MATH. SCAND. 49 (1981), 181–210

SCISSORS CONGRUENCES, I
THE GAUSS-BONNET MAP

CHIH-HAN SAH*

Hilbert's Third Problem was affirmatively solved by Max Dehn a few months after it was posed. For $n>2$, Dehn's solution showed that it is not possible to have an elementary theory of volume for polytopes formed out of geodesic simplices in n-dimensional Euclidean, spherical, elliptic, or hyperbolic spaces by using the idea of scissors congruences (finite cutting and pasting together with isometries). In [24], the works of many authors were incorporated into a theory of scissors congruences. In this framework, a number of algebraic structures (Hopf algebras, comodules, Eilenberg–MacLane homology of classical groups) appeared rather naturally from geometric considerations. Furthermore, a number of unresolved problems were posed to serve as guides for the theory. Some of these problems appear to have connections with subjects that are of current interests.

Among the various suggestions made in [24] was the vague feeling that there ought to be a Gauss–Bonnet map relating the scissors congruence groups in even dimensional hyperbolic and spherical spaces with the preceding odd dimension. It was also suggested that scissors congruences ought to be considered for extended hyperbolic spaces. These are the principal goals of the present paper. After a short summary of some of the basic results of [24] in section 1, we describe the Gauss–Bonnet map in section 2. This is the extended to the "closed" hyperbolic spaces in section 3. Section 4 deals with a few divisibility results. Appendix 1 deals with scissors congruences in an axiomatic setting. This repairs some of the inaccuracies in [24; Chap. 1]. Appendix 2 is a short exposition of an unpublished result of W. Thurston on the definition of Dehn invariants for extended hyperbolic spaces. It also shows that the Gauss–Bonnet map can be viewed as a "lifted" form of a Dehn invariant. Section 4 and Appendix can be viewed as "applications" of the Gauss–Bonnet map.

It is perhaps reasonable to give a status report on some of the problems mentioned in [24]. The syzygy problem [24, Chap. 5] concerning the higher order relations among the Hadwiger invariants used in the theorem of Jessen–

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* This work was partially supported by a granted from the National Science Foundation.
Received July 10, 1980; in revised form November 10, 1980.

Thorup [15] can be considered as solved by the homological approach developed by Johan Dupont [7]. The questions related to Gauss–Bonnet maps [24; Chap. 6, 7, 8] are considered in the present paper. Other questions concerning torsions, isomorphisms, etc., will be considered in a join work with Dupont. The large hyperbolic simplex problem has been affirmatively and elegantly solved by U. Haagerup and H. Munkholm [10]. The rational simplex problem remains open. The status of the various scissors congruence problems has not changed since [24]. Specifically, the translational scissors congruence problem is solved in all dimensions by Jessen–Thorup [15]. The Euclidean scissors congruence problem is solved for dimensions $n \leqq 4$ by Sydler [28] or [12] and Jessen [13]. The spherical and hyperbolic scissors congruence problems were solved classically in dimensions $n \leqq 2$. In all other cases, Dehn invariants (including volume) furnish necessary conditions, but the sufficiency remains open, see [24].

It is our pleasure to acknowledge our debt for the many useful comments communicated to us in conversations and/or writings from many people. Aside from our colleagues in Stony Brook, we would like to thank Robert Connelly, Johan Dupont, Børge Jessen, John Milnor and Anders Thorup for substantial contributions to our thinking process. Thanks are also due to the referee for helpful suggestions in streamlining the organization of the present work.

1. Preliminaries.

We first recall a result of Tits characterizing the classical geometries, see [8, 30].

FREE MOBILITY THEOREM. *Let X be a locally compact, connected metric space. Then X is isometric to an n-dimensional real Euclidean, elliptic, spherical, or hyperbolic space for a suitable natural number n if and only if X enjoys the following property:*

(FM) *For any two sufficiently small congruent triangles in X, there is a global isometry of X extending the congruence.*

A triangle is understood to be any ordered triple of points of X (degeneracies allowed) and a congruence between two such triples is understood to be a distance preserving map (necessarily unique) on the vertices.

We will be dealing with scissors congruence data based on one of these classical geometries. As has been mentioned in [24], the elliptic case can be absorbed by the spherical case. Let $X = X^n(\varepsilon)$ denote n-dimensional real Euclidean, spherical, or hyperbolic space according to $\varepsilon = 0$, 1 or -1. Let G denote the full group of isometries of X. The group $\mathscr{P}(X, G)$ of scissors congruence classes of polytopes is generated by $[A]$ (mod G) with A ranging over the small geodesic n-simplices of X satisfying the defining relations:

(RS) $[A]=[B]+[C]$ where $A=B\coprod C$ ranges over all simple subdivisions of small geodesic n-simplices A, B, C in X.

(RC) $[\sigma A]=[A]$ where σ ranges over G and A ranges over all small geodesic n-simplices.

In the cases of interest to us, an n-simplex is called small if its diameter is less than the diameter of X (this is a condition only when X is spherical). A more detailed discussion concerning the various formulations of $\mathscr{P}(X,G)$ can be found in Appendix 1. We use $\coprod$ to denote interior disjoint union as well as direct sum.

In order to define the Gauss–Bonnet map, we need to recall the notation of [24]. $\mathsf{R}^{1,\infty}$ will denote an R-vector space of countable dimension with R-basis $e_0, e_1, \ldots$ $\langle\cdot,\cdot\rangle_{1,\infty}$ will denote the inner product so that:

$\langle e_i, e_j\rangle_{1,\infty}=0, -1, 1$ *respectively when* $i\neq j$, $i=j=0$, $i=j>0$. $\langle\cdot,\cdot\rangle_{1+\infty}$ will denote the usual positive definite inner product:

$$\langle e_i, e_j\rangle_{1+\infty} = 0,\ 1 \textit{ respectively when } i\neq j,\ i=j\ .$$

R^∞ will denote the subspace spanned by $e_1, e_2, \ldots$. For $n\geqq 0$, $\mathsf{R}^{1,n}$ will denote the subspace spanned by $e_0,\ldots,e_n$ while R^n will denote the subspace spanned by $e_1,\ldots,e_n$.

$X^n(0)$ can be taken to be R^n and will be identified with the standard column vectors of length n. $G=G^n(0)$ is then the semidirect product of the translation group $T(n;\mathsf{R})$ by the rotation group $O(n;\mathsf{R})$. $X^n(1)$ can be taken to be the set $S(\mathsf{R}^{n+1})$ of all unit vectors in R^{n+1}. $G=G^n(1)$ is then $O(n+1;\mathsf{R})$. $X^n(-1)$ can be taken to be the hyperbolic space $\mathscr{H}^n$ modelled on the rays R^+x in $\mathsf{R}^{1,n}$ with $\langle \mathsf{R}^+x, \mathsf{R}^+x\rangle_{1,n}=\mathsf{R}^-$ and $\langle \mathsf{R}^+x, \mathsf{R}^+e_0\rangle_{1+n}=\mathsf{R}^+$. $G=G^n(-1)$ is the subgroup $\Omega(1,n;\mathsf{R})$ of index 2 in $O(1,n;\mathsf{R})$ formed by all elements with spinor norm 1 i.e., preserving the forward light cone associated to the quadratic form: $-X_0^2 +\sum_{1\leqq i\leqq n} X_i^2$. Contrary to the usual convention, signature (p,q) means p minus signs and q plus signs.

An Eulidean n-simplex A is determined by $n+1$ affinely independent vertices $x_0,\ldots,x_n$ (viewed as points in R^∞):

$$A = \operatorname{ccl}\{x_0,\ldots,x_n\} = \{\textstyle\sum_i \alpha_i x_i \mid \alpha_i\geqq 0 \text{ in } \mathsf{R} \text{ with } \sum_i\alpha_i=1\}\ .$$

A spherical $(n-1)$-simplex B (of degree $n=1+\dim B$) is determined by n R-linearly independent unit vectors $y_1,\ldots,y_n$ (in $S(\mathsf{R}^\infty)$):

$$B = \operatorname{sccl}\{y_1,\ldots,y_n\} = \{\textstyle\sum_i \alpha_i y_i \in S(\mathsf{R}^\infty) \mid \alpha_i\geqq 0 \text{ in } \mathsf{R}\}\ .$$

A hyperbolic n-simplex C is determined by $n+1$ independent rays $\mathsf{R}^+z_0,\ldots,\mathsf{R}^+z_n$ in $\mathscr{H}^\infty(\mathsf{R})$:

$$C = \operatorname{hccl}\{\mathsf{R}^+z_0,\ldots,\mathsf{R}^+z_n\} = \{\mathsf{R}^+\cdot\textstyle\sum_i\alpha_i z_i \in \mathscr{H}^\infty(\mathsf{R}) \mid \alpha_i\geqq 0 \text{ in } \mathsf{R}\}\ .$$

We could have identified each ray $\mathbf{R}^+ z_i$ with the unique vector z_i with $\langle z_i, e_0\rangle_{1+\infty} = 1$. $\mathscr{H}^n(\mathbf{R})$ is then identified with an open n-ball of radous 1 in an affine n-space and the geodesic hyperbolic n-simplices are represented by Euclidean n-simplices. Under this interpretation, the combinatorial aspects of cutting and pasting has not changed. However, the group of motions has changed so that the concepts of distance and angle have changed. Other models of hyperbolic spaces are sometimes more convenient. We refer to Thurston [29] for more detailed discussions.

The target group of the Gauss–Bonnet map will be $\mathscr{P}S^n = \mathscr{P}(S(\mathbf{R}^n), O(n; \mathbf{R}))$. We emphasize that the superscript n denotes degree rather than dimension when we are dealing with spherical cases. By definition, $\mathscr{P}S^0 = \mathbf{Z}\cdot[\varnothing] \cong \mathbf{Z}$ where $\varnothing$ denotes the unique (-1)-simplex—the empty set. We can also use the ray model for spherical spaces; $\varnothing$ then corresponds to the origin of $\mathbf{R}^\infty$. Similarly, $\mathscr{P}S^1 = \mathbf{Z}[\text{point}] \cong \mathbf{Z}$. We form the graded abelian group:

$$\mathscr{P}S = \coprod_{i \geqq 0} \mathscr{P}S^i .$$

$\mathscr{P}S$ is given the product structure through the orthogonal join (corresponding to orthogonal Minkowski sum in the ray model). More precisely, let P and Q be spherical simplices in $S(\mathbf{R}^\infty)$ with vertices $x_1, \ldots, x_p$ and $y_1, \ldots, y_q$. Select σ in $O(\infty; \mathbf{R})$ so that $\langle x_i, \sigma y_j\rangle_\infty = 0$ holds for all i, j. Then the product is defined by:

$$[P] * [Q] = [P * \sigma Q], \qquad P * \sigma Q = \operatorname{sccl}\{x_1, \ldots, x_p, \sigma y_1, \ldots, \sigma y_q\} .$$

This product is independent of the choice of σ as long as the orthogonality condition is satisfied. With this product, $\mathscr{P}S$ becomes a commutative ring graded by degree and $[\varnothing]$ is the unit element. The graded ideal generated by [point] is denoted by $\mathscr{C}S$ and its ith power is denoted by $\mathscr{C}_i S$; these powers define the decreasing point-adic filtration of $\mathscr{P}S$. We note that $[S(\mathbf{R}^n)] = 2^n[\text{point}]^{*n}$. Since we are over $\mathbf{R}$, the spherical $(n-1)$-simplex P has an invariant $(n-1)$-dimensional volume: $\text{vol}_{n-1}(P) > 0$. Following Schäfli, this volume is normalized so that $\text{vol}_{n-1}(S(\mathbf{R}^n)) = 2^n$. We then have a graded ring homomorphism:

$$\text{gr. vol.}: \mathscr{P}S \to \mathbf{R}[T], \ \text{gr. vol. } [P] = \text{vol}_{n-1}(\text{P})\cdot \text{T}^n .$$

In general, $\mathscr{P}(X^n(\varepsilon), G)$ is 2-divisible when $n \geqq 1$, [24; Prop. 1.4.3, p. 17]. Using this, $\mathscr{P}S^{2i+1} = \mathscr{P}S^{2i} * [\text{point}]$, [24; Prop. 6.2.2, p. 105]. The evenly graded commutative ring $\mathscr{P}S/\mathscr{C}S$ is actually a Hopf algebra over $\mathbf{Z}$. $\mathscr{P}S/\mathscr{C}S$ has additive torsion (at least in degree 2). From Hopf–Leray theorem, $\mathscr{P}S/\mathscr{C}S$ mod torsion is an integral domain. Using the volume msp, $\mathscr{P}S/\text{Tor}(\mathscr{P}S)$ is an integral domain. It is unknown if $\text{Tor}(\mathscr{P}S)$ is zero or not. For more details, see [24; Chap. 6].

