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MINIMAL AND DISTAL FUNCTIONS ON SOME NON-ABELIAN GROUPS

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Abstract.

A minimal, distal action of the plane's euclidean group G on the torus is exhibited, and it gives rise to right uniformly continuous, right distal functions on G that are not left uniformly continuous and not even left point-distal on the discrete group G_d. This answers a question of A. W. Knapp. Somewhat similar considerations on the discrete affine group of the line produce functions that are left minimal but have a constant function in the pointwise closure of the set of right translates.

Let G be a locally compact group and let X be a compact Hausdorff space. The pair (G, X) is called a *flow* provided there is a continuous map

$$G \times X \to X, \qquad (s, x) \to sx$$

satisfying $ex = x$ for all $x \in X$, where e is the identity of G, and $s(tx) = (st)x$ for all $s, t \in G$ and $x \in X$. The flow (G, X) is called *minimal* if $Gx = \{sx \mid s \in G\}$ is dense in X for all $x \in X$. A flow (G, X) is called *point-distal*, if there is at least one special point $x \in X$ such that Gx is dense in X and the equality

$$\lim_\alpha \, (s_\alpha \lim_\beta t_\beta x) \;=\; \lim_\alpha s_\alpha x$$

for nets $\{s_\alpha\}$ and $\{t_\beta\}$ in G always implies

$$\lim_\beta t_\beta x \;=\; x \, .$$

Point-distal flows are always minimal [1, 10]. A flow (X, G) is called *distal* if it has the property that the equality

$$\lim_\alpha s_\alpha x \;=\; \lim_\alpha s_\alpha y$$

for any x and y in X and net $\{s_\alpha\}$ in G always implies $x = y$. A distal flow need not be minimal, but a minimal distal flow is point-distal.

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A flow of one of these kinds gives rise to functions on G of the same kind. For example, if (G, X) is a distal flow, $x \in X$ and $F \in C(X)$, the C*-algebra of bounded, continuous, complex-valued functions on X, then the definition

$$f(s) = F(sx), \qquad s \in G$$

gives a distal function f on G; also f is *right uniformly continuous*, i.e., for any $\varepsilon > 0$ there is a neighbourhood V of $e \in G$ such that $|f(s) - f(t)| < \varepsilon$ whenever $st^{-1} \in V$. To get a point-distal function from a point-distal flow (G, X), the point $x \in X$ that is used must be a special point. Alternately, one could adopt the following equivalent definition: a right uniformly continuous function $f \in C(G)$ is point-distal if the equality

$$\lim_{\alpha} \lim_{\beta} f(ss_\alpha t_\beta) = \lim_{\alpha} f(ss_\alpha), \qquad s \in G,$$

for nets $\{s_\alpha\}$ and $\{t_\beta\}$ in G always implies

$$\lim_{\beta} f(st_\beta) = f(s), \qquad s \in G$$

(i.e., the action of G by right translation on the pointwise closure in $C(G)$ of the right translates of f gives a point-distal flow).

Point-distal functions as just defined might more accurately be called *right point-distal* functions since right uniform continuity and right translation are involved. Functions in $C(G)$ that are *left point-distal* would be defined analogously using left uniform continuity and left translates. Similar remarks hold for distal and minimal functions. Thus, if $f \in C(G)$ is left uniformly continuous, then f is *left minimal* provided, whenever

$$\lim_{\alpha} f(s_\alpha s) = h(s), \qquad s \in G,$$

for some net $\{s_\alpha\} \subset G$, there is another net $\{t_\beta\} \subset G$ such that

$$\lim_{\beta} h(t_\beta s) = f(s), \qquad s \in G.$$

The reader is referred to the literature for further details and information; see especially [1, 8, 14], but also [2, 4, 5, 6, 9, 10, 13].

We wish to consider the euclidean group G of the plane first. We represent it as the set $[0, 2\pi) \times \mathbf{C}$, where $[0, 2\pi)$ is the circle group and $\mathbf{C}$ is the complex plane; multiplication in G is given by the formula

$$(\theta, z)(\theta_1, z_1) = ((\theta_1 + \theta_2) \bmod 2\pi,\ z + e^{i\theta} z_1).$$

(When dealing with the group $[0, 2\pi)$, we will generally omit the "mod 2π", as, for example, in the next sentence.) If we let X be the torus, $X = [0, 2\pi)^2$, the pair (G, X) becomes a flow with the definition

$$(\theta, z)(\varphi, \psi) = (\varphi + x\cos(\theta+\psi) + y\sin(\theta+\psi), \theta+\psi)$$

for $(\theta, z) = (\theta, x+iy) \in G$ and $(\theta, \psi) \in X$. We state

THEOREM 1. *The flow (G, X) is minimal, distal and not equicontinuous.*

It follows from [3; Theorem II.3.8] that the enveloping semigroup E of (G, X), which is the closure of G in X^X, is a compact, right topological, non-topological group in the relative topology from X^X. Some "bad" properties E must have are established in [11].

