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# ON TRINOMIALS OF TYPE $x^n + Ax^m + 1$

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## 1. Introduction.

1.1. K. Th. Vahlen [19] and Capelli [2] have given a simple criterion for the reducibility over $\mathbf{Q}$ of the binomial $x^n + A$, $A \in \mathbf{Z}$. The corresponding question of reducibility of trinomials $x^n + Ax^m + B$, and of quadrinomials, has been studied by several authors (Bremner [1], Jonassen [3], Ljunggren [4–5], Schinzel [6–14], Selmer [15], and Tverberg [18]), and many questions have been raised. For trinomials $x^n + Ax^m + 1$, Ljunggren [5] and Tverberg [18] deal with the case $|A| = 1$, and Schinzel [6] with the case $|A| = 2$. Schinzel [12] asks the question:

> Does there exist a reducible trinomial of the form $x^n + Ax^m + 1$ with $n > m > 0$, $n \neq 2m$, $A \in \mathbf{Z}$, $|A| > 2$?

The answer is affirmative; Coray in a letter to Schinzel furnished the examples

$$x^{13} - 3x^4 + 1 = (x^3 - x^2 + 1)(x^{10} + \ldots + 1)$$
$$x^{13} + 3x^7 + 1 = (x^4 - x^3 + 1)(x^9 + \ldots + 1)\,.$$

It is the purpose of this paper to show there are only finitely many trinomials $x^n + Ax^m + 1$ which possess an irreducible cubic factor, and to give them explicitly.

1.2. We suppose that $x^n + Ax^m + 1$ is divisible by the irreducible cubic $c(x) \in \mathbf{Z}[x]$, $c(0) = \pm 1$. Define the cubic irrational $\theta = \theta_1$ by $c(\theta) = 0$, $\theta$ having conjugates $\theta_2$, $\theta_3$. Then $\theta_i^n + A\theta_i^m + 1 = 0$, $i = 1, 2, 3$.

Inductively we have $\theta^r = A_r\theta^2 + B_r\theta + C_r$, $A_r, B_r, C_r \in \mathbf{Z}$, where

$$(A_0, B_0, C_0) = (0,0,1), \quad (A_1, B_1, C_1) = (0,1,0), \quad (A_2, B_2, C_2) = (1,0,0)$$

and

(1)
$$A_r = \alpha_1\theta_1^r + \alpha_2\theta_2^r + \alpha_3\theta_3^r, \quad B_r = \beta_1\theta_1^r + \beta_2\theta_2^r + \beta_3\theta_3^r, \quad C_r = -c(0)A_{r-1}\,,$$

with

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$$\alpha_1 = (-\theta_2+\theta_3)/\Delta,\ldots,\ \beta_1 = (\theta_2^2-\theta_3^2)/\Delta,\ldots,$$
$$\Delta = (\theta_1-\theta_2)(\theta_2-\theta_3)(\theta_3-\theta_1)\ .$$

We thus seek solutions of

$$(A_n+A\cdot A_m)\theta^2+(B_n+A\cdot B_m)\theta+(C_n+A\cdot C_m+1) = 0$$

i.e. of $A_n+A\cdot A_m=B_n+A\cdot B_m=C_n+A\cdot C_m+1=0$.

Eliminating $A$,

$$A_mB_n = A_nB_m \tag{2}$$

$$A_nC_m = A_m(C_n+1) \tag{3}$$

$$B_nC_m = B_m(C_n+1)\ . \tag{3'}$$

We shall use a $p$-adic technique of Skolem to solve for $m,n$ these simultaneous equations.

## 2. The case $n$ odd.

We shall initially suppose that $m,n$ are not both divisible by 3.

