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SOME REMARKS ON LARGEST SUBHARMONIC MINORANTS

P. J. RIPPON

1. Introduction.

Let E be an open connected subset of k-dimensional Euclidean space R^k and let $F\colon E \to [0,\infty]$ be an upper semicontinuous function. Following Domar [1] we denote by $\{F\}$ the family of all functions u subharmonic in E which satisfy

$$u(x) \leqq F(x), \qquad x \in E\,,$$

and we put

$$M(x) = \sup_{u \in \{F\}} u(x), \qquad x \in E\,.$$

In this paper we examine the sharpness of the restriction on F under which the following theorem holds.

THEOREM 1. *Let φ be a non-negative, increasing function on $[0,\infty]$ and suppose that*

$$\int_1^\infty \frac{dt}{[\varphi(t)]^{1/(k-1)}} < \infty\,. \tag{1.1}$$

If

$$\int_E \varphi(\log^+ F(x))\,dx < \infty\,, \tag{1.2}$$

then $M(x)$ is bounded above on each compact subset of E, and is therefore subharmonic in E.

A proof of this result for the case $\varphi(t)=t^{k-1+\varepsilon}$, $\varepsilon>0$, was given by Domar [1, Theorem 2] and only minor modifications to his argument yield the above slight generalisation. Domar showed [1, Theorem 4] that the result is false if $\varphi(t)=t^{k-1}$, but it seems worthwhile to point out that (at least when $k=2$) the condition (1.1) is best possible for a wide class of functions φ.

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THEOREM 2. *Suppose that* φ *is a strictly positive, increasing function on* $(0,\infty)$ *and that*

$$\int_1^\infty \frac{dt}{\varphi(t)} = \infty .$$

If φ *is* $C^{(1)}$ *on* $(0,\infty)$ *and* $\varphi'(t)=o(\varphi(t))$ *as* $t \to \infty$, *then Theorem* 1 *is false in* $\mathbf{R}^2$ *for this* φ.

To prove Theorem 2 we construct an example which is closely related to both [1, Theorem 4] and [2]. It will be clear from the method that the restriction $\varphi'(t)=o(\varphi(t))$ could be relaxed a little, but it would of course be desirable to omit it altogether.

2. Proof of Theorem 2.

In [2] we used a theorem of Warschawski to obtain a certain harmonic function in a strip-like domain. Since this is needed in the present construction we recall it here.

LEMMA. *Let* θ *be strictly positive and* $C^{(1)}$ *on* $(-\infty,\infty)$, *and satisfy* $|\theta'(x)| \leqq m$, $-\infty < x < \infty$. *If*

$$S = \{x+iy : \ -\infty < x < \infty,\ 0 < y < \theta(x)\} , \tag{2.1}$$

then there is a positive harmonic function h in S such that

(i) *h vanishes continuously at finite points of* ∂S,

(ii) *for* $z=x+iy \in S$, $x>0$,

$$h(z) \leqq A \exp\left[B \int_0^x \frac{dt}{\theta(t)}\right],$$

where $A=\exp[8\pi(1+4m^2/3)]$, $B = \pi(1+m^2/3)$,

and

(iii) *there is an* arc C *in S tending to the boundary point at* $+\infty$ *such that*

$$h(z) \to \infty, \quad \text{as } z \to +\infty \text{ along } C .$$

We begin the proof of Theorem 2 by putting

$$\psi(t) = \varphi(t)\left(1+\int_1^t \frac{du}{\varphi(u)}\right), \qquad t \geqq 1 ,$$

which is a $C^{(1)}$ majorant for φ in $(1,\infty)$. It is easy to check that ψ/φ is increasing and unbounded in $(1,\infty)$, that

$$\int_1^\infty \frac{dt}{\psi(t)} = \infty ,$$

and that

$$\psi'(t) = o(\psi(t)), \qquad t \to \infty .$$

In particular there is a number $a \geqq 1$ such that

$$\frac{\psi'(t)}{\psi(t)} \leqq \frac{\sqrt{3}}{2\pi}, \qquad t \geqq a . \tag{2.2}$$

The significance of this bound will become apparent in a moment.

