_0.*

Moreover, if X is locally compact and f_0 is closed, then f can be chosen to be closed.

PROOF. Since X_0 is locally compact, it is closed in an open set V of X. We may replace X by V and thus assume that X_0 is closed in X. Similarly we may assume that f_0 is closed. By the same argument as in the proof of Theorem 3.3, we may assume that X is locally compact. Since M is an ANR, f_0 has a continuous extension $g: W \to M$ to an open neighborhood W of X_0. By [2, Lemma 3.2] there is a closed neighborhood $U \subset W$ of X_0 such that $g \mid U$ is proper. Theorem 3.6 then implies the existence of a closed embedding $f: U \to M$ with $f \mid X_0 = f_0$.

The next extension result generalizes the first part of Theorem 3.1, Theorem 3.3, and Theorem 3.11 for $n=0$.

3.12. THEOREM. *Let X be a zero-dimensional separable metric space, M a non-empty m-manifold, $m \geqq 1$, X_0 a locally compact closed subset of X and $f_0: X_0 \to M$ an embedding. Then there exists an embedding $f: X \to M$ extending f_0.*

Moreover, if X is locally compact and f_0 is closed, then f can be taken to be closed, provided that M is non-compact if X is non-compact.

PROOF. As in the proof of Theorem 3.11, we may assume that X is locally compact and that f_0 is closed. Theorem 3.11 then gives a closed neighborhood U of X_0 and a closed embedding $f_1: U \to M$ that extends f_0. By [13, E), p. 15] we may assume that U is open, too. Now $f_1 U$ is closed and nowhere dense in M. It follows that there is an embedding $f_2: X \setminus U \to M$ with

$$\overline{f_2(X \setminus U)} \cap f_1 U = \varnothing . \tag{3.13}$$

Then $f=f_1 \cup f_2$: $X \to M$ is an embedding extending f_0.

Suppose now that X and M are non-compact. We show that f_2 can then be chosen to be closed, in which case f is closed, too. Obviously, there is a disjoint, in M locally finite family $(A_j)_{j\in \mathbb{N}}$ of subsets of $M \setminus f_1 U$ homeomorphic to I^m. There is a cover $(B_j)_{j\in \mathbb{N}}$ of $X \setminus U$ by compact open sets. Set

$$C_1 = B_1 \quad \text{and} \quad C_j = B_j \setminus \bigcup_{i=1}^{j-1} B_i \quad \text{for } j>1 .$$

Then $(C_j)_{j\in \mathbb{N}}$ is a cover of $X \setminus U$ by disjoint compact open sets. For each j choose an embedding g_j: $C_j \to A_j$. Define f_2: $X \setminus U \to M$ by $f_2 | C_j = g_j$ for all j. Then f_2 is a closed embedding satisfying (3.13).

4. Approximating proper maps of locally compact metric spaces into LIP manifolds.

This section is devoted to proving the following theorem.

4.1. THEOREM. *Let X be a locally compact separable metric space with* $\dim X \leqq n$ *and let M be a* LIP *m-manifold, $m \geqq 2n+1$. Then closed embeddings are dense in* Prop (X, M).

More precisely, CEmb (X, M) *and* $\text{CEmb}^{\text{int}}(X, M)$ *are dense G_δ-sets in* Prop (X, M).

In Theorem 5.1 we will drop the LIP supposition on M from Theorem 4.1.

4.2. LEMMA. *If X is a metric space and M a manifold, then* $\text{Prop}^{\text{int}}(X, M)$ *is open and dense in* Prop (X, M).

PROOF. The openness is easy to see, and the denseness follows from Lemma 2.3. Alternatively, one could use Lemma 2.4.

4.3. NOTATION AND LEMMA. *Let X and Y be metric spaces and let $\mathcal{U}$ be an open cover of X. Then the set* $\text{Prop}_{\mathcal{U}}(X, Y)$ *of all proper $\mathcal{U}$-maps of X into Y is open in* Prop (X, Y).

PROOF. This follows from Lemma 2.6.

4.4. LEMMA. *Let X and Y be metric spaces and define the open covers $\mathcal{U}_j = (B(x, 1/j))_{x\in X}$ of X for $j \in \mathbb{N}$. Then*

$$\text{CEmb}(X, Y) = \bigcap_{j\in \mathbb{N}} \text{Prop}_{\mathcal{U}_j}(X, Y) .$$

PROOF. If $f\colon X \to Y$ is a closed embedding, then f is clearly proper and a $\mathscr{U}$-map for every open cover $\mathscr{U}$ of X. Conversely, if a proper map $f\colon X \to Y$ is a $\mathscr{U}_j$-map for every $j \in \mathbf{N}$, then f is injective and thus a closed embedding.