If $F \in C(X)$ is defined by $F(\varphi, \psi) = e^{i\theta}$, then

$$f(\theta, z) = F((\theta, z)(0,0)) = e^{i(x\cos\theta + y\sin\theta)}$$

defines a distal function $f \in C(G)$ that has some interesting properties.

1. f is not left uniformly continuous. (This is easily checked.) Hence f is not almost periodic (in the usual sense, e.g., as in [3; § III.9]).

2. The translates (left and right) of f separate the points of G. Hence the distal functions on G_1 separate the points of G. (We recall that the almost periodic functions on G can only separate points (θ_1, z_1) and (θ_2, z_2) in G if $\theta_1 \neq \theta_2$ [12; Lemma 2.4].)

3. If h is a minimal function on a locally compact abelian group G', then h obeys spectral synthesis, i.e., considered as a member of $L_\infty(G')$, h is weak * approximable by almost periodic functions taken from the weak * closed subspace of $L_\infty(G')$ generated by the translates of h [13]. Clearly f can not obey spectral synthesis.

Since the next assertions about f require proof, we state them as:

THEOREM 2. *f is not left distal, and this does not come about merely because f is not left uniformly continuous. In fact, when regarded as a function on the discrete group G_d, f is not even left point-distal. However, f is left minimal.*

PROOF. We prove first that f is left minimal. Suppose $\{(\theta_\alpha, z_\alpha)\}$ is a net in G_d such that

$$\lim_\alpha f((\theta_\alpha, z_\alpha)(\theta, z)) = h(\theta, z), \qquad (\theta, z) \in G_d\,.$$

Since

$$\begin{aligned} f((\theta_\alpha, z_\alpha)(\theta, z)) &= f(\theta_\alpha + \theta, z_\alpha + e^{i\theta_\alpha} z) \\ &= e^{ix_\alpha \cos(\theta_\alpha+\theta)} e^{iy_\alpha \sin(\theta_\alpha+\theta)} e^{i(x\cos\theta + y\sin\theta)}\,, \end{aligned}$$

we can assume $h(\theta,z)=g_1(\theta)h_1(\theta)f(\theta,z)$, where

$$g_1(\theta) = \lim_\alpha e^{ix_\alpha \cos(\theta_\alpha+\theta)} \quad \text{and} \quad h_1(\theta) = \lim_\alpha e^{iy_\alpha \sin(\theta_\alpha+\theta)}\,.$$

Now, we must find a net $\{(\theta_\beta, z_\beta)\} \subset G_d$ such that

$$\begin{aligned}&\lim_\beta h((\theta_\beta, z_\beta)(\theta,z))\\ &\quad = \lim_\beta g_1(\theta_\beta+\theta)h_1(\theta_\beta+\theta)^{ix_\beta\cos(\theta_\beta+\theta)}e^{iy_\beta\sin(\theta_\beta+\theta)}f(\theta,z)\\ &\quad = f(\theta,z), \qquad (\theta,z)\in G_d\,.\end{aligned}$$

We collect some information we need in a lemma.

LEMMA 3. *Let β be a finite subset of $[0,2\pi)$ and, for each $\theta'\in\beta$, let $m(\theta')=m'$ be an integer.*

(i) *The function*

$$f_\beta : \theta \to \sum_{\theta'\in\beta} m'\cos(\theta'+\theta)$$

is either the zero function or is not constant in any subinterval of $[0,2\pi)$. f_β is the zero function if and only if the function

$$g_\beta : \theta \to \sum_{\theta'\in\beta} m'\sin(\theta'+\theta)$$

is the zero function.

(ii)$_1$ *There is a maximal subset γ of β such that the function*

$$f_\gamma : \theta \to \sum_{\theta'\in\gamma} m'\cos(\theta'+\theta)$$

is the zero function on $[0,2\pi)$ if and only if $m'=0$ for all $\theta'\in\gamma$. Thus, if $\theta''\in\beta\setminus\gamma$, the function $\theta\to\cos(\theta''+\theta)$ is a rational linear combination of the functions

$$\{\theta\to\cos(\theta'+\theta) \mid \theta'\in\gamma\}\,.$$

Simultaneously, the function

$$g_\gamma : \theta \to \sum_{\theta'\in\gamma} m'\sin(\theta'+\theta)$$

is the zero function if and only if $m'=0$ for all θ' in γ and the function $\theta\to\sin(\theta''+\theta)$ is a rational linear combination of the functions

$$\{\theta\to\sin(\theta'+\theta) \mid \theta'\in\gamma\}$$

for $\theta''\in\beta\setminus\gamma$.