Set

$$\mathcal{P}E^n = \mathcal{P}(X^n(0), G) = \mathcal{P}(\mathrm{R}^n, T(n; \mathrm{R})O(n; \mathrm{R})) \quad \text{for } n>0$$

and set $\mathcal{P}E^0 = \mathrm{R}$. As in the spherical case, form the graded abelian group:

$$\mathcal{P}E = \coprod_{n \geqq 0} \mathcal{P}E^n .$$

One of the consequences of the theorem of Jessen–Thorup, see [15] or [24], is the fact that $\mathcal{P}E^n$ has an R-vector space structure (in a very complicated way). This can be incorporated into $\mathcal{P}E$ together with the product arising from orthogonal Minkowski sum. $\mathcal{P}E$ then becomes a commutative R-algebra graded by dimension. It is unknown if $\mathcal{P}E$ is an integral domain. However, we always have a graded R-algebra homomorphism through volume:

$$\text{gr. vol.}: \mathcal{P}E \to \mathrm{R}[T], \ \text{gr. vol.}\ [P] = \mathrm{vol}_n (P)\cdot T^n ,$$

where $\mathrm{vol}_n (P) > 0$ is the absolute volume of the n-simplex P.

It should be noted that the volume is definable purely algebraically in the present case.

Set

$$\mathcal{P}\mathcal{H}^n = \mathcal{P}(X^n(-1), G) = \mathcal{P}(\mathcal{H}^n, \Omega(1,n; \mathrm{R})), \quad n \geqq 0 .$$

$$\mathcal{P}\mathcal{H}^0 = \mathrm{Z}\cdot[\text{point}] \cong \mathrm{Z} .$$

As before, form the graded abelian group:

$$\mathcal{P}\mathcal{H} = \coprod_{n \geqq 0} \mathcal{P}\mathcal{H}^n .$$

We only have a homomorphism of graded abelian group:

$$\text{gr. vol.}: \mathcal{P}\mathcal{H} \to \mathrm{R}[T] .$$

In all cases, $\mathcal{P}S$, $\mathcal{P}E$, $\mathcal{P}\mathcal{H}$ are right comodules for the Hopf algebra $\mathcal{P}S/\mathcal{C}S$ with structure map given by the total Dehn invariant. There are natural compatibility results with the product structures, see [24; Chap. 6, 7, 8].

2. The Gauss–Bonnet map.

The basic idea of a general Gauss–Bonnet map was already present in Poincaré [23], Dehn [5], Hopf [11], see Klein [16; pp. 200–205]. Instead of the usual procedure of using analysis to measure angles, we lift the definition to the level of the scissors congruence groups. The free mobility theorem allows us to replace integration by a finite sum. See [1, 2, 3, 4, 6] among others for discussions and uses of the Gauss–Bonnet formula.

Let A be a small geodesic n-simplex in $X^n(\varepsilon)$, $n>0$. Let F denote an i-dimensional face of A, $0\leqq i\leqq n$. Let $x(F)$ be any interior point of F (e.g., the barycenter of F through the underlying linear structure). Define $\theta(F,A)$ to be the $(n-1)$-dimensional spherical polytope formed by the closure of the set of all interior unit vectors with origin at $x(F)$. Recall that we also have the interior angle $\theta_A(F)$ at the face F of A formed by all the interior unit vectors normal to the face F with origin $x(F)$. We have:

$$(2.1)\quad [\theta(F,A)] = [\theta_A(F)]*[S(\mathrm{R}^i)] = 2^i[\text{point}]^{*i}*[\theta_A(F)], \qquad i=\dim F\,.$$

Using free mobility property (FM), $[\theta(F,A)]$ does not depend on the choice of $x(F)$. To the n-simplex A, associate the element $\mathfrak{e}(A)\in\mathscr{P}S^n$ as follows:

$$(2.2)\qquad \mathfrak{e}(A) = \sum_F (-1)^{\dim F}[\theta(F,A)], \qquad F \text{ a nonempty face of } A\,.$$

Leaving out the terms corresponding to $\dim F=n-1$ and n, composing the result with the volume function, we then have the generalized angle sum, see Klein [16; p. 204]. The volume of A is related to $\mathfrak{e}(A)$ by:

$$\mathrm{vol}_{n-1}(\mathfrak{e}(A)) = \varkappa_n\cdot\mathrm{vol}_n(A), \qquad \varkappa_n \text{ is a universal constant}\,.$$

In this formula, volume has to be normalized appropriately. The universal constant $\varkappa_n$ is proportional to the product of ε and the Euler characteristic of an n-sphere. In view of Dehn's solution to Hilbert's third problem, $\mathfrak{e}(A)$ carries more information than $\mathrm{vol}_{n-1}(\mathfrak{e}(A))$; see Dupont [6] for other uses of $\mathrm{vol}_{n-1}(\mathfrak{e}(A))$. We will call $\mathfrak{e}$ the Gauss–Bonnet map. When $n=0$, set $\mathfrak{e}$ to be an isomorphisms for $\varepsilon\neq 0$ and 0 when $\varepsilon=0$.

THEOREM 2.3. *Let $n>0$ and let $\mathscr{P}X^n(\varepsilon)=\mathscr{P}(X^n(\varepsilon),G)$. The Gauss–Bonnet map $\mathfrak{e}$ induces an additive homomorphism*:

$$\mathfrak{e}\colon \mathscr{P}X^n(\varepsilon) \to \mathscr{P}S^n = \mathscr{P}X^{n-1}(1)\,.$$

When $\varepsilon=0$, $\mathfrak{e}=0$.

PROOF. From its definition, $\mathfrak{e}(A)$ depends only on the congruence class of the n-simplex A. Let $A=B\coprod C$ be a simple subdivision. We assert that $\mathfrak{e}(A)=\mathfrak{e}(B)+\mathfrak{e}(C)$.

Let H be the hyperplane subdividing A into B and C so that H contains all but two of the vertices of A. We group together the terms in $\mathfrak{e}(A)-\mathfrak{e}(B)-\mathfrak{e}(C)$ in accordance with the following cases:

CASE 1. All vertices of F lie in H.

F is therefore a face of A, B as well as C. $\theta_A(F)=\theta_B(F)\coprod\theta_C(F)$ with H doing the subdivision. We obtain from (2.1):

$$(-1)^{\dim F}([\theta(F,A)]-[\theta(F,B)]-[\theta(F,C)]) = 0 .$$

CASE 2. Exactly one vertex of F lies outside of H.

F is therefore a face of B or of C according to the exceptional vertex of F is part of B or part of C. $\theta(F,A)$ is then equal to $\theta(F,B)$ or $\theta(F,C)$ and we have cancellation in either case.

CASE 3. Exactly two of the vertices of F lie outside of H.

F is therefore subdivided by H into $B\cap F$ and $C\cap F$ and $B\cap C\cap F=H\cap F$ is a common $(i-1)$-dimensional face of B and C. We note that $i\geqq 1$ and $H\cap F$ is not a face of A. We have the following terms:

$$(-1)^{\dim F}\{[\theta(F,A)]-[\theta(F\cap B,B)]-[\theta(F\cap C,C)] \\ +[\theta(H\cap F,B)]+[\theta(H\cap F,C)]\} .$$

The interior point $x(F)$ can be moved to an interior point of $B\cap F$ or of $C\cap F$ so that $\theta_B(F\cap B) = \theta_A(F)=\theta_C(F\cap C)$. Similarly, if $x(F)$ is moved to a point of $H\cap F$, then $\theta(F,A)$ is subdivided by H into $\theta(H\cap F,B)$ and $\theta(H\cap F,C)$. We have cancellation again.

These cases partition all the terms in $\mathfrak{e}(A)-\mathfrak{e}(B)-\mathfrak{e}(C)$ so that the map $\mathfrak{e}$ is additive with respect to simple subdivisions of n-simplices. The functorial properties of $\mathscr{P}X^n(\varepsilon)$ imply that $\mathfrak{e}$ induces an additive homomorphism.

Assume $\varepsilon=0$. The group $\mathscr{P}X^n(0)=\mathscr{P}E^n$ is a direct sum of weight spaces for weights i with $l\leqq i\leqq n$ and $n-i$ even, see [24; Chap. 7]. This means that $\mathscr{P}E^n$ is an $\mathbf{R}$-vector space and a vector in $\mathscr{P}E^n$ has weight i if the homothety by λ in $\mathbf{R}^+$ multiplies this vector by λ^i. We note that the existence of the $\mathbf{R}$-vector space structure is quite difficult whole the existence of a $\mathbf{Q}$-vector space structure on $\mathscr{P}E^n$ is relatively easy. For our purposes, the existence of the $\mathbf{Q}$-vector space structure and the associated $\mathbf{Q}$-weight structure is enough. Breaking up $[A]$ into its weight components and applying the homothety λI_n to A to get $\lambda\circ A$, the components of $[\lambda\circ A]$ are obtained from those of $[A]$ through multiplication by appropriate powers of λ. The map $\mathfrak{e}$ is clearly of weight 0. Since $n>0$, such maps must be 0 from weight considerations. We note that $\mathfrak{e}$ has been defined to be 0 when $n=0=\varepsilon$.

When $n=2$, angles can be measured in radians and $\mathfrak{e}(A)$ is just $\alpha+\beta+\gamma-3\pi+2\pi$. This is just ε times the area of A. In general, if n is even, then we need a positive constant times $\varepsilon^{n/2}$ for the factor. The positive constant depends on the normalization of the invariant volume of the unit sphere.

The vanishing of $\mathfrak{e}$ when $\varepsilon=0$ is equivalent to a relation among the various face angles of a polytope in Euclidean space. After composing with the volume function, this has been studied extensively, see [9, 19, 20, 22, 26, 27] for example. Its possible use in the study of scissors congruences in Euclidean spaces has not as yet received much attention.

For the remaining part of this section, we consider the cases $\varepsilon = \pm 1$. When $n=0$, $\mathfrak{e}[\text{point}] = [\varnothing]$ and $\mathscr{P}S^1 \cong \mathscr{P}S^0 \cong \mathscr{P}\mathscr{H} \cong \mathbf{Z}$ through $\mathfrak{e}$. When $n=1$, $\mathfrak{e}$ can be seen to be 0 from its definition.

In the spherical case, $\mathscr{P}S^{2i+1} = [\text{point}] * \mathscr{P}S^{2i}$, [24; Prop. 6.2.2, p. 105]. The question of isomorphism between $\mathscr{P}S^{2i+1}$ and $\mathscr{P}S^{2i}$ was left open. This will now be settled in the affirmative.