Let $b = [\psi(a)]^{-1}$ and put

$$\alpha(y) = \psi^{-1}\left(\frac{1}{y}\right), \qquad 0 < y \leqq b ,$$

and

$$\mu(y) = \frac{1}{2\pi}\int_a^{\alpha(y)} \frac{dt}{\psi(t)}, \qquad 0 < y \leqq b ,$$

so that μ is decreasing on $(0, b]$, $\mu(b) = 0$ and $\mu(y) \to \infty$ as $y \to 0$. It follows that $y = \theta(x) = \mu^{-1}(x)$ is decreasing on $[0, \infty)$, $\theta(0) = b$ and, for $x > 0$,

$$\begin{aligned} |\theta'(x)| &= -\frac{1}{\mu'(y)} = -\frac{2\pi\psi(\alpha(y))}{\alpha'(y)} \\ &= -\frac{2\pi}{y\alpha'(y)} = 2\pi y \psi'(\alpha(y)) \\ &= \frac{2\pi\psi'(\alpha(y))}{\psi(\alpha(y))} \leqq \sqrt{3} , \end{aligned}$$

by (2.2).

Now extend θ to $\mathbf{R}$ so that $|\theta'| \leqq \sqrt{3}$ there, and consider the corresponding strip S defined by (2.1). The harmonic function h given by the lemma can be so normalised that

$$h(z) \leqq \exp\left[2\pi \int_0^x \frac{dt}{\theta(t)}\right], \qquad z = x + iy \in S,\ x > 0 .$$

Since

$$\int_0^x \frac{dt}{\theta(t)} = \int_b^{\theta(x)} \frac{\mu'(s)}{s}\, ds$$

$$= \frac{1}{2\pi}\int_b^{\theta(x)} \alpha'(s)\,ds$$

$$\leqq \frac{\alpha(\theta(x))}{2\pi},$$

we have

$$h(z) \leqq \exp[\alpha(\theta(x))], \quad z = x+iy \in S,\ x>0\,. \tag{2.3}$$

After these preliminaries we show that Theorem 1 is false in

$$E = \{x+iy : |x|<1, |y|<1\}\,.$$

Let x_n, $n=1,2,\ldots$, be an unbounded, increasing sequence of positive numbers and for $n=1,2,\ldots$, put

$$\theta_n(x) = \theta(x+x_n+1), \quad |x|<1\,,$$

$$E_n = \{x+iy : |x|<1,\ 0<y_n<\theta_n(x)\}$$

and

$$h_n(z) = h(z+x_n+1), \quad z \in E_n\,.$$

In view of parts (i) and (iii) of the lemma the functions

$$u_n(z) = \begin{cases} h_n(z), & z \in E_n\,, \\ 0, & z \in E\setminus E_n\,, \end{cases}$$

are each subharmonic in E, but are not uniformly bounded near, for instance, the origin.

Since $\varphi(t)/\psi(t)=o(1)$, as $t \to \infty$, it is possible to choose the numbers x_n such that, if

$$y_n = \theta(x_n) \quad \text{and} \quad \delta_n = \varphi(\alpha(y_n))/\psi(\alpha(y_n))\,,$$

then

$$\sum_{n=1}^{\infty} \delta_n < \infty\,. \tag{2.4}$$

Now, by (2.3),

$$\int_E \varphi\,(\log^+ u_n(z))\,dxdy \leqq \int_E \varphi(\alpha(\theta_n(x)))\,dxdy$$

$$= \int_{-1}^{1} \theta_n(x)\varphi(\alpha(\theta_n(x)))\,dx$$

$$\leqq \delta_n \int_{-1}^{1} \theta_n(x)\psi(\alpha(\theta_n(x)))\,dx$$

$$= 2\delta_n ,$$

since ψ/φ is monotonic and $\alpha(y)=\psi^{-1}(1/y)$.

Hence the upper semicontinuous function

$$F(z) = \begin{cases} \sup_n u_n(z), & z \in E \setminus \{y=0\} , \\ \infty, & z \in E \cap \{y=0\} , \end{cases} \tag{2.5}$$

satisfies

$$\int_E \varphi(\log^+ F(z))\,dx\,dy \leqq \sum_{n=1}^{\infty} \int_E \varphi(\log^+ u_n(z))\,dx\,dy$$

$$< \infty ,$$

by (2.4).

Thus if φ satisfies the hypotheses of Theorem 2, then we can construct F in E by (2.5) so that (1.2) is satisfied and yet the functions u_n, which belong to $\{F\}$, are not uniformly bounded above on any compact set containing the origin. The proof of Theorem 2 is complete.

ADDED IN PROOF. I am grateful to Professor Domar for pointing out that Theorem 2 holds also when $k>2$. A similar construction of the desired example is possible using a version of the lemma based on Harnack's inequality.

REFERENCES

1. Y. Domar, *On the existence of a largest subharmonic minorant for a given function*, Ark. Mat. 3 (1954–58), 429–440.
2. P. J. Rippon, *On a growth condition related to the MacLane class*, J. London Math. Soc. (2), 18 (1978), 94–100.

DEPARTMENT OF MATHEMATICS
UNIVERSITY COLLEGE
CORK
IRELAND