2.1. We consider the case when the cubic factor $c(x)$ is of the type $x^3+bx+1$. It now follows immediately from (1) that $B_n=A_{n+1}$ so that the equations (2), (3), (3′) become

$$A_mA_{n+1} = A_nA_{m+1} \tag{4}$$

$$A_nA_{m-1} = A_m(A_{n-1}-1) \tag{5}$$

$$A_{n+1}A_{m-1} = A_{m+1}(A_{n-1}-1)\ . \tag{5'}$$

CASE I: $|b|>1$. Suppose that

$$\theta^3+b\theta+1 = 0,\ \theta^n+A\theta^m+1 = 0,\ |b| > 1\ . \tag{6}$$

CASE I(i): $n\equiv 0 \pmod 3$. Put $n=3N$, $N$ odd, and suppose

$$b^\lambda \,\|\, N,\ \lambda \geqq 0\ . \tag{7}$$

Then

$$A_n = \sum_{i=1}^{3} \alpha_i\theta_i^{3N} = -\sum_{i=1}^{3} \alpha_i(1+b\theta_i)^N = -A_0-bNA_1-b^2\binom{N}{2}A_2-\ldots$$

$$= -b^2\binom{N}{2}+b^5\left[\binom{N}{4}+\binom{N}{5}\right]+b^8(\cdot)+\ldots \tag{8}$$

$$=b^2\binom{N}{2}[-1+b\varphi_1+b^2\varphi_2+\dots]$$

where $\varphi_1\,\varphi_2,\dots$ are $b$-adic integers; and similar expansion gives

$$A_{n+1} = bN[-1+b(\cdot)+b^2(\cdot)+\dots]\,. \tag{9}$$

Now from (7) and (8), $b^{\lambda+2}\,|\,A_nA_{m+1}$, so by (4) and (9), we have $b\,|\,A_m$. But if $m=3M+k$, $k=1,2$, then as above $A_m\equiv(-1)^MA_k \bmod b$, whence $b\,|\,A_k$, which forces $k=1$. We now have the expansions

$$A_{m-1} = (-1)^Mb^2\binom{M}{2}[-1+b(\cdot)+\dots]$$

$$A_m = (-1)^MbM[-1+b(\cdot)+\dots]$$

$$A_{m+1} = (-1)^M[-1+b(\cdot)+\dots]$$

and $$A_{n-1} = 1-b^3\binom{N}{3}[1+b(\cdot)+\dots]\,.$$

Substituting into (4) gives after rearrangement, and removing a factor $N$,

$$\left(M-\frac{N-1}{2}\right)+b(\cdot)+b^2(\cdot)+\dots = 0\,, \tag{10}$$

and substitution in (5′), after removing the factor $N$,

$$\left(\frac{M(M-1)}{2}-\frac{(N-1)(N-2)}{6}\right)+b(\cdot)+b^2(\cdot)+\dots = 0\,. \tag{11}$$

Now the simultaneous equations (10) and (11), have for a given $b$, at most two solutions in $b$-adic integers $M,N$ (see the proof of Satz 12, Skolem [16]). But for any given $b$, there are two solutions to these equations, given by $(M,N)=(0,1),(-1,-1)$ corresponding to $\theta^3+b\theta+1=0$ and $\theta^{-3}+b\theta^{-2}+1=0$. Consequently there are no further solutions with $n\equiv0 \bmod 3$, $m\not\equiv0 \bmod 3$.

CASE I(ii): $n\not\equiv0 \bmod 3$. From (6) we have

$$\theta_2^{n-m}+\theta_2^{-m} = -A = \theta_3^{n-m}+\theta_3^{-m}\,. \tag{12}$$

Let $n-m=6R+\varrho$, $-m=6S+\sigma$, where $0\leqq\varrho$, $\sigma<6$.

Since $n=6(R-S)+(\varrho-\sigma)$ we are assuming that $\varrho$, $\sigma$ have different parity and that $\varrho\not\equiv\sigma \bmod 3$.

