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EXTENDING TRACES TO FACTORIAL TRACES[1]

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Abstract.

A tracial state of a C*-algebra can be extended to a factorial tracial state of some larger C*-algebra.

1. Introduction.

Is a countably decomposable finite von Neumann algebra contained in a finite factor? This question arose in a conversation between one of us and B. Fuglede. It was understood that both von Neumann algebras should act on the same (given) Hilbert space.

While this spatial form of the question has not yet been completely answered, this note describes a solution in the case that the commutant of the given finite von Neumann algebra is properly infinite, and the Hilbert space is separable. The first step is to answer the nonspatial (or algebraic) form of the question, that is, to show that a countably decomposable finite W*-algebra can be embedded as a sub-W*-algebra of a finite factor. This is done in Section 2. The second step is to show that the factor can be constructed to have the same infinite number of generators as the given von Neumann algebra; this is done in Section 3. That these two steps are sufficient follows from a well known theorem on the spatial realization of isomorphisms — see 3.4.

Steps one and two above can be reformulated in terms of C*-algebras, and it is in fact in this way that we proceed. Thus, we show that a tracial state can be extended to a factorial tracial state.

2. Factorial extensions.

2.1. LEMMA. *Let A be a C*-algebra and let τ be a tracial state of A which is a finite convex combination of factorial tracial states, $\tau = \sum \lambda_i \tau_i$. Then there exist a*

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finite factor M and a morphism $\varphi: A \to M$ such that $\tau = \mathrm{tr}_M \circ \varphi$, where tr_M denotes the tracial state of M.

PROOF. Denote by $\varphi_i: A \to M_i$ the factorial representation determined by τ_i (so that M_i is a finite factor and $\tau_i = \mathrm{tr}_{M_i} \circ \varphi_i$). Denote the W*-algebra tensor product $\otimes M_i$ by P, and the composition $A \to M_i \to M_i \otimes 1$ by ψ_i, so that $\tau_i = \mathrm{tr}_P \circ \psi_i$. Choose some continuous finite factor N, and set $N \otimes P = M$. Since $\sum \lambda_i = 1$ there exist orthogonal projections e_i in N with $\sum e_i = 1$ and $\mathrm{tr}_N(e_i) = \lambda_i$. Denote by φ the morphism $\sum e_i \otimes \psi_i: A \to M$, that is, $\varphi(a) = \sum e_i \otimes \psi_i(a)$. Then

$$\mathrm{tr}_M(\varphi(a)) = \sum \mathrm{tr}_N(e_i)\, \mathrm{tr}_P(\psi_i(a)) = \sum \lambda_i \tau_i(a) = \tau(a).$$

2.2. THEOREM. *Let A be a C*-algebra and let τ be a tracial state of A. Then there exist a finite factor M and a morphism $\varphi: A \to M$ such that $\mathrm{tr}_M \circ \varphi = \tau$.*

PROOF. Clearly it is enough to consider the case that A has a unit, and in this case the set of tracial states of A is compact and convex, so by the Krein–Mil'man theorem, τ is the limit of a net $(\tau_j)_{j \in J}$ where each τ_j is a finite convex combination of extremal tracial states. Since extremal tracial states are factorial, by 2.1 there exist morphisms $\varphi_j: A \to M_j$ of A into finite factors such that $\tau_j = \mathrm{tr}_{M_j} \circ \varphi_j$.

Choose an ultrafilter ω in the directed index set J containing all the final segments, so that $\tau_j \to \tau$ along ω. Denote by M the ultraproduct of the M_j with respect to ω, and by φ the ultraproduct of the φ_j (see, for example, [2, page 451]). Then M is a finite factor, and

$$\mathrm{tr}_M \circ \varphi = \lim_{j \to \omega} \mathrm{tr}_{M_j} \circ \varphi_j = \lim_{j \to \omega} \tau_j = \tau.$$

3. Preserving the number of generators.

3.1. THEOREM. *Let M be a finite factor and let A be a sub-C*-algebra of M. Then there exists a sub-C*-algebra B of M containing A, with the same infinite number of generators as A, and with a unique tracial state.*

PROOF. It is enough to construct a sub-C*-algebra B_1 of M containing A, with the same infinite number of generators as A, and such that all tracial states of B_1 agree on A. For then we may repeat this construction with A replaced by B_1, and so on, to obtain an increasing sequence $A \subset B_1 \subset B_2 \subset \ldots$ of C*-algebras all with the same infinite number of generators, and such that all tracial states of B_{k+1} agree on B_k. Clearly then for B we may take the closure of $\bigcup B_k$.