(ii)$_2$ *There is a* $\theta \in [0, 2\pi)$ *for which the sets*

$$\{\cos(\theta' + \theta) \mid \theta' \in \gamma\} \quad \text{and} \quad \{\sin(\theta' + \theta) \mid \theta \in \gamma\}$$

are each linearly independent over the rationals. (In fact, these sets fail to be linearly independent only for θ in a countable set.)

PROOF. (i) follows from the fact that f_β and g_β are the real and imaginary parts, respectively, of the function

$$\theta \to e^{i\theta} \sum_{\theta' \in \beta} m' e^{i\theta'} ;$$

(ii)$_1$ also follows readily. To prove (ii)$_2$ one need only note that the set of functions that are not identically zero from γ to the integers is countable, and that each corresponding function (i.e., f_γ or g_γ) is rational only at a countable number of points in $[0, 2\pi)$.

Returning to the proof of the theorem, let β be a finite subset of $[0, 2\pi)$ and let $|\beta|$ be its cardinality. By the lemma, there is a subset $\gamma \subset \beta$ and $\theta_\beta \in [0, 2\pi)$ such that

$$\{\cos(\theta' + \theta_\beta) \mid \theta' \in \gamma\} \quad \text{and} \quad \{\sin(\theta' + \theta_\beta) \mid \theta' \in \gamma\} ,$$

are maximal rationally independent subsets of

$$\{\cos(\theta' + \theta_\beta) \mid \theta' \in \beta\} \quad \text{and} \quad \{\sin(\theta' + \theta_\beta) \mid \theta' \in \beta_\beta\} ,$$

respectively. Now a direct application of Kronecker's theorem [7; Theorem 26.15] gives integers n_β and m_β such that

$$\left.\begin{aligned} |e^{2\pi i n_\beta \cos(\theta + \theta_\beta)} - (g_1(\theta + \theta_\beta))^{-1}| &< |\beta|^{-1} \\ |e^{2\pi i m_\beta \sin(\theta + \theta_\beta)} - (h_1(\theta + \theta_\beta))^{-1}| &< |\beta|^{-1} \end{aligned}\right\}, \qquad \theta \in \gamma .$$

Then, if $\theta' \in \beta \setminus \gamma$, we have, for example, that

$$e^{2\pi i n_\beta \cos(\theta' + \theta_\beta)} = \prod_{\theta \in \gamma} e^{2\pi i n_\beta a \cos(\theta + \theta_\beta)} ,$$

where each $a = a(\theta)$ is a rational number, and also

$$(g_1(\theta' + \theta_\beta))^{-1} = \left(\prod_{\theta \in \gamma} (g_1(\theta + \theta_\beta)^a \right)^{-1} .$$

It follows that n_β and m_β can be chosen so the inequalities above hold for all $\theta \in \beta$. Thus,

$$\{(\theta_\beta, z_\beta)\} = \{(\theta_\beta, 2\pi(n_\beta + i m_\beta))\}$$

is the required net.

To see f is not left point-distal, let $\{n_\alpha\}$ be a net of integers such that

$$g_1(\theta) = \lim_\alpha e^{in_\alpha \cos\theta} = \begin{cases} e^{ia\pi}, & \text{if } \cos\theta = a, \text{ a rational number ,} \\ 1, & \text{otherwise .} \end{cases}$$

Kronecker's theorem assures the existence of such a net. Then

$$h(\theta, z) = \lim_\alpha f((0, n_\alpha)(\theta, z)) = g_1(\theta) f(\theta, z) ,$$

and, for any finite $\beta \subset [0, 2\pi)$, there is a

$$\theta_\beta \in \{\theta \in [0, 2\pi) \mid 0 < \theta < |\beta|^{-1}\}$$

such that $\cos(\theta' + \theta_\beta)$ is not rational for any $\theta' \in \beta$. It follows that

$$\lim_\beta f((\theta_\beta, 0)(\theta, z)) = f(\theta, z) = \lim_\beta h((\theta_\beta, 0)(\theta, z)) ,$$

as required.

REMARKS. 1. Knapp [8; pp. 5–6] asked if right distal and left distal are the same. The existence of the function f above indicates the answer is "no".