Now $\theta_i^6=1+b\xi_i$ where $\xi_i=2\theta_i+b\theta_i^2$, $i=1,2,3$, so (12) can be written as

$$\theta_2^\varrho[1+b\xi_2]^R-\theta_3^\varrho[1+b\xi_3]^R+\theta_2^\sigma[1+b\xi_2]^S-\theta_3^\sigma[1+b\xi_3]^S = 0$$

and expanded to give

$$\left(\frac{\theta_2^\varrho-\theta_3^\varrho}{\theta_2-\theta_3}\right)+bR\left(\frac{\xi_2\theta_2^\varrho-\xi_3\theta_3^\varrho}{\theta_2-\theta_3}\right)+b^2(\cdot)+\dots$$
$$+\left(\frac{\theta_2^\sigma-\theta_3^\sigma}{\theta_2-\theta_3}\right)+bS\left(\frac{\xi_2\theta_2^\sigma-\xi_3\theta_3^\sigma}{\theta_2-\theta_3}\right)+b^2(\cdot)+\cdots = 0\,.$$

So if

$$T_k = \frac{\theta_2^k-\theta_3^k}{\theta_2-\theta_3} \quad \text{then} \quad T_\varrho+T_\sigma \equiv 0 \bmod b\,. \tag{13}$$

We calculate the various possibilities:

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $T_k$ | 0 | 1 | $-\theta$ | $-b$ | $b\theta-1$ | $b^2+\theta$ |

| $\varrho$ | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma$ | 1 | 5 | 0 | 2 | 1 | 3 | 2 | 4 | 3 | 5 | 0 | 4 |
| $T_\varrho+T_\sigma \bmod b$ | 1 | $\theta$ | 1 | $1-\theta$ | $1-\theta$ | $-\theta$ | $-\theta$ | $-1$ | $-1$ | $-1+\theta$ | $\theta$ | $-1+\theta$ |

(13) implies from the above table that $1/b$, $\theta/b$, or $(1-\theta)/b$ is an integer of $\mathbf{Q}(\theta)$; the first two cases are trivially impossible for $|b|>1$, and since $(1-\theta)$ has trace 3 and norm $b+2$, the latter instance would imply $b\mid 3$, $b^3\mid(b+2)$, again impossible for $|b|>1$.

Thus the factor $x^3+bx+1$ can arise only if $b=\pm 1$.

CASE II(i): $b=1$. We work 47-adically, noting that if $\theta^3+\theta+1=0$ then $\theta^{46}=1+47\xi$, with $\xi=-27+13\theta+77\theta^2$.

Put $m=46M+r$, $n=46N+s$, $0\leqq r$, $s<46$, $s$ odd. Then for $\varepsilon=-1,0,1$, we have

$$A_{n+\varepsilon} = A_{s+\varepsilon}+47N(-27A_{s+\varepsilon}+13A_{s+\varepsilon+1}+77A_{s+\varepsilon+2})+47^2\binom{N}{2}(\cdot)+\dots \tag{14}$$

$$A_{m+\varepsilon} = A_{r+\varepsilon}+47M(-27A_{r+\varepsilon}+13A_{r+\varepsilon+1}+77A_{r+\varepsilon+2})+47^2\binom{M}{2}(\cdot)+\dots \tag{15}$$

Equations (4) and (5) modulo 47 give the congruences

$$A_r A_{s+1} \equiv A_s A_{r+1} \tag{16}$$

$$A_s A_{r-1} \equiv A_r(A_{s-1}-1) \tag{17}$$

A simple (machine!) calculation shows that the only solutions of (16) and (17) in the range $0 \leqq r,\ s<46$, $s$ odd, are the following:

$$(r,s) = (1,3), (2,7), (2,45), (3,1), (11,33), (24,13), (41,39), (44,43)\,. \tag{18}$$

Consider, for example, $(r,s)=(11,33)$. From (14), (15) we have

$$\begin{aligned}
A_{n-1} &= 135+47N(\ \ 1)+47^2(\cdot)+\dots\\
A_n &= -201+47N(-3)+47^2(\cdot)+\dots\\
A_{n+1} &= -335+47N(-8)+47^2(\cdot)+\dots\\
A_{m-1} &= -2+47M(\ \ 8)+47^2(\cdot)+\dots\\
A_m &= 3+47M(\ \ 1)+47^2(\cdot)+\dots\\
A_{m+1} &= 5+47M(-12)+47^2(\cdot)+\dots
\end{aligned}$$

Substituting into (4), (5′) gives respectively

$$\begin{aligned}
(-21M-9N)+47(\cdot)+47^2(\cdot)+\dots &= 0\\
(-9M-11N)+47(\cdot)+47^2(\cdot)+\dots &= 0\,.
\end{aligned}$$

The same result as in case I(i) (Satz 12, Skolem [16]; alternatively Satz 11, Skolem [17]) shows that there is at most one solution in 47-adic integers $M,N$, to this system of equations, which has to satisfy $M\equiv N\equiv 0 \bmod 47$. But $M=N=0$ is a solution! Indeed one readily checks that $\theta^{33}+67\theta^{11}+1=0$.