PROPOSITION 2.4. *Let P and Q be spherical simplices of dimensions p and q, respectively. Assume $\langle \mathrm{Lsp}\,(P), \mathrm{Lsp}\,(Q)\rangle_\infty = 0$ where* Lsp *denotes the linear span. Then,*

$$\mathfrak{e}[P*Q] \;=\; \mathfrak{e}[P]*[Q] + [P]*\mathfrak{e}[Q] - 2[\text{point}]*\mathfrak{e}[P]*\mathfrak{e}[Q]\,.$$

PROOF. A face F of $P*Q$ has the form $P'*Q'$ where P' and Q' are faces (possibly empty) of P and Q respectively. We have the following cases:

CASE 1. P' and Q' are both nonempty.
From the orthogonality assumption, we have:

$$\theta_{P*Q}(P'*Q') \;=\; \theta_P(P')*\theta_Q(Q')\,.$$

From (1.1), we conclude:

$$[\theta(P'*Q', P*Q)] \;=\; 2[\text{point}]*[\theta(P', P)]*[\theta(Q', Q)]\,.$$

CASE 2. Exactly one of P', Q' is empty, say $Q' = \varnothing$.
We therefore have:

$$\theta_{P*Q}(P') \;=\; \theta_P(P')*Q\,.$$

From (1.1), we conclude:

$$[\theta(P', P*Q)] \;=\; [\theta(P', P)]*[Q]\,.$$

Breaking up $\mathfrak{e}[P*Q]$ into three groups of terms according to the face $P'*Q'$ is such that $Q' = \varnothing$, $P' = \varnothing$ or both P' and Q' are nonempty, we obtain the desired assertion. We note that the join product is compatible with the degree gradation, not the dimension gradation. This explains the factor of -1. The join product with 2[point] is just orthogonal suspension. The definition of $\mathfrak{e}$ together with the preceding remark explain the presence of this factor in our formula.

Except for differences in notation, the next result goes back to Poincaré [23].

PROPOSITION 2.5. *Let A be any n-simplex in $X^n(\varepsilon)$. Then,*

$$2[\text{point}] * \mathfrak{e}(\mathfrak{e}(A)) = 0$$

PROOF. $\mathfrak{e}$ is understodd to be the zero map on $\mathscr{P}S^0$. We already noted that $\mathfrak{e}=0$ if either $\varepsilon=0$ or $n=1$. We may therefore assume $n>1$ and $\varepsilon=\pm 1$. Actually, our argument is formal and applies to the case $\varepsilon=0$.

Combining (2.1) and Proposition 2.4, we have:

$$\mathfrak{e}(\theta(F,A)) = \mathfrak{e}(\theta_A(F)) * [S(\mathbf{R}^{\dim F})], \quad \text{provided that } \dim F \text{ is even;}$$

$$\mathfrak{e}(\theta(F,A)) = 2\{[\theta_A(F)] - \mathfrak{e}(\theta_A(F)) * [\text{point}] * [S(\mathbf{R}^{\dim F-1})]\}, \quad \text{provided that } \dim F \text{ is odd}.$$

The interior angle $\theta_A(F)$ at the face F of A is a spherical simplex of degree (number of vertices) equal to the codimension of F in A. In fact, if $\mathbf{R}^+x$ is a vertex of A not in F (in the ray model when $\varepsilon \neq 0$), then the unit vector along the normal projection of x to the linear subspace Lsp (F) leads to a vertex of $\theta_A(F)$. It follows that a face of $\theta_A(F)$ is uniquely represented as the normal projection to Lsp (F) of a face G of A with G containing F. This yields:

$$\mathfrak{e}(\theta_A(F)) = \sum_{G>F} (-1)^{\dim G - \dim F - 1}[\theta_A(G)] * [S(\mathbf{R}^{\dim G - \dim F - 1})],$$

where the sum extends over all faces G of A properly containing F.

Combining the preceding calculations, we have:

$$\begin{aligned} 2[\text{point}] * \mathfrak{e}(\mathfrak{e}(A)) &= \sum_{\dim F \text{ even}} \sum_{G>F} (-1)^{\dim G - \dim F - 1}[\theta(G,A)] \\ &\quad + \sum_{\dim F \text{ odd}} \sum_{G>F} (-1)^{\dim G - \dim F - 1}[\theta(G,A)] \\ &\quad -2 \sum_{\dim F \text{ odd}} [\theta(F,A)] \\ &= -\left\{\sum_F (-1)^{\dim F} \sum_{G \geqq F} (-1)^{\dim G}[\theta(G,A)]\right\} \\ &\quad + \sum_F [\theta(F,A)] - 2 \sum_{\dim F \text{ odd}} [\theta(F,A)]. \end{aligned}$$

The last two sums can be combined to yield $\mathfrak{e}(A)$. In the remaining double sum, we reverse the summation. Leaving aside the factor -1 at the front, the coefficient of $(-1)^{\dim G}[\theta(G,A)]$ is just $\sum_{H \leqq G}(-1)^{\dim H}$, where H ranges over the nonempty faces of G. Since $\dim H$ is 1 less than the number of vertices of H, binomial theorem shows that:

$$\sum_{H \leqq G} (-1)^{\dim H} = 1 .$$

As a consequence, the double sum in question is just:

$$-\sum_{G} (-1)^{\dim G}[\theta(G, A)] = -\mathfrak{e}(A) .$$

We therefore have cancellation.

THEOREM 2.6. *Let* $\varepsilon = \pm 1$. *The following assertions hold*:

(a) $\mathfrak{e}\colon \mathscr{P}S^{2i+1} \to \mathscr{P}S^{2i}$ *is an isomorphism inverse to the map*: $[\text{point}] *\colon \mathscr{P}S^{2i} \to \mathscr{P}S^{2i+1}$

(b) $\mathfrak{e}\colon \mathscr{P}(X^{2i+1}(\varepsilon)) \to \mathscr{P}S^{2i+1}$ *is the zero map.*

PROOF. The assertions have already been noted for $i=0$. We will now verify the assertions for the case $\varepsilon = 1$ (essentially due to Poincaré) by complete induction on the degree n.

CASE 1. $n = 2i+1$ is odd so that $i > 0$.

Consider the composition of the maps in the following sequences:

$$\mathscr{P}S^{2i} \xrightarrow{[\text{point}]*} \mathscr{P}S^{2i+1} \xrightarrow{\mathfrak{e}} \mathscr{P}S^{2i} .$$

Since $\mathfrak{e}[\text{point}] = [\varnothing]$, Proposition 2.4 and induction hypothesis imply that the composition is the identity map. Since $i > 0$, the first map is surjective, [24; Prop. 6.2.2, p. 105]. Thus (a) holds.

CASE 2. $n = 2i+2$ is even so that $i \geqq 0$.

Let A be any spherical simplex of degree n. By Proposition 2.5, $2[\text{point}] * \mathfrak{e}(\mathfrak{e}(A)) = 0$. Now $\mathfrak{e}(A)$ has odd degree $n-1$. Induction hypothesis together with (a) imply that $2\mathfrak{e}(A) = [\text{point}] * \mathfrak{e}(2(\mathfrak{e}(A))) = 0$. If $n = 2$, then $\mathscr{P}S^{n-1} = \mathscr{P}S^1 \cong \mathbf{Z}$ is torsion free so that $\mathfrak{e}(A) = 0$ (this can also be checked directly). If $n > 2$, then $\mathscr{P}S^n$ is 2-divisible, [24; Prop. 1.4.3, p. 17]. It follows that $\mathfrak{e}(A) = 0$ in all cases when n is even and $\varepsilon = 1$. We have (b) for $\varepsilon = 1$.

Suppose that $\varepsilon = -1$ and that A is a hyperbolic $(2i+1)$-simplex. It follows that $\mathfrak{e}(A) \in \mathscr{P}S^{2i+1}$. From assertion (a), we have:

$$\mathfrak{e}(A) = [\text{point}] * \mathfrak{e}(\mathfrak{e}(A)) .$$

Proposition 2.5 shows that $2\mathfrak{e}(A) = 0$. Just as in the spherical case, $\mathscr{P}\mathscr{H}^{2i+1}$ is 2-divisible when $i \geqq 0$, [24; Prop. 1.4.3, p. 17]. It follows that $\mathfrak{e}$ is the zero map. This is just (b) for $\varepsilon = -1$.

COROLLARY 2.7. *Let* $\mathscr{P}S^{ev} = \coprod_{i \geq 0} \mathscr{P}S^{2i}$ *and let* $\mathscr{P}S^{odd} = \coprod_{i \geq 0} \mathscr{P}S^{2i+1}$. *Then* $[\text{point}]*: \mathscr{P}S^{ev} \to \mathscr{P}S^{odd}$ *is an isomorphism with* $\mathfrak{e}$ *as inverse. They are isomorphisms of free* $\mathscr{P}S^{ev}$ *modules of rank* 1 *as well as right comodules for the Hopf algebra* $\mathscr{P}S/\mathscr{C}S$.

COROLLARY 2.8. *The additive homomorphism* $[\text{point}]*: \mathscr{P}S^{odd} \to \mathscr{P}S^{ev}$ *has kernel equal to* $\text{Tor}(\mathscr{P}S^{odd}) \cong \text{Tor}(\mathscr{P}S^{ev})$. (*It is unknown if* $\text{Tor}(\mathscr{P}S)$ *is zero or not.*)

COROLLARY 2.9. $\mathscr{P}S/\mathscr{C}S \cong \mathscr{P}S^{ev}/[\text{point}]^{*2} * \mathscr{P}S^{ev} \cdot [\text{point}]^{*2} * \mathscr{P}S^{ev}$ *is torsion free and we have*:

$$\text{Tor}(\mathscr{P}S/\mathscr{C}S) \cong (\mathbf{Q}/\mathbf{Z}) \coprod \left(\coprod_{i>1} \text{Tor}(\mathscr{P}S^{2i}) \right) = (\mathbf{Q}/\mathbf{Z}) \coprod \text{Tor}(\mathscr{P}S).$$

The arguments can be easily supplied from the results in [24; Chap. 6]. We omit the details. The torsion $\mathbf{Q}/\mathbf{Z}$ comes from the fact that $(\mathscr{P}S/\mathscr{C}S)^2$ is isomorphic to $\mathbf{R}/\mathbf{Z}$ through the length map. This is the trivial torsion in $\mathscr{P}S/\mathscr{C}S$.

3. Extended Gauss–Bonnet map.

In [24; Chap. 8], we presented some vague ideas suggesting that it might be interesting to study the scissors congruence problem for the extended hyperbolic n-space $\bar{\mathscr{H}}^n = \mathscr{H}^n \cup \partial\mathscr{H}^n$ where the geometry of $\partial\mathscr{H}^n$ is that of the conformal geometry of $(n-1)$-sphere. We will now pursue some of these vague ideas. It should be noted that $\partial\mathscr{H}^n$ will not be considered by itself. In particular, when $n \leqq 3$, the multiple transitivity of the conformal group trivializes the scissors congruence problem on $\partial\mathscr{H}^n$ when $\partial\mathscr{H}^n$ is viewed as an $(n-1)$-dimensional space. Aside from this, the absence of an invariant metric makes it rather difficult to talk about geodesic $(n-1)$-simplices. However, $\partial\mathscr{H}^n$ can be used to construct an n-dimensional scissors congruence problem in the sense of Dupont [7]. More precisely, the simple subdivision process can be replaced by the homological subdivision process. This possibility will be considered elsewhere with Dupont. In the present section, we will have occasion to consider a natural homomorphic image sitting inside the scissors congruence group arising from $\bar{\mathscr{H}}^n$. According to Dupont, using his homological approach, the scissors congruence group $\mathscr{P}\mathscr{H}^n$ is actually isomorphic to the scissors congruence group $\mathscr{P}\bar{\mathscr{H}}^n$, $n \geqq 2$. However, there is a subtle difference between the geometric interpretations of these two groups. The difference is that Zylev's cancellation theorem holds for scissors congruence in $\mathscr{H}^n$, but fails for $\bar{\mathscr{H}}^n$ when $n \geqq 1$. For more details on this point, we refer to Appendix 1.