In Lemma 4.9 we will show that in the situation of Theorem 4.1, $\mathrm{Prop}_{\mathscr{U}}(X, M)$ is dense in $\mathrm{Prop}(X, M)$ for every open cover $\mathscr{U}$ of X. For this we need some concepts and auxiliary results.

4.5. HAUSDORFF MEASURE AND METRIC DIMENSION. Let X and Y denote separable metric spaces. For every real number $p \geqq 0$ we let $\mathscr{H}^p(X)$ denote the p-dimensional Hausdorff measure of X; see [13, Definition VII.1]. We denote by $\dim_H X$ the Hausdorff dimension of X. For the definition and the inequality $\dim_H X \geqq \dim X$, see [13, p. 107]. If $f\colon X \to Y$ is LIP, then obviously $\dim_H fX \leqq \dim_H X$.

Suppose that X is totally bounded. Let

$$N(X, \varepsilon) = \min\{k \in \mathbf{N} \mid X = A_1 \cup \ldots \cup A_k,\ d(A_j) < \varepsilon\}$$

for $\varepsilon > 0$. We define the *metric dimension* $\dim_{\mathrm{met}} X$ of X by

$$\dim_{\mathrm{met}} X = \limsup_{\varepsilon \to 0+} \log N(X, \varepsilon) / \log(1/\varepsilon);$$

cf. [19, p. 156] and [27, p. 68]. If X is not totally bounded, we define $\dim_{\mathrm{met}} X = \infty$. Then $\dim_{\mathrm{met}} X \geqq \dim_H X$; cf. [27, p. 68]. Thus, $\dim_{\mathrm{met}} X \geqq \dim X$. Clearly, $\dim_{\mathrm{met}} A \leqq \dim_{\mathrm{met}} X$ for $A \subset X$.

The following lemma gives an estimate from above for the Hausdorff dimension of a cartesian product. Here, we take d,

$$d((x_1, x_2), (y_1, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2)),$$

as the metric for the cartesian product $X_1 \times X_2$ of two metric spaces (X_1, d_1) and (X_2, d_2).

4.6. LEMMA. *For all separable metric spaces X and Y,*

$$\dim_H X \times Y \leqq \min(\dim_H X + \dim_{\mathrm{met}} Y, \dim_{\mathrm{met}} X + \dim_H Y).$$

PROOF. This follows from [27, Satz 7′ and p. 68] and is easy to check.

4.7. LEMMA. *Every separable metric space X can be remetrized by a totally bounded metric for which $\dim_{\mathrm{met}} X = \dim X$.*

PROOF. We may assume that $\dim X = n < \infty$. Since X has an n-dimensional metric compactification ([13, Theorem V 6]), we may assume that X is compact. Hence, by [19, pp. 161–162] X can be embedded as a subspace of $\mathbf{R}^{2n+1}$ in such a way that

$$N(X,\varepsilon) \leqq (1/\varepsilon)^{n+\delta(\varepsilon)} \quad \text{for every } \varepsilon>0$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then $\dim_{\text{met}} X \leqq n$, and so $\dim_{\text{met}} X = n$.

The proof in [19] is based on a rather difficult polyhedral approximation of X. Therefore we give yet another proof for the lemma, assuming, as above, that $\dim X = n < \infty$. In [3] one constructs for every sequence $(\varkappa_j)_{j\in \mathbf{N}}$ of integers $\geqq 3$ an n-dimensional compact set $U^n \subset \mathbf{R}^{2n+1}$ such that every n-dimensional separable metric space can be embedded into U^n. (In this construction the case $n=0$ can be included though it was excluded in [3, p. 209].) Hence it suffices to check that for the sequence $\varkappa_j = 2^{j+1}$ we have $\dim_{\text{met}} U^n \leqq n$. Since in this case $U^n = P^n_{2n+1}$ in terms of [26], this is easily done by the aid of the facts about the function $\varepsilon \mapsto N(P^n_{2n+1}, \varepsilon)$ given in [26].