In exactly the same manner, one obtains for each pair $(r,s)$ at (18) a unique solution for $(m,n)$ corresponding to the vanishing of the following functions:

$$\begin{gathered}
\theta^3+\theta+1,\ \theta+\theta^3+1,\ \theta^{-1}+\theta^2+1,\ \theta^{-3}+\theta^{-2}+1;\\
\theta^7-2\theta^2+1,\ \theta^{-7}-2\theta^{-5}+1;\quad \theta^{33}+67\theta^{11}+1,\ \theta^{-33}+67\theta^{-22}+1\,.
\end{gathered} \tag{19}$$

Case II(ii): $b=-1$. With $\theta^3-\theta+1=0$, we have $\theta^{58}=1+59\xi$, where $\xi\equiv -11+19\theta-13\theta^2 \bmod 59$. Mutatis mutandis, the 59-adic calculation is similar to the case $b=+1$, so we omit further details. One finds precisely the solutions:

$$\begin{gathered}
\theta^3-\theta+1,\ \theta^{-3}-\theta^{-2}+1;\\
\theta^5+\theta^4+1,\ \theta+\theta^{-4}+1,\ \theta^{-5}+\theta^{-1}+1,\ \theta^{-1}+\theta^{-5}+1;\\
\theta^7+2\theta^4+1,\ \theta^{-7}+2\theta^{-3}+1;\quad \theta^{13}-3\theta^9+1,\ \theta^{-13}-3\theta^{-4}+1\,.
\end{gathered} \tag{20}$$

2.2. We now consider the case when the cubic factor has the form $x^3+bx-1$.

We now have from (1), $B_n=A_{n+1}$, $C_n=A_{n-1}$, so that equations (2), (3), and (3′) become

$$A_mA_{n+1} = A_nA_{m+1} \tag{21}$$

$$A_nA_{m-1} = A_m(A_{n-1}+1) \tag{22}$$

$$A_{n+1}A_{m-1} = A_{m+1}(A_{n-1}+1) \tag{22'}$$

Put $n=3N+s$, $m=3M+r$, $r,s=0,1,2$.

CASE I: $|b|>1$. With $\theta^3+b\theta-1=0$ we have $\theta^3\equiv 1 \bmod b$, and equations (21), (22), and (22′) give the congruences (recalling $(A_{-1},A_0,A_1,A_2)=(1,0,0,1)$):

$$\begin{aligned}
\text{if } s=1: \quad & A_r \equiv 0 \bmod b\\
& A_{r-1} \equiv A_{r+1} \bmod b\,,\\
\text{and if } s=2: \quad & A_{r+1} \equiv 0 \bmod b\\
& A_{r-1} \equiv A_r \bmod b\,.
\end{aligned}$$

For any $r=0,1,2$ we deduce $1\equiv 0 \bmod b$, impossible for $|b|>1$. Thus $s=0$, and we have the expansions

$$A_n = b^2\binom{N}{2}[1+b(\cdot)+\ldots]$$

$$A_{n+1} = bN[-1+b(\cdot)+\ldots]\,.$$

(21) implies $b\,|\,A_m$, forcing $r=1$; and we have modulo $b^3$ that $A_{m+1}\equiv 1$, $A_{m-1}\equiv b^2\binom{M}{2}$, $A_{n+1}\equiv -bN$, $A_{n-1}\equiv 1$, so that in particular, $A_{m+1}(A_{n-1}+1)\equiv 2 \bmod b^3$, and $A_{n+1}A_{m-1}\equiv 0 \bmod b^3$. But then (22′) implies $2\equiv 0 \bmod b^3$, impossible for $|b|>1$.