As indicated in the preceding section, $\mathscr{H}^n$ can be modelled either on the rays in $\mathbf{R}^{1,n}$ interior to the forward light cone or on an open n-ball. For the description of $\bar{\mathscr{H}}^n$, it is more convenient to use the ball model. A point of $\bar{\mathscr{H}}^n$ is identified with a vector $v \in \mathbf{R}^{1,n}$ satisfying the conditions:

$$\langle v, e_0 \rangle_{1+n} = 1 \quad \text{and} \quad \langle v, v \rangle_{1,n} \leqq 0 .$$

In this identification, v lies on $\partial\mathscr{H}^n$ if and only if $\langle v, v\rangle_{1,n} = 0$. As before, we allow n to go to ∞ and view everything as part of $\bar{\mathscr{H}}^\infty$. Points of $\partial\mathscr{H}^n$ are called infinite while points of $\mathscr{H}^n$ are called finite. A geoedesic n-simplex in $\bar{\mathscr{H}}^n$ is represented by an affine n-simplex contained in a closed n-ball. Except for vertices, all its points are finite. Such a simplex will be called i-asymptotic if exactly i of its vertices are on $\partial\mathscr{H}^n$. An $(n+1)$-asymptotic n-simplex will also be called totally asymptotic (or ideal by other authors). All such simplices have finite invariant volume. Simple subdivisions are defined as usual. In a proper simple subdivision, the division point must be finite. The group $\mathscr{P}\bar{\mathscr{H}}^n = \mathscr{P}(\bar{\mathscr{H}}^n, \Omega(1,n;\mathbf{R}))$ is defined in a manner completely analogous to that of $\mathscr{P}\bar{\mathscr{H}}^n$. We note that $\Omega(1,n;\mathbf{R})$ has a subgroup of index 2 preserving the orientation of $\mathscr{H}^n$. When $n>1$, this subgroup is transitive on both $\mathscr{H}^n$ and $\partial\mathscr{H}^n$.

Continuing with the ball model, fix the origin e_0 of $\mathscr{H}^n$. Each geodesic ray in $\bar{\mathscr{H}}^n$ starting from e_0 "terminates" at a unique point e_0+v in $\partial\mathscr{H}^n$ with v in $S(\mathbf{R}^n)$. In this manner, $S(\mathbf{R}^n)$ can be identified with the unit tangent sphere at e_0 as well as with $\partial\mathscr{H}^n$. It is important to note the identification with $\partial\mathscr{H}^n$ depends on the choice of the origin e_0. Two different choices differ by a conformal automorphism of $\partial\mathscr{H}^n$, not necessarily an isometry of $S(\mathbf{R}^n)$. For obvious reasons, we will also use "visual sphere at e_0" interchangeably with "unit tangent sphere at e_0". The isotropy group of e_0 in $\Omega(1, n; \mathbf{R})$ is just $O(n; \mathbf{R})$ so that the visual sphere at e_0 yields the spherical scissors congruence data in dimension $n-1$. This is the manner $\theta(F, A)$ is defined in the Gauss–Bonnet map of the preceding section. We note that an infinite geodesic can never be cut up into a finite number of pieces and placed inside $\mathscr{H}^n$, $n>0$. As a consequence, stable scissors congruence and scissors congruence are definitely not equivalent for $(\bar{\mathscr{H}}^n, \Omega(1,\mathrm{n};\mathbf{R}))$, $n>0$. In this respect, we have a more natural geometric example than the usual example of Hilbert based on non-Archimedean ordered fields.

Each n-simplex of $\mathscr{H}^n$ is also one of $\bar{\mathscr{H}}^n$, we therefore have the natural inclusion homomorphism:

$$\iota_n \colon \mathscr{P}\mathscr{H}^n \to \mathscr{P}\bar{\mathscr{H}}^n, \qquad n \geqq 0 .$$

ι_0 is an isomorphism of $\mathbf{Z}$ because $\mathscr{H}^0$ and $\bar{\mathscr{H}}^0$ reduce to points. We have:

PROPOSITION 3.1. *ι_1 is the zero map. $\mathscr{P}\bar{\mathscr{H}}^1$ is isomorphic to* **Z** *with the class of a* 1*-asymptotic* 1*-simplex (an infinite half line in $\bar{\mathscr{H}}^1$) as a generator. $\mathscr{P}\mathscr{H}^1$ is isomorphic to the multiplicative group* $\mathbf{R}^+$ *of positive real numbers.*

The straightforward proof will be omitted. We note that $\mathscr{H}^1$ can be identified with the connected component of $\Omega(1,1;\mathbf{R})$, hence with $\mathbf{R}^+$.

At an infinite point p_∞ (of $\partial\mathscr{H}^n$), we can speak of the collection of geodesics of $\mathscr{H}^n$ ending at p_∞ (or incident to p_∞, or emanating from p_∞). This collection is called the horosphere at p_∞ and its geometry is that of the similarity geometry in $(n-1)$-dimensional Euclidean space — $\mathbf{R}^n$ equipped with the group of all similarity maps (the semidirect product of $T(n-1;\mathbf{R})$ by the direct product of $O(n-1;\mathbf{R})$ and $\mathbf{R}^+\cdot I_{n-1}$). In the upper half space model for $\mathscr{H}^n$, p_∞ is placed at ∞ above the horizontal hyperplane (identified with $\partial\mathscr{H}^n-\{p_\infty\}$). The geodesics ending at p_∞ are the vertical half lines and the remaining geodesics are semicircles orthogonal to the horizontal hyperplane. The geometry of the horosphere is now faithfully reproduced on the horizontal hyperplane $\mathbf{R}^{n-1}$ with its similarity group of motions identified with the isotropy group of p_∞ in $\Omega(1,n;\mathbf{R})$. We now recall that $\mathscr{P}E^{n-1}$ is a direct sum of weight spaces for $\mathbf{R}^+$-weights i, $1\leqq i\leqq n-1$ and $n-1-i$ even. As a consequence, the scissors congruence group associated to the horosphere (i.e., $\mathbf{R}^{n-1}$ with the similarity group of motions) is 0 when $n>1$.

We now define the extension $\bar{\mathfrak{e}}$ of the Gauss–Bonnet map $\mathfrak{e}$. Let F be any nonempty face of an n-simplex A in $\bar{\mathscr{H}}^n$. When F is not an infinite vertex, the barycenter $x(F)$ lies in $\mathscr{H}^n$ and $\theta(F,A)$ makes perfectly good sense. When F is an infinite vertex, we define $\theta(F,A)$ to be 0. $\bar{\mathfrak{e}}(A)$ is then defined by formula (2.2). $\bar{\mathfrak{e}}$ evidently extends $\mathfrak{e}$ on n-simplices of $\mathscr{H}^n$. We note that $\theta(F,A)$ can be thought of as local curvature dara and summation plays the role of integration. The proof of Theorem 2.3 extends without difficulty to yield:

THEOREM 3.2. *The extended Gauss–Bonnet map $\bar{\mathfrak{e}}$ is an additive homomorphism and we have the following commutative diagram of maps:*

$$\begin{array}{ccc} \mathscr{P}\mathscr{H}^n & \xrightarrow{\iota_n} & \mathscr{P}\bar{\mathscr{H}}^n \\ & {\scriptstyle \mathfrak{e}}\searrow \quad \swarrow{\scriptstyle \bar{\mathfrak{e}}} & \\ & \mathscr{P}S^n = \mathscr{P}X^{n-1}(1) & \end{array}$$

When $n=0$, all three maps are isomorphisms (of **Z***). When $n=1$, all three maps are* 0.

PROPOSITION 3.3. *Let $n>1$. The map $\iota_n\colon \mathscr{P}\mathscr{H}^n\to\mathscr{P}\bar{\mathscr{H}}^n$ is surjective. For any fixed i, $0\leqq i\leqq n$, $\mathscr{P}\bar{\mathscr{H}}^n$ can be generated by the classes of all i-asymptotic n-simplices. In particular, $\bar{\mathfrak{e}}=0$ for odd n.*

PROOF. We first take care of the generation assertion for $1 \leqq i \leqq n$. Let A be any i-asymptotic n-simplex with $i<n$. A then has at least two finite vertices. Extend the edge joining two finite vertices (in either direction) to an infinite point and superdivide at this point. $[A]$ then becomes $[B]-[C]$, where B and C are $(i+1)$-asymptotic n-simplices. Conversely, repeated simple subdivisions at finite points lying on edges joining 2 infinite vertices imply that each i-asymptotic n-simplex with $i>1$ is an interior disjoint union of 1-asymptotic n-simplices.

The surjectivity of ι_n is equivalent with the generation assertion with $i=0$. For this, it suffices to show that the quotient group $\mathscr{P}\bar{\mathscr{H}}^n/\iota_n(\mathscr{P}\mathscr{H}^n)$ is 0. This quotient group can be generated by the cosets of the classes of 1-asymptotic n-simplices in $\bar{\mathscr{H}}^n$. Since $\Omega(1,n;\mathrm{R})$ is transitive on $\partial\mathscr{H}^n$, we can take the generating set of 1-asymptotic n-simplices to have the common infinite vertex p_∞. We now use the upper half space model for $\bar{\mathscr{H}}^n$ with the horizontal hyperplane identified with R^{n-1}. Let B be any Euclidean $(n-1)$-simplex with vertices $w_1,\ldots,w_n$ in R^{n-1}. On the vertical half lines starting from $w_1,\ldots,w_n$ (corresponding to geodesics ending at p_∞) select random points $v_1,\ldots,v_n$ at "finite" positive heights corresponding to points of $\mathscr{H}^n$. The hyperbolic convex closure of $p_\infty=v_0,v_1,\ldots,v_n$ is then a 1-asymptotic n-simplex B'. If we alter the choices of $v_1,\ldots,v_n$ (one at a time) to get B'', then $[B']-[B'']$ lies in $\iota_n(\mathscr{P}\mathscr{H}^n)$. It follows that the coset $[B']$ mod $\iota_n(\mathscr{P}\mathscr{H}^n)$ depends only on B and we have a well defined map from the set of all $(n-1)$-simplices in the Euclidean space R^{n-1} to the group $\mathscr{P}\bar{\mathscr{H}}^n/\iota_n(\mathscr{P}\mathscr{H}^n)$. This map is evidently additive with respect to simple subdivision and the image covers a set of generators for the group $\mathscr{P}\bar{\mathscr{H}}^n/\iota_n(\mathscr{P}\mathscr{H}^n)$ in accordance with the generation assertion of the preceding paragraph. Since the isotropy group of p_∞ in $\Omega(1,n;\mathrm{R})$ is just the group of all similarity transformations of R^{n-1}, our map is constant on similarity classes of $(n-1)$-simplices in R^{n-1}. As remarked earlier, the scissors congruence group based on similarity group of motions in R^{n-1} is 0 when $n>1$. This means that our map induces a surjective map from the 0 group to $\mathscr{P}\bar{\mathscr{H}}^n/\iota_n(\mathscr{P}\mathscr{H}^n)$. In other words, ι_n is surjective for $n>1$. The vanishing of $\bar{\mathrm{e}}$ for odd n now follows from the surjectivity of ι_n and the corresponding result for e in Theorem 2.6 (the case $n=1$ is trivial).