4.8. Notation and Lemma ([15, Corollary 5.18]). *If X is a metric space and M a* LIP *manifold, the set* LIP (X, M) *of all* LIP *maps of X into M is dense in* $C_s(X, M)$.

4.9. Lemma. *Let X and M be as in Theorem* 4.1 *and let $\mathscr{U}$ be an open cover of X. Then* $\text{Prop}_{\mathscr{U}}(X, M)$ *is dense in* $\text{Prop}(X, M)$.

Proof. The proof is based on the ideas of [15, Theorem 6.18]. By Lemma 4.7 we may assume that $\dim_{\text{met}} X \leqq n$. Since $\text{Prop}(X, M)$ is open in $C_s(X, M)$, Lemmas 4.2 and 4.8 imply that the set $\text{Prop}^{\text{int}}(X, M) \cap \text{LIP}(X, M)$ is dense in $\text{Prop}(X, M)$. Thus it suffices to prove the following: Let $f\colon X \to M$ be a proper LIP map with $fX \subset \text{int}\, M$ and let $\varepsilon \in C_+(X)$. Then there exists a proper $\mathscr{U}$-map $g\colon X \to M$ in $U(f, \varepsilon)$.

It is easy to find points $a_j \in X$, numbers $\delta_j \in (0,1)$, open sets $U_j \subset M$ and LIP homeomorphisms

$$\psi_j\colon U_j \to \mathbf{R}^m \quad \text{for } j \in J$$

where either $J = \mathbf{N}$ or $J = \{1, \ldots, k\}$ for some $k \in \mathbf{N}$ such that the following conditions hold:

(i) The balls $A_j = B(a_j, \delta_j)$, $j \in J$, form a cover $\mathscr{A}$ of X which is a pointwise star-refinement of $\mathscr{U}$.

(ii) $B_j = \bar{B}(a_j, 2\delta_j)$ is compact for every $j \in J$.

(iii) $(B_j)_{j\in J}$ is locally finite in X.

(iv) $fB_j \subset U_j$ for every $j \in J$.

Then $\eta_j = d(fB_j, M \setminus U_j) > 0$. Hence by (iii) and [28, Lemma 5.1] there is $\eta \in C_+(X)$ with $\max \eta B_j < \eta_j$ for all $j \in J$. We may assume that $\varepsilon \leqq \eta$ and that $U(f, \varepsilon) \subset \mathrm{Prop}\,(X, M)$.

We next construct recursively maps f_j: $X \to M, j \in J \cup \{0\}$, starting with $f_0 = f$, such that, for every $j \in J$,

$(1)_j$ f_j is LIP,
$(2)_j$ $f_j \in U(f, \varepsilon)$,
$(3)_j$ $f_j = f_{j-1}$ on $X \setminus A_j$,
$(4)_j$ $f_j A_j \cap f_{j-1}(X \setminus A_j) = \varnothing$.

Suppose that $i \in J$ and that the maps $f_0, \ldots, f_{i-1}$ have been defined satisfying $(1)_j$–$(4)_j$ for $j = 1, \ldots, i-1$. Then $f_{i-1}B_i \subset U_i$. Define a LIP map $\alpha = \psi_i f_{i-1}$: $B_i \to \mathbf{R}^m$. Let

$$t = \min\{\varepsilon(x) - d(f_{i-1}(x), f(x)) \mid x \in B_i\} \quad (>0)\,.$$

There is $s > 0$ such that, if $x \in \alpha B_i$ and $y \in \mathbf{R}^m$ with $|x - y| < s$, then $d(\psi_i^{-1}(x), \psi_i^{-1}(y)) < t$. Define a LIP map φ: $A_i \times \mathbf{R}^m \to \mathbf{R}^m$ by

$$\varphi(x, y) = (y - \alpha(x))/(\delta_i - d(x, a_i))\,.$$

Let $Y = \psi_i(f_{i-1}(X \setminus A_i) \cap U_i)$ and $Z = \varphi(A_i \times Y)$. Then by Lemma 4.6,

$$\begin{aligned} \dim_H Z &\leqq \dim_H A_i \times Y \leqq \dim_{\mathrm{met}} A_i + \dim_H Y \\ &\leqq 2 \dim_{\mathrm{met}} X \leqq 2n < m\,. \end{aligned}$$