CASE II: $b=\pm 1$. For $b=+1$ a 47-adic calculation as in 2.1, II(i), offers no difficulties; there are no solutions with $n$ odd. Similarly for $b=-1$; working 59-adically, the only solutions are given by

$$\theta-\theta^3+1,\ \theta^{-1}-\theta^2+1;\quad \theta^7-2\theta^5+1,\ \theta^{-7}-2\theta^{-2}+1\,. \tag{23}$$

2.3. Consider a cubic factor of type $x^3+ax^2\pm 1$. Transforming by $x\to 1/x$, we see that $x^n+Ax^m+1\equiv 0 \bmod (x^3+ax^2+1)$ if and only if $x^n+Ax^{n-m}+1\equiv 0 \bmod (x^3+ax+1)$; and $x^n+Ax^m+1\equiv 0 \bmod (x^3+ax^2-1)$ if and only if

$x^n+Ax^{n-m}+1\equiv 0 \bmod (x^3-ax-1)$. So all solutions can be determined from those of sections 2.1 and 2.2.

2.4. Consider a cubic factor of type $x^3+ax^2+bx+1$, $ab\neq 0$. Transforming by $x\to 1/x$ if necessary, it suffices to find those trinomials $x^n+Ax^m+1$ in which $m$ is also odd.

Now $(\theta_2^n+\theta_3^n)+A(\theta_2^m+\theta_3^m)+2=0$, so that $2\equiv 0 \bmod (\theta_2+\theta_3)$ that is, $v=2/(a+\theta)$ is an integer of $\mathbf{Q}(\theta)$. But $v$ satisfies the equation

$$v^3-\frac{2(a^2+b)}{(ab-1)}v^2+\frac{8a}{(ab-1)}v-\frac{8}{(ab-1)} = 0\,,$$

so necessarily $(ab-1)$ divides h.c.f. $(8, 2(a^2+b))$. For irreducibility, we further require $a+b\neq -2$, and $a\neq b$; and since we are assuming $ab\neq 0$, there arise the following finitely many possibilities listed in the first line of the table.

| $(a,b)$ | $(-9,-1)$ | $(-1,-9)$ | $(-5,-1)$ | $(-1,-5)$ | $(-3,-1)$ | $(-1,-3)$ | $(-2,-1)$ | $(-1,-2)$ |
|---|---|---|---|---|---|---|---|---|
| $p$ | 13 | 13 | 67 | 67 | 67 | 67 | 13 | 13 |

| $(a,b)$ | $(5,1)$ | $(1,5)$ | $(3,1)$ | $(1,3)$ | $(2,1)$ | $(1,2)$ | $(7,-1)$ | $(-1,7)$ | $(3,-1)$ | $(-1,3)$ | $(1,-1)$ | $(-1,1)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | 31 | 31 | 157 | 157 | 101 | 101 | | | | | | |

For $\theta^3+a\theta^2+b\theta+1=0$, let $p$ be an odd rational prime with the order of $\theta$ modulo $p$ equal to $k$, where $k$ is even. Suppose the only solution $(m,n)=(r,s)$ of equations (2), (3), and (3′) taken modulo $p$, in the range $0\leqq r,\ s<k$, is $(r,s)=(0,0)$. Then writing $m=Mk+r$, $n=Nk+s$, we have $A_m\equiv A_r$, $A_n\equiv A_s \bmod p$, and so $(m,n)$ a solution of (2), (3), and (3′) forces $(r,s)=(0,0)$, whence $n$ (and $m$) is even. Thus there can be no solutions with $n$ odd. The second line of the above table gives for certain of the listed possibilities $(a,b)$ a prime $p$ such that the above congruence condition is valid; so these possibilities cannot occur. We must dispose of the six remaining cases.