REMARK 3.4. Proposition 3.3 does not say that every polytope in $\bar{\mathscr{H}}^n$, $n>1$, is $\Omega(1,n;\mathrm{R})$-scissors congruent to a polytope in $\mathscr{H}^n$. It only asserts stable $\Omega(1,n;\mathrm{R})$-scissors congruence. The bijectivity of ι_n simply means that stable $\Omega(1,n;\mathrm{R})$-scissors congruence in $\bar{\mathscr{H}}^n$ is equivalent with $\Omega(1,n;\mathrm{R})$-scissors congruence in $\mathscr{H}^n$, $n>1$. In any event, the volume map is compatible with ι_n, $n>1$, so that no information is lost on the level of volume if we work with $\mathscr{P}\bar{\mathscr{H}}^n$

for $n>1$. This was the idea behind Gauss' proof of the defect formula for the area of a hyperbolic triangle.

It seems plausible that $\bar{\mathfrak{e}}$ is an isomorphism for n even. With this possibility in mind, we will carry out a simple geometric construction to get a "partial inverse" of $\bar{\mathfrak{e}}$. As it will become clear, this construction uses $\partial\mathscr{H}^n$ in an essential way.

We already noted that orthogonal join does not make sense in hyperbolic spaces. Nevertheless, geometric join can be formed through geodesic convexity. Recall that $S(\mathsf{R}^n)$ can be identified with the visual sphere at R^+e_0 of $\mathscr{H}^n$: the unit vector $x \in S(\mathsf{R}^n)$ is identified with $\mathsf{R}^+(e_0+x)$ or e_0+x in $\partial\mathscr{H}^n$. In general, let $A = \operatorname{sccl}\{x_1,\ldots,x_n\}$ be a spherical $(n-1)$-simplex on the visual sphere of R^+e_0. Define $\mathfrak{f}(A)$ to be the n-asymptotic n-simplex given by:

$$\begin{aligned}(3.5)\qquad \mathfrak{f}(A) &= \operatorname{hccl}\{\mathsf{R}^+e_0, \mathsf{R}^+(e_0+x_1),\ldots,\mathsf{R}^+(e_0+x_n)\}\\ &= \operatorname{ccl}\{e_0, e_0+x_1,\ldots,e_0+x_n\} \text{ in the ball model .}\end{aligned}$$

Evidently, $\mathfrak{f}(A)$ is the cone (geometric join) with apex R^+e_0 and base a totally asymptotic $(n-1)$-simplex: $\operatorname{hccl}\{\mathsf{R}^+(e_0+x_1),\ldots,\mathsf{R}^+(e_0+x_n)\}$. We already noted that the isotropy group of R^+e_0 in $\Omega(1,n;\mathsf{R})$ is $O(n;\mathsf{R})$ so that the construction $\mathfrak{f}$ is equivariant with the action of $O(n;\mathsf{R})$. It follows that we have a bijective correspondence between the $O(n;\mathsf{R})$-congruence classes of spherical $(n-1)$-simplices and the $\Omega(1,n;\mathsf{R})$-congruence classes of n-asymptotic n-simplices in $\bar{\mathscr{H}}^n$. However, $\mathfrak{f}$ is not additive with respect to simple subdivision. Suppose that $A = B\coprod C$ is a simple subdivision at the point $z=\alpha x_1 + \beta x_n \in S(\mathsf{R}^n)$, $n>1$, $\alpha,\beta \in \mathsf{R}^+$ and $\alpha+\beta>1$. Then:

(3.6)

$$\mathfrak{f}(A) \coprod \operatorname{hccl}\{\mathsf{R}^+(e_0+z), \mathsf{R}^+(e_0+x_1),\ldots,\mathsf{R}^+(e_0+x_n)\} = \mathfrak{f}(B) \coprod \mathfrak{f}(C)\,.$$

PROPOSITION 3.7. *Let $\mathscr{P}\bar{\mathscr{H}}^n_\infty$ be the subgroup of $\mathscr{P}\bar{\mathscr{H}}^n$ generated by the classes of all the totally asymptotic n-simplices in $\bar{\mathscr{H}}^n$. Then the extended Gauss–Bonnet homomorphism $\bar{\mathfrak{e}}\colon \mathscr{P}\bar{\mathscr{H}} \to \mathscr{P}S^n$ carries $\mathscr{P}\bar{\mathscr{H}}^n_\infty$ into* 2[point] $*\,\mathscr{P}S^{n-1}$ *and induces isomorphisms*:

$$\bar{\mathfrak{e}}\colon \mathscr{P}\bar{\mathscr{H}}^n/\mathscr{P}\bar{\mathscr{H}}^n_\infty \to \mathscr{P}S^n/2[\text{point}]*\mathscr{P}S^{n-1}, \qquad n>0\,.$$

In particular, $\mathscr{P}\bar{\mathscr{H}}/\mathscr{P}\bar{\mathscr{H}}_\infty \cong \mathscr{P}S/2\mathscr{C}S$ so that:

$$\mathscr{P}\bar{\mathscr{H}}^1/\mathscr{P}\bar{\mathscr{H}}^1_\infty \cong \mathsf{Z}/2\mathsf{Z} \quad\text{and}\quad \mathscr{P}\bar{\mathscr{H}}^{2i+1} = \mathscr{P}\bar{\mathscr{H}}^{2i+1}_\infty \quad \text{for } i>0;$$

$$\dim_{\mathsf{Q}} \mathsf{Q}\otimes(\mathscr{P}\bar{\mathscr{H}}^{2i}/\mathscr{P}\bar{\mathscr{H}}^{2i}_\infty) = |\mathsf{R}| \quad \text{for } i>0\,.$$

PROOF. Let A be any totally asymptotic n-simplex in $\bar{\mathscr{H}}^n$. For any vertex F of A, $\theta(F,A)=0$ so that

$$\bar{\mathfrak{e}}(A) \in [S(\mathsf{R})] * \mathscr{P}S^{n-1} = 2[\text{point}] * \mathscr{P}S^{n-1} .$$

As a consequence, $\bar{\mathfrak{e}}$ induces a homomorphism on the quotient groups as exhibited. The generation assertion of Proposition 3.3 shows that the induced homomorphism is surjective. The equivariance and (3.6) shows that $\mathfrak{f}$ induces a homomorphism:

$$(3.8) \qquad \bar{\mathfrak{f}}: \mathscr{P}S^n \to \mathscr{P}\bar{\mathscr{H}}^n/\mathscr{P}\bar{\mathscr{H}}^n_\infty .$$

Let $A = \mathrm{sccl}\{x_1,\ldots,x_n\}$ be any spherical $(n-1)$-simplex in $S(\mathsf{R}^n)$ and define $\tau_i A$ to be $\mathrm{sccl}\{x_1,\ldots,-x_i,\ldots,x_n\}$. We then have:

$$(3.9) \quad \mathfrak{f}(A) \coprod \mathfrak{f}(\tau_i A) = \mathrm{hccl}\{\mathsf{R}^+(e_0-x_i),\mathsf{R}^+(e_0+x_1),\ldots,\mathsf{R}^+(e_0+x_n)\} .$$

In fact, the left hand side of (3.9) results from a simple subdivision of the right hand side at the point R^+e_0. If x_1 is orthogonal to x_i for $i>1$, then A and $\tau_1 A$ are $O(n;\,\mathsf{R})$-congruent so that $\mathfrak{f}(A)$ and $\mathfrak{f}(\tau_1 A)$ are $\Omega(1,n;\,\mathsf{R})$-congruent. (3.8) and (3.9) together imply that we have an induced homomorphism:

$$\bar{\mathfrak{f}}: \mathscr{P}S^n/2[\text{point}] * \mathscr{P}S^{n-1} \to \mathscr{P}\bar{\mathscr{H}}^n/\mathscr{P}\bar{\mathscr{H}}^n_\infty .$$

Checking on the generating set of n-asymptotic n-simplices, $\mathfrak{f}\circ\bar{\mathfrak{e}} = \mathrm{Id}$. Since $\mathfrak{e}$ is surjective, it is an isomorphism with $\bar{\mathfrak{f}}$ as its inverse. The remaining assertions follow from the discussions in [24; Chapt. 6 and App. 2].

REMARK 3.10. The equality $\mathscr{P}\bar{\mathscr{H}}^{2i+1} = \mathscr{P}\bar{\mathscr{H}}^{2i+1}_\infty$, $i>0$, may be verified directly from Proposition 3.3 and (3.9) as in [24; Prop. 6.2.2, p. 105].

COROLLARY 3.11. *When* $n=2$, ι_2, $-\mathfrak{e}$ *and* $-\bar{\mathfrak{e}}$ *in Theorem* 3.2 *are isomorphisms of* R.

PROOF. If we compose $-\mathfrak{e}$ with the length map from $\mathscr{P}S^2$ to R, then we have the area map from $\mathscr{P}\mathscr{H}^2$ to R. The surjectivity is clear from continuity. The injectivity is just the classical theorem of Bolyai, see Moise [21; pp. 334–336]. Theorem 3.2 shows that the bijectivity of ι_2 is equivalent with the bijectivity of $-\bar{\mathfrak{e}}$. Proposition 3.7 reduces the bijectivity of $\bar{\mathfrak{e}}$ to the bijectivity of its restriction:

$$\bar{\mathfrak{e}}: \mathscr{P}\bar{\mathscr{H}}^2_\infty \to 2[\text{point}] * \mathscr{P}S^1 = \mathbf{Z}\cdot 2[\text{point}]^{*2} .$$

Since the action of $\Omega(1,2;\,\mathsf{R})$ on $\partial\mathscr{H}^2$ is just the action of PGL $(2,\mathsf{R})$ on $\boldsymbol{P}^1(\mathsf{R})$, this action is triply transitive. It follows that $\mathscr{P}\bar{\mathscr{H}}^2_\infty$ is a cyclic group generated by the class of any totally asymptotic 2-simplex. Direct calculation shows that the restriction is surjective. Since the image is infinite cyclic, we must have bijectivity.

4. Some Divisibility Results.

An examination of the major steps leading to the Dehn–Sydler and Jessen–Thorup theorems shows that divisibility of the relevant scissors congruence groups played a decisive role. From the divisibility followed the absence of torsion. As a result, the vector space approach became possible. For these cases, the divisibility depended on the existence of homotheties. These homotheties are not available in either the spherical or the hyperbolic case. As in the Euclidean case, the known Dehn invariants all have values in vector spaces of characteristic 0. Evidently, divisibility and torsion questions are connected with the determination of the image and the kernel of the Dehn invariants. We recall that $\mathscr{P}\mathscr{H}^n$ and $\mathscr{P}S^{n+1}$ are both 2-divisible when $n>0$, [24; Prop. 1.4.3, p. 17]. The same holds for $\mathscr{P}\bar{\mathscr{H}}^n$, $n>1$. In all these cases, the proof is based on the existence of an inscribed sphere in an n-simplex. Connected with this construction is an open question related to the possible existence of 2-torsion in $\mathscr{P}S^{n+1}$, $n\geqq 3$. We will have occasion to use this construction again and therefore will repeat the construction.