Let S be a dense subset of A of minimal cardinality $\aleph$, and for each finite subset F of S and each $n=1,2,\ldots$ consider the set $T(F,n)$ of all states of M which on each element of F differ by at least $1/n$ from tr_M. Then $T(F,n)$ is compact, and each state in $T(F,n)$ is nonzero on some commutator $xy-yx$ in M, so by compactness there are finitely many commutators in M such that each state in $T(F,n)$ is nonzero on one of these. Thus, for each F and n there is a finite subset $M(F,n)$ of M such that no state in $T(F,n)$ is zero on all commutators of elements in $M(F,n)$. Hence no state of M which is different from tr_M on A (and so belongs to some $T(F,n)$) is zero on all commutators of elements in $\bigcup M(F,n)$. The sub-C*-algebra B_1 of M generated by A and $\bigcup M(F,n)$ has $\aleph$ generators, and any tracial state f of B_1 is zero on all commutators of elements in $\bigcup M(F,n)$, and so must agree with tr_M on A. (Extend f to a state of M; this can belong to no $T(F,n)$, which just says that it agrees with tr_M on A.)

3.2. Theorem. *Let A be a* C*-*algebra, and τ a tracial state of A. Then there exist a* C*-*algebra B containing A, with the same infinite number of generators as A, and a factorial tracial state of B extending τ.*

Proof. It is enough to consider the case that τ is faithful, since if B_1 contains the quotient $A/\ker\tau$ and has a factorial tracial state f extending τ on $A/\ker\tau$, then $A\oplus B_1=B$ contains A, via the natural embedding $\pi\colon A\to A\oplus A/\ker\tau$, and the factorial tracial state f on B extends τ (i.e. satisfies $f\circ\pi=\tau$). If, moreover, B_1 has the same number of generators as $A/\ker\tau$, then B has the same number of generators as A.

If τ is faithful, then by 2.2 A is contained in a finite factor M whose tracial state extends τ. By 3.1 there is a sub-C*-algebra B of M containing A, with the same number of generators as A, and with a unique tracial state, which of course is factorial.

3.3. Corollary. *Let N be a countably decomposable finite* W*-*algebra. Then N is a sub-*W*-algebra of a finite factor M, with the same infinite number of generators as N.*

Proof. Choose a faithful normal trace τ on N, and denote by A the sub-C*-algebra of N generated by a generating set for the W*-algebra N which has minimal infinite cardinality. By 3.2, A is contained in a C*-algebra B with a generating set of the same cardinality and with a factorial tracial state extending $\tau\,|\,A$. With M the corresponding finite factor, we have an embedding of A in M preserving τ, and since τ is faithful on N this extends to a normal embedding of N in M.

3.4. COROLLARY. *Let N be a countably decomposible finite von Neumann algebra, acting on the Hilbert space H. Assume that N is countably generated, and that the commutant N' is infinite of homogeneous order. Then N is contained in a finite factor* (*acting on H*).

PROOF. By 3.3, N is a sub-W*-algebra of a finite factor M, such that M has the same infinite number of generators as N, and therefore such that M has a normal representation on a separable Hilbert space K. If N' on H is of order $\aleph$, replace M by $M \otimes 1$ on the tensor product of K with a Hilbert space of dimension $\aleph$. The resulting action of N has commutant also of order $\aleph$, and therefore (by [1, Chapitre III, § 8.6, Corollaire 7]) is unitarily equivalent to the given action of N. This shows that N is contained in a finite factor on H.

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