2. The action of G by right translation on the pointwise closure in $C(G)$ of the set of right translates of f gives a distal flow that is isomorphic to the original flow (G, X).

3. It turns out that f is also an example of a function that is right almost periodic in the sense of Bohr, but not left almost periodic in the sense of Bohr. Wu [15] gives examples of this phenomenon and has written to the author that the final paragraph in his paper refers to work which he did not publish and in which he was considering functions like the present function f as further such examples. We refer the reader to [4] or [15] for definitions and prove f is right almost periodic in the sense of Bohr. Given $\varepsilon > 0$, one readily checks that

$$|f((\theta, z)(\theta', z')) - f(\theta, z)| < \varepsilon$$

for all $(\theta, z) \in G_1$ if and only if $\theta' = 0$ and the real part x' of z' satisfies $|e^{ix'} - 1| < \varepsilon$. Let A_ε be the set of such $(\theta', z') \in G_1$. The compact set

$$K = \{(\theta, z) = (\theta, x + iy) \in G_1 \mid 0 \leqq |x^2 + y^2|^{1/2} \leqq 2\pi\}$$

then yields the equality $KA_\varepsilon = G$. (We note that sets K' smaller than K will also satisfy $K'A_\varepsilon = G$, even for all $\varepsilon > 0$.) That f is not left almost periodic in the sense of Bohr follows from the fact that f is not left uniformly continuous [6; Theorem 4.61]. The reader is warned that, in [6], "uniformly almost periodic" means "almost periodic in the sense of Bohr". Another observation that follows from the theorem just quoted is that, if g is any function that is right, but not left, almost periodic in the sense of Bohr on a topological group G, then g is not right almost periodic in the sense of Bohr on the discrete group G_d. This is unusual behaviour; functions of most kinds (e.g., any other kind mentioned in this paper) on a topological group give functions of the same kind when regarded as functions on the discrete group.

Let h be the character

$$h(0; z) = h(0, x+iy) = e^{ix}$$

defined on the closed normal subgroup $\{0\} \times \mathbf{C} \subset G$. It follows from [12; Lemma 2.4] that h cannot be extended to a function almost periodic (in the usual sense) on G. However, the function f is an extension of h to a (right) distal function on G.

A similar extension problem arises for the affine group of the line

$$G_1 = \{(x,y) \in \mathbf{R}^2 \mid x > 0\} ,$$

whose multiplication is given by the formula

$$(x,y)(x',y') = (xx', y+xy') .$$

Again, by [12; Theorem 2.10], for example, the character

$$h(1,y) = e^{iy}$$

defined on the closed normal subgroup $\{(1,y) \mid y \in \mathbf{R}\}$ cannot be extended to an almost periodic function on G_1. We pose the

QUESTIONS. Does h extend to a distal function on G_1? More generally, do the distal functions on G_1 separate the points of G_1?

Since a function g on G_1 is right uniformly continuous if, for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|g(x',y') - g(x,y)| < \varepsilon$$

whenever $|x'x^{-1} - 1| < \delta$ and $|y' - x'y/x| < \delta$, one is led to consider the extension of f of h defined by

$$f(x,y) = e^{iy/x} .$$

This extension fails dramatically to provide a positive anwer to the first question, but does have some interesting properties.

1. f is neither left nor right uniformly continuous; hence we regard f as a function on the discrete group $(G_1)_d = G_{1d}$ for the rest of the discussion.

2. f has a net of right translates converging pointwise to 1 on G_{1d}; hence f is not even minimal. (This is proved by a Kronecker theorem argument.)

3. f is left minimal, but not left point-distal on G_{1d}. (This is proved by arguments as in the proof of Theorem 2.)

CONCLUDING REMARK. In view of the characterizations of minimal and point-distal functions [9; Corollary 3.5 and 10; Proposition 1.1, respectively], it seems the existence of "unsymmetrical" functions on a discrete group G, as established in this paper, ought to yield some interesting information about structure in the minimal ideal of the Stone–Čech compactification βG of G. (βG is a compact right topological semigroup: see [3] for definitions and basic results.) All we have managed to glean, however, is that the existence of the function f on G_1 defined above implies that, in each minimal left ideal L of βG_{1d} there is an idempotent f and another member x such that $xG_{1d}f=\{xsf \mid s \in G_{1d}\}$ is not dense in L. (We are regarding G_{1d} as a subset of βG_{1d}.) We are unable to decide whether the maximal subgroup of L containing x is dense in L or not. For abelian G, and x and f in minimal left ideal L of βG, $xGf=Gxf$ is always dense in L.

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