Let A be any n-simplex (Euclidean, spherical or extended hyperbolic) with ordered vertices $v_0,\ldots,v_n$. Let z be the center of the inscribed sphere. Let z_i be the foot of the perpendicular from z to the hyperplane spanned by the codimensional 1 face A_i of A opposite the vertex v_i of A, $0\leqq i\leqq n$. For $0\leqq i\neq j\leqq n$, let $P_{i,j}$ denote the n-simplex spanned by z, z_i and v_s, $s\neq i$ or j. The hyperplane reflection with respect to the hyperplane determined by z and $A_i\cap A_j$ therefore carries $P_{i,j}$ onto $P_{j,i}$. On the visual sphere $S(\mathsf{R}^n)$ at z, $P_{i,j}$ determines a spherical $(n-1)$-simplex $\theta_{i,j}$. Evidently, $S(\mathsf{R}^n)=\coprod_{i\neq j}\theta_{i,j}$ and $\theta_{i,j}$ is $O(n;\mathsf{R})$-congruent to $\theta_{j,i}$. It follows that

$$2\sum_{i<j}[\theta_{i,j}] = [S(\mathsf{R}^n)]\,.$$

PROBLEM:

(4.1) Is it true that $\coprod_{i<j}\theta_{i,j}$ is $O(n;\mathsf{R})$ scissors congruent to a hemisphere when $n\geqq 4$?

We already know that $\mathscr{P}S^n$ is torsion free when $n<4$ so that (4.1) has an affirmative answer for $n<4$. If (4.1) has a negative answer, then $\mathscr{P}S^n$ has 2-torsion.

In view of Theorem 2.6, we can limit ourselves to even n in (4.1). Related to this, we note the following result: (cf. Corollaries 2.8 and 2.9)

PROPOSITION 4.2. Tor $(\mathscr{P}S)$ is equal to the annihilator ideal of $[\text{point}]^{*2}$ in the commutative, graded ring $\mathscr{P}S$.

In [24; p. 128], we proposed some candidates for possible torsions in $\mathscr{PS}$ by using the fundamental domains of finite groups acting orthogonally on the sphere. Using his homological approach [7], Dupont has shown that these candidates can be dismissed in dimension 3 (corresponding to $n=4$). By using results from finite group theory, we have been able to eliminate these candidates for all n. The details will be given elsewhere in a joint work with Dupont. We note that (4.1) can be viewed as a question concerning the fundamental domain of a self scissors congruence of order 2. General questions can then be raised. They will be considered elsewhere. For the rest of this section, we concentrate on divisibility questions.

We recall that an n-simplex A with ordered vertices $v_0, \ldots, v_n$ is called orthogonal (or an orthoscheme) if for each i, $0<i<n$, the face spanned by $v_0, \ldots, v_i$ is orthogonal to the face spanned by $v_i, \ldots, v_n$. This definition makes sense for $n \geqq 2$ and for Euclidean, spherical, or extended hyperbolic spaces. We note that when A lies in $\bar{\mathscr{H}}^n$, then v_i, $1 \leqq i \leqq n-1$, must lie in $\mathscr{H}^n$. In general, the orthogonality of A determines the ordering of the vertices up to a complete reversal. The following result is wellknown:

PROPOSITION 4.3. *The scissors congruence groups can be generated by the classes of orthogonal simplices.*

We recall the argument. Let A be any n-simplex. Let $x(F)$ be a point on the i-dimensional hyperplane determined by the i-dimensional face F of A. Using $x(F)$ in place of the barycenter of F, we can carry out a fake barycentric subdivision of A. The class $[A]$ is then a sum and difference of the classes of the n-simplices in this fake barycentric subdivision (the class of a degenerate simplex is taken to be 0). We can keep track of the signs by using the ordering of the vertices and the orientation of the underlying space. In particular, the resulting $(n+1)!$ n-simplices in the fake barycentric subdivision may be labelled by the $(n+1)!$ ordering of the vertices of A. To be precise, for a permutation σ of $0, \ldots, n$, $(v_{\sigma(0)}, \ldots, v_{\sigma(n)})$ denotes the n-simplex (possibly degenerate) with vertices $x(F)$, F ranges over the faces of A having vertices $v_{\sigma(0)}, \ldots, v_{\sigma(i)}$, $0 \leqq i \leqq n$. If $x(A)$ is selected at random and $x(A_i)$ is taken to be the foot of the perpendicular from $x(A)$ to the hyperplane determined by the codimensional face A_i opposite v_i, then we can repeat this process with each A_i starting from $x(A_i)$. This construction will be called the orthogonal fake barycentric subdivision starting from $x(A)$. We note that $x(v_i)$ is always v_i. At the end of this orthogonal fake barycentric subdivision, the resulting n-simplices are either orthogonal or degenerate. Proposition 4.3 now follows easily. Orthogonal n-simplices can also be characterized as follows:

PROPOSITION 4.4. *Let A be an n-simplex with ordered vertices $v_0,\dots,v_n$. For $0<i<n$, let $F(i)$ and $B(i)$ be the faces of A determined respectively by $v_0,\dots,v_i$ and $v_i,\dots,v_n$. Then A is an orthogonal n-simplex in the given ordering if and only if:*

$$\theta_A(v_i) = \theta_{F(i)}(v_i) * \theta_{B(i)}(v_i), \qquad 0<i<n .$$

Here $*$ denotes the orthogonal join.

The proof is straightforward and is reduced to the case $n=3$. For $n=3$, it amounts to the statement that two "non-coplanar" geodesics have a unique common perpendicular. We omit further details.

THEOREM 4.5. *Let $n \geqq 3$. Then $\mathscr{P}\bar{\mathscr{H}}^n_\infty$ is 2-divisible.*

PROOF. For odd n, the assertion follows from Proposition 3.7. We now give a uniform proof using only $n \geqq 3$. Let A be any totally asymptotic n-simplex with ordered vertices $v_0,\dots,v_n$. Carry out the orthogonal fake barycentric subdivision from the vertex v_n. This yields $n!$ orthogonal 2-asymptotic n-simplices of the form: $(v_{\sigma(n)},\dots,v_{\sigma(n-1)},v_n)$, σ a permutation of $0,\dots,n-1$.

We next examine the step where we carry out the perpendicular construction from a fake barycenter of a 2-dimensional face of A to the three 1-dimensional edges. This step amounts to an orthogonal bisection leading to pairs of n-simplices of the form $(v_{\sigma(0)},v_{\sigma(1)},\dots)$ and $(v_{\sigma(1)},v_{\sigma(0)},\dots)$ where the undisplayed vertices are identical in the order given. It follows that these two n-simplices are congruent under a hyperplane reflection and that they appear with the same sign when $[A]$ is written as a sum and difference of $[(v_{\varrho(0)},\dots,v_{\varrho(n-1)},v_n)]$, ϱ a permutation of $0,\dots,n-1$.

From the description in the preceding paragraph, we can write $[A]$ as twice the sum and difference of $[(v_{\varrho(0)}, v_{\varrho(1)},\dots, v_{\varrho(n-1)}, v_n]$, ϱ a permutation of $0,\dots,n-1$ satisfying the restriction that $\varrho(0)<\varrho(1)$. Call this restricted sum z. We claim that z lies in $\mathscr{P}\bar{\mathscr{H}}^n_\infty$ when $n \geqq 3$. In view of Proposition 3.7, it is enough to show that $\mathfrak{e}(z)$ lies in $2[\text{point}] * \mathscr{P}S^{n-1}$. For this, we only need to compute the angle at the vertices. The vertices of the n-simplex $(v_{\varrho(0)},\dots,v_{\varrho(n-1)},v_n)$ are the fake barycenters and are finite except for $v_{\varrho(0)}$ and v_n. At an infinite vertex, the angle is defined to be 0. At a fake barycenter of an i-dimension face F of A, $1<i<n-1$, we can consider all the terms in z of the form $\pm[(v_{\varrho(0)},\dots,v_{\varrho(i)},\dots)]$ where $\varrho(j)$ for $i<j \leqq n-1$ are fixed. The sum of the vertex angles at this fake barycenter is therefore of the form:

$$[\theta] * [\theta'] ,$$

where $2[\theta] = [S(\mathsf{R}^i)]$ and $[\theta'] \in \mathscr{P}S^{n-i}$. Since $n-i>1$, $\mathscr{P}S^{n-i}$ is 2-divisible. It follows that the sum of the vertex angles lies in $2[\text{point}] * \mathscr{P}S^{n-1}$ as long as we are not at a fake barycenter of an i-dimensional face, $1 < i < n-1$. If $i = 1$ or $n-1$, the vertex angle for each individual n-simplex already lies in

$$[\text{point}] * \mathscr{P}S^{n-1} = 2[\text{point}] * \mathscr{P}S^{n-1}$$

because $\mathscr{P}S^{n-1}$ is 2-divisible when $n \geqq 3$. Finally, we started from v_n so that there is only one nondegenerate n-simplex at the fake barycenter $x(A)$. Since v_n is infinite, the vertex angle is 0. These arguments imply that $\mathfrak{e}(z)$ lies in 2[point] so that $z \in \mathscr{P}\bar{\mathscr{H}}_n^\infty$ and $[A] \in 2\mathscr{P}\bar{\mathscr{H}}_n^\infty$.

REMARK 4.6. The proof shows that the possible existence of 2-torsion posed by (4.1) is bypassed because of the presence of 2-divisibility. If (4.1) had an affirmative answer, then the preceding argument can be shortened through the inscribed sphere construction.

So far, our divisibility results are connected with the prime 2 and with the existence of hyperplane reflections. We next consider a special results. It is based on the study of the volume function in $\bar{\mathscr{H}}^3$ carried out by Lobatchevskii. A simpler treatment can be found in Milnor [18] or Thurston [29; Chap. 7].

Let $0 < \theta < \pi/2$ be identified with a spherical 1-simplex (an arc) in radian measurement. Define $\mathcal{L}(\theta) = [\mathfrak{f}(\{\text{point}\} * 2\theta)]$. Note $\mathfrak{f}(\{\text{point}\} * 2\theta)$ is a 3-asymptotic 3-simplex so that it and its mirror image with respect to the base: $\mathfrak{f}(2\theta)$ together form a totally asymptotic 3-simplex having the dihedral angles 2θ, $\pi/2 - \theta$, $\pi/2 - \theta$ along any three incident edges. We now extend $\mathcal{L}$ to a map:

(4.7) $\mathcal{L}\colon \mathsf{R} \to \mathscr{P}\bar{\mathscr{H}}^3$, $\mathcal{L}(0) = \mathcal{L}(\pi/2) = 0$, $\mathcal{L}$ is odd and periodic with period π.

Let $A(\alpha, \beta, \gamma)$ be a totally asymptotic 3-simplex with dihedral angles α, β, γ so that $0 < \alpha, \beta, \gamma$ and $\alpha + \beta + \gamma = \pi$. We recall the important fact that opposite dihedral angles in a totally asymptotic 3-simplex are equal so that the parameters: α, β, γ do not depend on the choice if an infinite vertex. Constructing a perpendicular from any vertex of $A(\alpha, \beta, \gamma)$ to the plane of the opposite face shows that the choice of (4.7) yields:

$$[A(\alpha, \beta, \gamma)] = \mathcal{L}(\alpha) + \mathcal{L}(\beta) + \mathcal{L}(\gamma) \,. \tag{4.8}$$

We already know that $\mathscr{P}\bar{\mathscr{H}}^3$ can be generated by totally asymptotic 3-simplices. (4.8) shows that $\mathcal{L}(\theta)$, $0 < \theta < \pi/2$, also form a set of generators. Divisibility questions concerning $\mathscr{P}\bar{\mathscr{H}}^3$ can be reduced to corresponding questions about $\mathcal{L}(\theta)$.