Thus $\mathscr{H}^m(Z) = 0$. Hence there is $y \in \mathbf{R}^m \setminus Z$ with $|y| < s$. Define a LIP map h: $B_i \to \mathbf{R}^m$ by

$$h(x) = \alpha(x) + \max\,(\delta_i - d(x, a_i), 0)y\,.$$

Then $Y \cap hA_i = \varnothing$, because $\varphi(x, h(x)) = y$ for every $x \in A_i$. We now define f_i: $X \to M$ setting $f_i = f_{i-1}$ on $X \setminus B_i$ and $f_i = \psi_i^{-1}h$ on B_i. Since $h = \alpha$ on $B_i \setminus A_i$, it follows that $f_i = f_{i-1}$ on $X \setminus A_i$, so f_i is LIP. We have

$$f_i A_i \cap f_{i-1}(X \setminus A_i) = \psi_i^{-1}(hA_i \cap Y) = \varnothing\,.$$

If $x \in A_i$, then

$$|\psi_i(f_i(x)) - \psi_i(f_{i-1}(x))| \leqq \delta_i |y| < s;$$

so $d(f_i(x), f_{i-1}(x)) < t$, which implies $d(f_i(x), f(x)) < \varepsilon(x)$. Thus f_i satisfies $(1)_i, \ldots, (4)_i$.

Let $x \in X$. Then there are a neighborhood V_x of x in X and $j(x) \in J$ such that $V_x \cap A_i = \varnothing$ if $i > j(x)$. By $(3)_i$ this implies that $f_i | V_x = f_{j(x)} | V_x$ for $i \geqq j(x)$. Hence the limit $g(x) = \lim_{j \to \infty} f_j(x)$ exists in the case $J = \mathbf{N}$. If $J = \{1, \ldots, k\}$, we set $g(x)=f_k(x)$. The map $g\colon X \to M$ is continuous (in fact, LIP) because $g=f_{j(x)}$ on V_x for all $x \in X$. By $(2)_j$ we have $g \in U(f,\varepsilon)$. This implies that g is proper. We conclude the proof by showing that g is a $\mathscr{U}$-map. Consider a point $p \in M$. We must construct a neighborhood W of p such that $g^{-1}W \subset U$ for some $U \in \mathscr{U}$. Since g is closed, we may assume that $p=g(x)$ for some $x \in X$. Choose $U \in \mathscr{U}$ with $\mathrm{st}\,(x,\mathscr{A}) \subset U$. Let $y \in g^{-1}(p)$. Let

$$j = \max\{i \in J \mid \{x,y\} \cap A_i \neq \emptyset\} .$$

Then $g(x)=f_j(x)$ and $g(y)=f_j(y)$. Since $f_jA_j \cap f_j(X \setminus A_j)=\emptyset$, it follows that $x,y \in A_j$, whence $y \in \mathrm{st}\,(x,\mathscr{A})$. Thus $g^{-1}(p) \subset U$. Therefore $W=M \setminus g(X \setminus U)$ is an open neighborhood of p with $g^{-1}W \subset U$.

4.10. Proof of Theorem 4.1. The theorem follows immediately from Lemmas 1.3, 4.3, 4.4, 4.9, and 4.2.

4.11. Lemma. *Let X be a separable metric space, Y an* ANR, *$\mathscr{U}$ an open cover of X, $f\colon X \to Y$ a continuous map and $\mathscr{V}$ an open cover of Y. Then there exist a polyhedron P with* $\dim P \leq \dim X$ *and continuous maps $g\colon X \to P$ and $h\colon P \to Y$ such that g is a $\mathscr{U}$-map and hg is $\mathscr{V}$-near to f.*

Moreover, if X and Y are locally compact and f is proper, then g and h can be chosen to be proper.

Proof. The first part is certainly known. One can prove it by slightly modifying the proof of [9, Theorem 6.1] (using [7, Dowker's Theorem 3.2.1]). In the second part we may assume that $N(f,\mathscr{V}) \subset \mathrm{Prop}\,(X,Y)$. Then the proof of [1, Lemma 1.1] shows that if P is replaced by a sufficiently small closed subpolyhedron $P' \supset gX$, then the maps $g'\colon X \to P'$, $g'(x)=g(x)$ for $x \in X$, and $h'=h \mid P'$ are proper. This proves the second part.