(i) Consider $\theta^3+7\theta^2-\theta+1=0$.
$\theta$ satisfies

$$\theta = \left(\frac{-\theta^2-6\theta+3}{4}\right)^7 = \varphi^7, \quad \text{where } \varphi^3-\varphi+1 = 0\,,$$

and consequently $\varphi^n+A\varphi^m+1=0$ implies $\theta^{n/7}+A\theta^{m/7}+1=0$, where $\theta^{1/7}=\varphi$. Since $1/7$ is $p$-adic integral for all $p\neq 7$, we can use a 59-adic method as in the previous particular examples, and one finds that the only 59-adic solutions are precisely those corresponding to the solutions listed at (20) (i.e. $\theta^{3/7}-\theta^{1/7}+1$, etc.). Thus there are certainly no solutions with $n$ a natural integer; and in exactly the same way there are no solutions corresponding to $\theta^3-\theta^2+7\theta+1=0$.

(ii) Consider $\theta^3+3\theta^2-\theta+1=0$.
We have $(\theta+1)^3=4\theta$, so that

$$\theta-2^{2/3}\theta^{1/3}+1 = 0, \quad \theta^{-1}-2^{2/3}\theta^{-2/3}+1 = 0 .$$

Working 53-adically confirms that these are the only 53-adic solutions, so no natural integer solutions. Similarly for $\theta^3-\theta^2+3\theta+1=0$.

(iii) Consider $\theta^3+\theta^2-\theta+1=0$.
Corresponding to the identities

$$x^2(x+1)^3+2 = (x^3+x^2-x+1)(x^2+2x+2)$$
$$(x^3+1)^3-2x^7 = (x^3+x^2-x+1)(x^6-x^5+x^3+x+1)$$

we obtain equations

$$\theta+2^{1/3}\theta^{-2/3}+1 = 0 = \theta^{-1}+2^{1/3}\theta^{-5/3}+1$$
$$\theta^3-2^{1/3}\theta^{7/3}+1 = 0 = \theta^{-3}-2^{1/3}\theta^{-2/3}+1$$

and a 47-adic argument shows that these are the only 47-adic solutions; in particular there are again no natural integer solutions.

2.5. For a cubic factor of type $x^3+ax^2+bx-1$, $ab\neq 0$, we obtain as above that $(ab+1)$ divides h.c.f. $(8, 2(a^2+b))$, and there are only finitely many possibilities for $(a,b)$. There is no difficulty with the $p$-adic treatment, and we omit details. No rational integral solutions arise, with $n$ odd.

2.6. We now have to turn to the case where $m\equiv n\equiv 0 \bmod 3$, and we show that are no solutions in this instance. For if $x^{3N}+Ax^{3M}+1$ has the irreducible factor $x^3+ax^2+bx+1$, then $y^N+Ay^M+1$ has the irreducible factor $y^3+(a^3-3ab+3)y^2+(b^3-3ab+3)y+1$. The coefficients of this latter cubic are all non-zero because $a^3-3ab+3$ has no zero mod 9. But for $3\nmid (M,N)$, $N$ odd, we have seen in sections 2.4 and 2.5 that no such factors can arise. Similarly for a factor $x^3+ax^2+bx-1$.

## 3. The case $n$ even.

Transforming by $x\to -x$ if necessary, it suffices to find those cubic factors with constant coefficient $+1$. Suppose in the first instance that $n\equiv 0 \bmod 2$, $m\equiv 1 \bmod 2$. If $\theta^3+a\theta^2+b\theta+1=0$, and $\theta^n+A\theta^m+1=0$, then $\theta^{n-m}+A=-\theta^{-m}$, and taking norms,

$$-(-1)^{-m} = (\theta_1^{n-m}+A)(\theta_2^{n-m}+A)(\theta_3^{n-m}+A)$$

that is

$$1 = -1-A\sum \theta_i^{m-n}+A^2\sum \theta_i^{n-m}+A^3$$

so

$$A(A^2+A\sum \theta_i^{n-m}-\sum \theta_i^{m-n}) = 2\,,$$

whence $A=\pm 1, \pm 2$. Now when $A=\pm 1$, Ljunggren [5] shows that $x^n+Ax^m+1=g(x)h(x)$, where $g(x)$ is irreducible and $h(x)=\prod(x-\lambda)$, with the product over roots of unity $\lambda$.