The volume of $\mathcal{L}(\theta)$, $0 < \theta < \pi/2$, is given by the formula:

$$\text{vol. } \mathcal{L}(\theta) = -\int_0^\theta \log|2 \sin t|\, dt \,. \tag{4.9}$$

The volume of $\mathcal{L}(\theta)$ is therefore a periodic function with period π from $\mathbf{R}$ to $\mathbf{R}$. It is known to satisfy the distribution relations:

$$f(n\theta) = n \sum_{0 \leq j < |n|} f(\theta + j\pi/n) . \tag{4.10}$$

The validity of this relation for the volume of $\mathcal{L}(\theta)$ can be checked by factorizing $Z^n - 1$ first and followed by taking logarithm of the absolute value of the imaginary part with $Z = \exp(-2\imath\theta)$. We note that if (4.10) holds for the group valued function $\mathcal{L}$, then the divisibility of $\mathcal{P}\bar{\mathcal{H}}^3$ would follow as a corollary. A formal exercise based on the unique factorization theorem in $\mathbf{Z}$ shows that the validity for arbitrary n in $\mathbf{Z}$ is a consequence of (4.10) for n ranging over primes as long as f is odd (corresponding to $n = -1$) and periodic with period π. For a general discussion on distribution relations, see Lang [17]. In view of the definition given in (4.7), the verification of (4.10) can always be reduced to the case where $0 < \theta < \pi/2n$.

PROPOSITION 4.11. *Let* $0 < \theta < \pi/4$. *Then* $2\mathcal{L}(\theta) = \mathcal{L}(2\theta) + 2\mathcal{L}(\frac{1}{2}\pi - \theta)$. *In particular,* $\mathcal{L}$ *satisfies the distribution relation* (4.10) *for* $n = 2$.

PROOF. Reflecting $\mathfrak{f}(\{\text{point}\} * (\pi - 2\theta))$ about the base $\mathfrak{f}(\pi - 2\theta)$ shows that $2\mathcal{L}(\frac{1}{2}\pi - \theta) = [A(\pi - 2\theta, \theta, \theta)]$. The desired equation follows from (4.7) and (4.8).

PROPOSITION 4.12. $\mathcal{L}$ *satisfies the distribution relation* (4.10) *for* $n = 3$. *In particular,* $\mathcal{P}\bar{\mathcal{H}}^3$ *is* 3-*divisible.*

PROOF. We adapt a construction of Thurston [29; Chap. 7]. We begin with an isosocles triangle in $\mathcal{H}^2$ with apex angle $2\pi/3$. By rotation, three of these then make up a regular hyperbolic triangle. Embed $\mathcal{H}^2$ in $\bar{\mathcal{H}}^3$ and construct orthogonal geodesics in both directions from the vertices of our regular hyperbolic triangle until we reach $\partial\mathcal{H}^3$. The convex closure of these 6 infinite vertices is then an "orthogonal" prism with a finite regular triangle for its "midsection". If the isosocles triangle had base angles equal to θ (we must have $0 < \theta < \pi/6$), then the dihedral angles as seen from any of the six vertices of the prism are: 2θ, α, α with $\alpha = \frac{1}{2}\pi - \theta$. Here 2θ is the interior dihedral angle along the three vertical edges of the prism while α is the interior dihedral angle along the six base edges of the prism. Construct the orthogonal geodesic through the center of the regular triangle (the apex of our isosocles triangles) and let this geodesic meet the top and bottom bases of our prism at p^+ and p^- respectively. Extend this geodesic until it meets $\partial\mathcal{H}^3$ at q^+ and q^- respectively. Let v_i, $1 \leq i \leq 3$, denote the finite vertices of our regular triangle and let v_i^+ and v_i^- denote the corresponding infinite vertices of our prism. If P denotes our prism and $Q = \text{hccl}\{p^+, p^-, v_1^+, v_1^-, v_2^+, v_2^-\}$, then rotational symmetric yields:

$$[P] = 3[Q]. \tag{4.13}$$

We next note that $R^+ = \text{hccl}\{q^+, p^+, v_1^+, v_2^+\}$ is congruent to $R^- = \{q^+, p^-, v_1^-, v_2^-\}$. Indeed, moving p^+, p^- in turn to $\mathbf{R}^+ e_0$, both R^+ and R^- are equal to 3-asymptotic 3-simplices of the form: $\mathfrak{f}(\{\text{point}\} * 2\pi/3)$. It is now evident that:

$$Q \coprod R^+ = R^- \coprod \text{hccl}\{q^+, v_1^+, v_2^+, v_1^-, v_2^-\}, \tag{4.14}$$

$\text{hccl}\{q^+, v_1^+, v_2^+, v_1^-, v^{-2}\}$ is just an infinite cone with apex q^+ and a "rectangular" base. The dihedral angles along the 4 base edges are: θ, $\alpha + \frac{1}{3}\pi$, θ, and $\alpha - \frac{1}{3}\pi$. In general, (4.8) can be extended to infinite cones provided that all the angles are interpreted as the interior dihedral angles along the edges at the base. The verification proceeds in the same manner: construct a perpendicular from the apex of the cone to the hyperplane determined by the base. The definition of $\mathcal{L}$ together with the basic property that opposite dihedral angles of a totally asymptotic 3-simplex are equal take care of the rest. As a consequence, we have:

$$[\text{hccl}\{q^+, v_1^+, v_2^+, v_1^-, v_2^-\}] = 2\mathcal{L}(\theta) + \mathcal{L}(\alpha + \tfrac{1}{3}\pi) + \mathcal{L}(\alpha - \tfrac{1}{3}\pi). \tag{4.15}$$

Since $2\alpha + 2\theta = \pi$, we can combine Proposition 4.11, (4.7), (4.13)–(4.15) to get:

(4.16)

$$3\left(\mathcal{L}(\alpha) + \mathcal{L}\left(\alpha + \frac{\pi}{3}\right) + \mathcal{L}\left(\alpha + \frac{2\pi}{3}\right)\right) = [P] + 3\mathcal{L}(\alpha) - 6\mathcal{L}(\theta) = [P] + 3\mathcal{L}(2\alpha) - 3\mathcal{L}(\alpha).$$

Viewing P as an abstract prism (i.e., a "product" of a 1-simplex with a 2-simplex), we can decompose P into a disjoint union of 3 totally asymptotic 3-simplices. Because of the symmetry, this can be carried out without losing track of the dihedral angles. (4.7) can be used to yield:

$$[P] = 3\mathcal{L}(2\theta) + 3\mathcal{L}(\alpha) + \mathcal{L}(\alpha - 2\theta). \tag{4.17}$$

Using (4.7), (4.16), (4.17), and the relation $2\alpha + 2\theta = \pi$, we have:

$$3\left(\mathcal{L}(\alpha) + \mathcal{L}\left(\alpha + \frac{\pi}{3}\right) + \mathcal{L}\left(\alpha + \frac{2\pi}{3}\right)\right) = \mathcal{L}(\alpha - 2\theta) = \mathcal{L}(3\alpha). \tag{4.18}$$

Since $0 < \theta < \pi/6$, we have $0 < \alpha < \pi/3$.

REMARK 4.19. Thurston's construction can be carried out with n in place of 3. However, the argument we used does not appear to generalize immediately. In several places, we have used the fact tha $3-1=2$. As mentioned earlier, Dupont had shown that $\mathscr{P}\bar{\mathscr{H}}^n$ and $\mathscr{P}\mathscr{H}^n$ are isomorphic for $n \geqq 3$. Using this and a continuity argument, Dupont obtained a Dehn invariant for $\mathcal{L}(\theta)$:

$$\text{(4.20)} \qquad \text{Dehn inv.}\ (\mathcal{L}(\theta)) = \log|\sin 2\theta| \otimes \theta\ .$$

Note the striking similarity between (4.20) and (4.9). By exactly the same reasoning, the Dehn invariant of $\mathcal{L}(\theta)$ also satisfies the distribution relations (4.10). Dupont's formula was already known (but not written down) to Thurston. Thurston's formulation is geometric and can be extended to higher dimensions with the help of the vanishing of $\mathfrak{e}$ for Euclidean spaces. Dupont's formula (4.20) connects up to the scissors congruence group $\mathscr{P}\bar{\mathscr{H}}^3$ and the algebraic K-group $K_2(\mathbf{C})^-$, see Sah–Wagoner [25], and answers a question raised by Dennis Sullivan, see [24; p. 148] and Dupont [7]. A short exposition of Thurston's definition of Dehn invariants can be found in Appendix 2.

REMARK 4.20. Most of the results in the present paper can be extended with minor modifications for spaces based on an ordered, square root closed fields, see [24]. When volume is needed, we need to impose the Archimedean axiom on the underlying field.

Appendix 1. Axiomatic Scissors Congruences.

Geometric scissors congruence data consist of a set X (usually nonempty) and a specified family of distinguished subsets (usually nonempty) of X called cells (or n-cells when n is understood to be the dimension in a suitable sense). With these primitive data, we define the concept of interior disjoint union:

Two cells A and B are said to be interior disjoint or to form an interior disjoint union when the following conditions hold:

(D1) $A \cap B$ contains no nonempty cells; and
(D2) If C is a cell contained in $A \cup B$, then $C \subset A$ if and only if $C \cap B$ contains no nonempty cells.

A finite union of pairwise interior disjoint cells will be called a polytope. Replacing the word cell by polytope throughout (D1) and (D2) yields the definition of interior disjoint union of polytopes. The empty set is allowed to be a polytope. $\coprod$ will be used to denote interior disjoint union. If $P=\coprod_i P_i$ with P_i denoting cells (respectively polytopes), then the finite collection of P_i's will be called a cell (respectively polytope) decomposition of P.

If $A=B \coprod C$ with A, B, C denoting cells, then we say A is simply subdivided into B and C, or that B and C form a simple pasting of A. If $P=\coprod_i P_i$ is a cell decomposition, then a simple subdivision (respectively pasting) of P is understood to be one involving one (respectively two) of the finite number of cells displayed. Any finite iteration of simple subdivisions (respectively pastings) beginning with $P=\coprod_i P_i$ is called a subdivision (respectively pasting) of $P=\coprod_i P_i$. A finite iteration of both subdivisions and pastings will be called a cut and paste. The cut and paste process evidently defines an equivalence relation among cell decompositions $P=\coprod_i P_i$ of any given polytope P. In order to assure ourselves an ample supply of small cells to carry out cut and paste processes, we impose the axiom:

(CP) Let $P=\coprod_i P_i$ and $Q=\coprod_j Q_j$ be cell decompositions. Then there exist cut and paste processes leading to cell decompositions $P=\coprod_u R_u$, $Q=\coprod_v S_v$ such that each R_u (respectively S_v) is either interior disjoint from Q (respectively P) or coincides with one of the S_v's (respectively) R_u's).

In the classical case of Euclidean, spherical, or hyperbolic spaces, we can interprete cells to mean convex closures of finite number of points having geometric dimension n. With the usual interpretation of interior disjoint union, axiom (CP) evidently holds. However, if we interprete cells to mean geodesic n-simplices, the validity of (CP) is no longer completely obvious. I am indebted to Robert Connolly for pointing this out and for informing me that a replacement of the simple subdivision process by a stellar subdivision process leads to difficult problems. However, the following unpublished "folklore" result was communicated to us by Thorup:

THEOREM. *Let X be the underlying space of one of the classical geometries of dimension n. Let cells on X be interpreted to mean geodesic n-simplices of dimension n having diameters strictly less than the diameter of X.* (CP) *then holds with the usual interpretation of interior disjoint union.*

A more homological proof of this result can be found in Dupont [7]. Thorup's proof is based on central projection from a vertex and works for any ordered, square root closed field k in place of $\mathbf{R}$. Indeed, the cut and paste processes in (CP) can even be replaced by subdivisions.