4.12. Remark. Applying Lemma 4.11 one can easily reduce Lemma 4.9 (and, hence, also Theorem 4.1) to the special case of a polyhedral X. Observe that in this special case, in the proof of Lemma 4.9 one could replace Lemma 4.7 by the fact that $\dim_{\mathrm{met}} P=\dim P$ for every compact subpolyhedron P of a euclidean space.

5. Approximating continuous maps of metric spaces into manifolds.

5.1. Theorem. *Let X be a locally compact separable metric space with* $\dim X \leq n$ *and let M be an m-manifold, $m \geq 2n+1$. Then closed embeddings are dense in* $\mathrm{Prop}\,(X,M)$.

More precisely, $\mathrm{CEmb}\,(X,M)$ *and* $\mathrm{CEmb}^{\mathrm{int}}\,(X,M)$ *are dense G_δ-sets in the Baire space* $\mathrm{Prop}\,(X,M)$.

PROOF. Since CEmb (X, M) is a G_δ-set in Prop (X, M) by Lemmas 4.3 and 4.4, by Lemma 4.2 it suffices to show that $\mathrm{CEmb}^{\mathrm{int}}(X, M)$ is dense in $\mathrm{Prop}^{\mathrm{int}}(X, M)$. Now $\mathrm{Prop}^{\mathrm{int}}(X, M)$ can be considered as a subset of Prop $(X, \mathrm{int}\, M)$ (i.e., of elements f with fX closed in M). This reduces the proof to the claim that CEmb $(X, \mathrm{int}\, M)$ is dense in Prop $(X, \mathrm{int}\, M)$. This claim then follows from Lemma 1.5 and Theorem 4.1 if $m \neq 4$. If $m=4$ and, hence, $n \leqq 1$, the claim follows from Theorem 3.6; in fact, it follows from Theorem 3.6 for $n \leqq 1$ and all $m \geqq 2n+1$. However, in the case $n=0$, $m \geqq 1$ there exists a simple direct proof for the theorem, which uses Remark 4.12. We present it in order to lessen the dependence of the proof of Theorem 5.1 on Theorem 3.1.

It suffices to show that if $\mathscr{U}$ is an open cover of X, then $\mathrm{Prop}_{\mathscr{U}}(X, M)$ is dense in Prop (X, M). So let $f \in \mathrm{Prop}(X, M)$ and $\varepsilon \in C_+(X)$. To simplify notations we assume that X is non-compact. There is a refinement $(V_j)_{j \in \mathbf{N}}$ of $\mathscr{U}$ by disjoint non-empty compact open sets such that $d(fV_j) < \varepsilon_j/2$ where $\varepsilon_j = \min \varepsilon V_j$, for all j; cf. the proof of Theorem 3.12. For each j choose $x_j \in V_j$. Set $y_1 = f(x_1)$ and then choose a point

$$y_j \in B(f(x_j), \quad \min(\varepsilon_j/2, 1/j)) \setminus \{y_1, \ldots, y_{j-1}\} \qquad \text{for } j > 1 .$$

Set $g(x) = y_j$ if $x \in V_j$ and $j \in \mathbf{N}$. Then $g \in \mathrm{Prop}(X, M)$. Since $g^{-1}(y_j) = V_j$, g is a $\mathscr{U}$-map. Finally, if $x \in V_j$, then

$$\begin{aligned} d(g(x), f(x)) &\leqq d(y_j, f(x_j)) + d(f(x_j), f(x)) \\ &< \varepsilon_j/2 + \varepsilon_j/2 = \varepsilon_j \leqq \varepsilon(x) . \end{aligned}$$

5.2. LEMMA. *If X is a locally compact separable metric space and M is a manifold with a non-compact component, then* Prop $(X, M) \neq \varnothing$.

PROOF. This follows from the facts Prop $(X, \mathbf{R}^1) \neq \varnothing$ and Prop $(\mathbf{R}^1, M) \neq \varnothing$. The first fact is well-known and is also shown in the proof of Corollary 2.11. The second is proved in [17, IV, p. 7-02].

5.3. COROLLARY. *Let $n \in \mathbf{N} \cup \{0\}$ and let M be a non-compact connected m-manifold, $m \geqq 2n+1$. Then every at most n-dimensional locally compact separable metric space can be closedly embedded into M.*

PROOF. This follows from Theorem 5.1 and Lemma 5.2.

5.4. REMARK. Recall from Introduction that the first part of Theorem 5.1 for a polyhedral X also follows from [5] and PL topology. One could then prove Theorem 5.1 like Theorem 4.1, an application of Lemma 4.11 reducing the proof of Theorem 5.1 to this special case.