Accordingly, since no root of an irreducible cubic can be a root of unity, we must have $g(x)=x^3+ax^2+bx+1$ so that $\deg h(x)$ is odd, forcing $h(\pm 1)=0$, an immediate contradiction.

For $A=\pm 2$, Schinzel [6] shows inter alia that if $n$ is even, $m$ odd, $n>m>0$, then $(x^n+2x^m+1)/(x^{(m,n)}+1)$ is irreducible. So we must have in our instance that

$$x^n+2x^m+1 = (x^{(m,n)}+1)(x^3+ax^2+bx+1)\,.$$

Then $(m,n)+3=n$, so $(m,n)$ divides 3. Clearly $(m,n)=3$ is impossible, and if $(m,n)=1$ it is easy to check that the only possibilities are

$$x^4+2x^3+1 = (x+1)(x^3+x^2-x+1) \tag{24}$$

$$x^4+2x+1 = (x+1)(x^3-x^2+x+1)\,.$$

Consider now $n=2N$, $m=2M$. Then $y^N+Ay^M+1$ has the irreducible factor $y^3-(a^2-2b)y^2+(b^2-2a)y-1$; but from the above, the only such factors for $2\nmid (M,N)$ are $y^3-y-1$, $y^3+y^2-1$, $y^3+y^2+y-1$, $y^3-y^2-y-1$, and modulo 4 there are no solutions to any of the possibilities $(a^2-2b, b^2-2a)=(0,-1)$, $(-1,0)$, $(-1,1)$, $(1,-1)$.

From (19), (20), (23), (24), we thus deduce

THEOREM. *Suppose* $x^n+Ax^m+1\in \mathbf{Z}[x]$, $n,m\in\mathbf{Z}$, $n\geqq 2m>0$, *has an irreducible factor in* $\mathbf{Z}[x]$ *of degree* 3. *Then, if* $n>3$, *the only possibilities are the following*:

$$x^4+2x+1 = (x+1)(x^3-x^2+x+1); \qquad x^4-2x+1 = (x-1)(x^3+x^2+x-1);$$

$$x^5+x+1 = (x^2+x+1)(x^3-x^2+1);$$

$$x^7-2x^2+1 = (x-1)(x^3+x^2-1)(x^3+x+1);$$

$$x^7+2x^3+1 = (x^3-x^2+1)(x^4+x^3+x^2+1);$$

$$x^{13}-3x^4+1 = (x^3-x^2+1)(x^{10}+\ldots+1);$$

$$x^{33}+67x^{11}+1 = (x^3+x+1)(x^{30}-\ldots-1).$$

## 4. Postscript.

It should be similarly straightforward to find all cubic factors of trinomials $x^n+Ax^m-1$, $n>m>0$. The previous sections will furnish all solutions if $n$ is odd, by considering $x \to -x$. When $n$ is even, Łutczyk (see Schinzel [12]) gave the example

$$x^8+3x^3-1 = (x^3+x-1)(x^5-x^3+x^2+x+1);$$

we note the further example

$$x^{14}+4x^5-1 = (x^3+x^2-1)(x^{11}-\ldots+1),$$

and also the one parameter family given by

$$x^6+(4\mu^4-4\mu)x^2-1 = (x^3+2\mu x^2+2\mu^2x+1)(x^3-2\mu x^2+2\mu^2x-1), \quad \mu \neq 0,1.$$

The following trinomials with their corresponding cubic factor also came to light whilst the above calculations were being carried out:

$$x^8+3x+2(x^3+x+1);\ x^8-7x-4(x^3+x^2+2x+1);$$

$$x^8-36x-13(x^3+x^2+3x+1);$$

$$x^{14}+4x+3(x^3-x^2+1);\ x^{16}+7x^3+3(x^3-x^2+1);\ x^{16}+7x^2-4(x^3-x^2+1);$$

$$x^{16}+7x^9-2(x^3-x+1);\ x^{16}-56x^3-9(x^3-x^2+x+1);$$

$$x^{17}+103x+56(x^3-x^2+x+1).$$

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