In order to speak of geometric scissors congruence, we specify a group G of bijective maps (called motions) of X. We then impose the axiom:

(C) The set of cells on X is closed under the action induced by G.

Axiom (C) implies the compatibility of the action of G with the concept of interior disjoint union. These axioms complete the geometric G-scissors congruence data and we abbreviate our data to (X, G).

A G-scissor between polytopes P and Q (written $P\sim_G Q$, or $P\sim Q$ mod G, or simply $P \sim Q$ when there is no chance of confusion) consists of cell decompositions: $P = \coprod_i P_i$, $Q = \coprod_i Q_i$, $1 \leqq i \leqq t$, so that P_i and Q_i are G-congruent for each i. Axiom (C) shows that the concept of G-scissors congruence between P and Q does not depend on the particular choices of cell decompositions of P and Q. The geometric G-scissors congruence problem is:

> Find reasonable necessary and sufficient conditions for two polytopes P and Q on X to be G-scissors congruent.

With the idea of volume as a guide, necessary conditions usually appear in the form of G-invariant Jessen functionals. These are abelian group valued functions on cells that are additive with respect to simple subdivisions and are invariant under the action of G. In general, these functionals can not be expected to solve the G-scissors congruence problem. They lead to the more complicated concept of stable G-scissors congruence. This is the equivalence relation on polytopes generated by the relation:

> The polytopes P and Q are G-scissors congruent by adjunction if there exist polytopes R and S interior disjoint from P and Q respectively so that: $R \sim_G S$ and $(P \coprod R) \sim_G (Q \coprod S)$.

We will use $\overset{s}{\sim}_G$ to signify stable G-scissors congruence. It is evident that G-invariant Jessen functionals are constant on stable G-scissors congruence classes. Using empty polytopes, G-scissors congruence implies stable G-scissors congruence. Note that the space X may not be large enough so that the relation of G-scissors congruence by adjunction is not necessarily transitive. However, in classical cases, we can use Jordan approximation to deduce the desired transitivity so that stable G-scissors congruence then becomes the same as G-scissors congruence by adjunction. For the same reason, neither $\sim_G$ nor $\overset{s}{\sim}_G$ necessarily lead to well defined operation of sum on the set of equivalence classes.

In general, an equivalence relation $\equiv$ on the set of polytopes of X is said to be cancellative or subtractive if it satisfies:

> If $P = P' \coprod P''$ and $Q = Q' \coprod Q''$ are polytope decompositions, then $P \equiv Q$ and $P' \equiv Q'$ together imply $P'' \equiv Q''$.

Neither $\overset{s}{\sim}_G$ nor $\sim_G$ is subtractive in general. In fact, $\sim_G$ is subtractive precisely when $\sim_G$ and $\overset{s}{\sim}_G$ coincide. Sufficient conditions for $\sim_G$ to be identical with $\overset{s}{\sim}_G$ are supplied by a theorem of Zylev, see [24; Thm. 1.3.1, p. 6]. In particular, these conditions are satisfied when we deal with classical scissors congruences. However, it was noted earlier that $\sim_G$ and $\overset{s}{\sim}_G$ are distinct concepts in the case of extended hyperbolic spaces.

The algebraic G-scissors congruence problem is the determination of all abelian group valued functions on cells so that they are G-invariant and that they are additive with respect to simple subdivisions. Since the target group is not specified in advance, this problem has to be understood in the sense of a Grothendieck group. To be precise, we form a universal group out of the polytopes.

We begin by ignoring the group G of motions. The group $\mathscr{P}(X,\{1\})$ of polytopes (on X) is defined to be the quotient group of the free abelian group based on cells modulo the subgroup generated by the relators:

(RS) $A-B-C$, with $A=B\coprod C$ range over simple subdivisions.

The coset of A will be denoted by $[A]$. We then let $\mathscr{P}(X, G)$ be the quotient group of $\mathscr{P}(X,\{1\})$ modulo the subgroup generated by the relators:

(RC) $[\sigma A]-[A]$, σ ranges over G and A ranges over cells.

When there is no chance of confusion, the coset of $[A] \bmod G$ in $\mathscr{P}(X, G)$ will be abbreviated to $[A]$. In general, if $P \overset{s}{\sim}_G Q$, then $[P]=[Q] \bmod G$. By an abuse of language, $\mathscr{P}(X, G)$ will be called the group of G-scissors congruence classes of polytopes on X.

The proof of the following result is not difficult but is tedious and will be omitted.

THEOREM. *Assume the scissors congruence data* (X, G) *is such that* $\sim_G$ *is subtractive. Then* $P\sim_G Q$ *holds for polytopes* P *and* Q *on* X *if and only if* $[P]=[Q] \bmod G$ *holds in* $\mathscr{P}(X, G)$.

We note that the main difficulty lies with the fact that $n[P]$ does not have a natural geometric interpretation for an arbitrary integer n. We need a bookkeeping device to construct a G-scissors congruence out of the assumption that $[P]=[Q] \bmod G$ in the abelian group $\mathscr{P}(X, G)$. This can not be done purely algebraically because the sum $[R]+[S]$ is not necessarily the class of a polytope T unless X is large enough with respect to the action of G.

We conclude this appendix with some remarks. First of all, the present axiomatization is intended to replace the inadequate axiomatization of [24]. Secondly, in the classical cases, Dupont [7] has given an equivalent axiomatization which is more homological. More precisely, the generators for $\mathscr{P}(X,\{1\})$ are taken to be the abstract n-simplices of diameter less than the diameter of X; the relators (RS) are replaced by:

(RD) $(a_0,\ldots,a_n)=0$ if $a_0,\ldots,a_n$ lie on a geodesic subspace of dimension less than n;

(RB) $\sum_i (-1)^i(a_0,\ldots,\hat{a}_i,\ldots,a_{n+1})=0$, where $(a_0,\ldots,a_{n+1})$ ranges over all small abstract $(n+1)$-simplices of X, i.e., $a_0,\ldots,a_{n+1}$ are any $n+2$ points of X such that the distance between a_i and a_j is less than the diameter of X.

The relation (RC) is modified to take into account the orientation:

(RC)° $(\sigma a_0,\ldots,\sigma a_n)=\det(\sigma)\cdot(a_0,\ldots,a_n)$, σ in G and $(a_0,\ldots,a_n)$ ranges over all small abstract n-simplices.

In the extended hyperbolic spaces, Dupont's scissors congruence group may differ a bit from the one used by us. Both of these are closely related to the scissors congruence groups considered by Thurston in unpublished works.

Appendix 2. Extended Dehn invariants.

The degeneracy of the map $\iota_1: \mathscr{P}\mathscr{H}^1 \to \mathscr{P}\bar{\mathscr{H}}^1$ prevents us from doing the obvious in trying to extend the definition of Dehn invariant to extended hyperbolic spaces. Actually, there is no problem as long as we avoid this case. In particular, there is no problem for $\mathscr{P}\bar{\mathscr{H}}^{2i}$ or for $\mathscr{P}\bar{\mathscr{H}}^{2i+1}$ provided that we avoid the codimensional $2i$ case. As mentioned earlier, Dupont found a definition extending the old definition for $\bar{\mathscr{H}}^3$. In an oral communication, Milnor told us of an earlier definition due to Thurston. We will now describe Thurston's definition and generalize it to $\bar{\mathscr{H}}^{2i+1}$ by using the Gauss–Bonnet map.

Let A be any $(2i+1)$-simplex in $\bar{\mathscr{H}}^{2i+1}$. For each infinite vertex v of A, delete from A a small horoball about v. Arrange the choice so that these horoballs are disjoint. Let A' be the compact set left over from A after this deletion process. A' then has a finite number of geodesic edges of finite length in bijective correspondence with those of A. A' also has a finite (possibly empty) set of horo-edges. If F' denotes one of the geodesic edges of A' lying on the geodesic edge F of A, then we have:

$$\theta_{A'}(F') = \theta_A(F).$$

We define the codimensional $2i$ Dehn invariant $H\Psi^{(2i)}$ of A by:

$$H\Psi^{(2i)}(A) = \sum_F [F']\otimes\overline{[\theta_A(F)]} \in \mathscr{P}\mathscr{H}^1\otimes(\mathscr{P}S^{2i}/\mathscr{C}S^{2i}).$$

If A lies in ω^{2i+1}, this is just the old definition, see [24; p. 157]. If A has infinite vertices, we must show that the definition is independent of the choices of the horoballs. To see this, we use the upper half space model and place the infinite vertex v at p_∞. Changing the horoball at p_∞ modifies A' by a "prism". This prism can be viewed as the orthogonal product of a 1-simplex with a Euclidean

$(n-1)$-simplex $B(v)$ lying on the horosphere of v. The ambiguity in the computations of $H\bar{\Psi}^{(2i)}(A)$ then has the form:

$$[L]\otimes\sum_{w}\overline{[\theta_{B(v)}(w)]}\ ,$$

where w ranges over the vertices of $B(v)$ and L is the hyperbolic 1-simplex orthogonal to and bounded by the two horospheres at v. The sum in the right hand factor lies in $\mathscr{P}S^{2i}/\mathscr{C}S^{2i}$ and is therefore equal to $\overline{\mathfrak{e}(B(v))}$. $B(v)$ is Euclidean, therefore $\mathfrak{e}(B(v))=0$ by Theorem 2.3. Since the horoballs can be changed one at a time, $H\bar{\Psi}^{(2i)}(A)$ is well defined. This definition is evidently invariant under isometries and is additive with respect to simple subdivisions (necessarily at finite points). It follows that we have an additive homomorphism:

$$H\bar{\Psi}^{(2i)}\colon \mathscr{P}\bar{\mathscr{H}}^{(2i+1)} \to \mathscr{P}\mathscr{H}^{1}\otimes(\mathscr{P}S^{2i}/\mathscr{C}S^{2i});\quad \text{and}$$
$$H\bar{\Psi}^{(2i)}\circ\iota_{2i+1} = H\Psi^{(2i)},\quad i>0\ .$$

In passing, we note that, for dimension $2i$, $\mathfrak{e}$ (and $\bar{\mathfrak{e}}$) can be viewed as a lifting of the codimension $2i$ Dehn invariant for either the spherical or the hyperbolic (respectively the extended hyperbolic) spaces. To be more precise, the codimensional $2i$ Dehn invariant can be recovered from the Gauss–Bonnet map through the composition with the natural projection from $\mathscr{P}S^{2i}$ to $\mathscr{P}S^{2i}/\mathscr{C}S^{2i}$. It should be noted that codimensional $2i$ Dehn invariant for $2i$ dimensional spaces was not defined in [24] because the analogous invariant for Euclidean spaces is always 0.

We note also that $\bar{\mathfrak{e}}(A)$ can be computed for a totally asymptotic 4-simplex A. Using essentially the observation that $\mathfrak{e}(B(v))=0$ for each vertex v of A, we have the following formula:

$$\bar{\mathfrak{e}}(A) = 2[\text{point}]^{*2}*\{2[S(\mathbf{R}^2)-\sum_{\dim F=2}[\theta_A(F)]\}\ ,$$

where F ranges over the codimensional 2 faces of A. Except for a constant of normalization, this formula was found by Thurston on the level of volume:

$\mathrm{vol}_4(A)=\frac{1}{3}\pi(4\pi-\text{sum of interior dihedral angels of } A)$,
A any totally asymptotic 4-simplex.

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DEPARTMENT OF MATHEMATICS
STATE UNIVERSITY OF NEW YORK
STONY BROOK, N.Y. 11794, U.S.A.