5.5. LEMMA. *In the situation of Lemma* 3.8, *if* (U_1, U_2) *is an open cover of* X, *then* $\hat{X}$ *can be chosen in such a way that there exists an open cover* (V_1, V_2) *of* $\hat{X}$ *with* $V_i \cap X = U_i$, $i=1,2$.

PROOF. Let $A_i = X \setminus U_i$, $i=1,2$. There is a continuous map φ: $X \to [1,2]$ with $\varphi^{-1}(i) = A_i$, $i=1,2$. Choose a topologically compatible totally bounded metric ϱ_0 on X. Then the formula

$$\varrho(x,y) = \varrho_0(x,y) + |\varphi(x) - \varphi(y)| \quad \text{for } x, y \in X$$

defines a topologically compatible totally bounded metric ϱ on X for which $\varrho(A_1, A_2) \geqq 1$. The proof of [7, Lemma 1.13.3] shows that the compact metric space $\tilde{X}$ in the proof of Lemma 3.8 can be chosen in such a way that the metric $\tilde{\varrho}$ of $\tilde{X}$ satisfies $\tilde{\varrho}(x,y) \geqq \varrho(x,y)$ for $x, y \in X$. Let $B_i = \mathrm{cl}_{\tilde{X}} A_i$, $i=1,2$. Then $\tilde{\varrho}(B_1, B_2) \geqq 1$, whence $B_1 \cap B_2 = \emptyset$. Thus $(V_1, V_2) = (\hat{X} \setminus B_1, \hat{X} \setminus B_2)$ is an open cover of $\hat{X}$ with $V_i \cap X = U_i$, $i=1,2$.

5.6. THEOREM. *Let* X *be a separable metric space with* $\dim X \leqq n$ *and let* M *be an* m*-manifold*, $m \geqq 2n+1$. *Then embeddings are dense in* $C(X, M)$.

More precisely, the set

$$E = \{f \in \mathrm{Emb}(X, M) \mid \dim \overline{fX} = \dim X\}$$

is dense in $C(X, M)$, *and* $\mathrm{Emb}^{\mathrm{int}}(X, M)$ *contains a dense* G_δ*-set in the Baire space* $C(X, M)$.

PROOF. The denseness of E follows immediately from Theorem 5.1 and Lemma 3.8; cf. the proof of Theorem 3.9. To prove that $\mathrm{Emb}^{\mathrm{int}}(X, M)$ contains a dense G_δ-set in $C(X, M)$, by Lemmas 1.3, 2.4, 2.6, and 2.7 it suffices to show that for every open cover $\mathscr{B} = (U_1, U_2)$ of X the set of all $\mathscr{B}$-maps f: $X \to M$ is dense in $C(X, M)$. To this end, consider a map $f \in C(X, M)$ and an open cover $\mathscr{U}$ of M. By Lemmas 3.8 and 5.5 there exists a locally compact separable metric space $\hat{X}$ containing X topologically as a subspace with $\dim \hat{X} \leqq n$ and such that f is extendable to a proper map

$$\hat{f}: \hat{X} \to M \quad \text{and} \quad U_i = V_i \cap X, \; i=1,2,$$

for some open cover $\hat{\mathscr{B}} = (V_1, V_2)$ of $\hat{X}$. Theorem 5.1 gives a closed embedding $\hat{g}$: $\hat{X} \to M$ in $N(\hat{f}, \mathscr{U})$. Clearly $\hat{g}$ is a $\hat{\mathscr{B}}$-map. Then the embedding $g = \hat{g} | X$ in $N(f, \mathscr{U})$ is a $\mathscr{B}$-map.

If M is embeddable into $\mathbb{R}^m$, Theorem 5.6 also follows from Theorem 2.1. Of course, the denseness of E in Theorem 5.6 is a corollary of Theorem E.

REFERENCES

1. B. J. Ball, *Alternative approaches to proper shape theory*, in *Studies in topology*, eds. N. M. Stavrakas and K. R. Allen, pp. 1–27, Academic Press, New York - San Francisco - London, 1975.
2. B. J. Ball and R. B. Sher, *A theory of proper shape for locally compact metric spaces*, Fund. Math. 86 (1974), 163–192.
3. H. G. Bothe, *Eine Einbettung m-dimensionaler Mengen in einen* $(m+1)$*-dimensionalen absoluten Retrakt*, Fund. Math. 52 (1963), 209–224.
4. H. G. Bothe, *Relative embeddability of compacta in euclidean spaces*, in *Topics in topology*, Colloq. Math. Soc. János Bolyai, Vol. 8, ed. Á. Császár, pp. 127–133, North-Holland Publishing Co., Amsterdam - London, 1974.
5. J. Dancis, *General position maps for topological manifolds in the* 2/3 *rds range*, Trans. Amer. Math. Soc. 216 (1976), 249–266.
6. J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, 1974.
7. R. Engelking, *Dimension theory*, North-Holland Publishing Co., Amsterdam - Oxford - New York, PWN - Polish Scientific Publishers, Warszawa, 1978.
8. R. Engelking, Personal communication.
9. O. Hanner, *Some theorems on absolute neighborhood retracts*, Ark. Mat. 1 (1951), 389–408.
10. R. E. Heisey and H. Toruńczyk, *On the topology of direct limits of* ANR's, Pacific J. Math. 93 (1981), 307–312.
11. M. W. Hirsch, *Differential topology* (Graduate Texts 33), Springer-Verlag, Berlin - Heidelberg - New York, 1976.
12. J. F. P. Hudson, *Piecewise linear topology*, W. A. Benjamin, Inc., New York - Amsterdam, 1969.
13. W. Hurewicz and H. Wallman, *Dimension theory* (Princeton Mathematical Series 4), Princeton University Press, Princeton, N.J., 1948.
14. R. C. Kirby and L. C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations* (Annals of Mathematics Studies 88), Princeton University Press and University of Tokyo Press, Princeton, N.J., 1977.
15. J. Luukkainen and J. Väisälä, *Elements of Lipschitz topology*, Ann. Acad. Sci. Fenn. Ser. A I Math. 3 (1977), 85–122.
16. E. E. Moise, *Geometric topology in dimensions* 2 *and* 3 (Graduate Texts 47), Springer-Verlag, Berlin - Heidelberg - New York, 1977.
17. C. Morlet, *Le lemme de Thom et les théorèmes de plongement de Whitney*, I–IV, Séminaire Henri Cartan, 14e année, 1961/62, *Topologie Différentielle*, nr. 4–7, Secrétariat Mathématique, Paris, 1964.
18. J.-I. Nagata, *Modern dimension theory*, P. Noordhoff N. V., Groningen, and North-Holland Publishing Co., Amsterdam, 1965.
19. L. Pontrjagin and L. Schnirelmann, *Sur une propriété métrique de la dimension*, Ann. of Math.(2) 33 (1932), 156–162.
20. J. H. Roberts, *A problem in dimension theory*, Amer. J. Math. 70 (1948), 126–128.
21. T. B. Rushing, *Topological embeddings*, Academic Press, New York - London, 1973.
22. L. C. Siebenmann, *Topological manifolds*, Actes du Congrès International des Mathématiciens, Nice, 1970, Tome 2, pp. 133–163, Gauthier–Villars, Paris, 1971.
23. D. Sullivan, *Hyperbolic geometry and homeomorphisms*, in *Geometric topology*, Proc. Georgia Topology Conf., Athens, Ga., 1977, ed. J. C. Cantrell, pp. 543–555, Academic Press, New York - San Francisco - London, 1979.
24. H. Toruńczyk, *A short proof of Hausdorff's theorem on extending metrics*, Fund. Math. 77 (1972), 191–193.

25. H. Toruńczyk, *No infinite-dimensional locally compact* ANR *is a topological group*, Proc. of the 1979 Topology Conf., Ohio Univ., Athens, Ohio, 1979, ed. W. Kuperberg, G. M. Reed, and P. Zenor, Topology Proc. 4 (1980), 173–176.
26. J. Väisälä, *Demension and measure*, Proc. Amer. Math. Soc. 76 (1979), 167–168.
27. H. Wegmann, *Die Hausdorff-Dimension von kartesischen Produkten metrischer Räume*, J. Reine Angew. Math. 246 (1971), 46–75.
28. J. H. C. Whitehead, *Manifolds with transverse fields in euclidean space*, Ann. of Math. (2) 73 (1961), 154–212.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HELSINKI
HALLITUSKATU 15
SF-00100 HELSINKI 10
FINLAND